

# Time Inconsistency and Reputation in Monetary Policy: A Strategic Modeling in Continuous Time\*

Jingyuan Li

Guoqiang Tian

Department of Economics

Department of Economics

Texas A&M University

Texas A&M University

College Station, Texas 77843

College Station, Texas 77843

## Abstract

This paper develops a reputation strategic model of monetary policy with a continuous finite or infinite time horizon. By using the optimal stopping theory and introducing the notions of sequentially weak and strong rational equilibria, we give the conditions under which the time inconsistency problem may be solved with trigger reputation strategies not only for stochastic but also for nonstochastic settings even with a finite horizon. We provide the conditions for the existence of stationary sequentially strong rational equilibrium, and also completely characterize the existence of stationary sequentially weak rational equilibrium. We show that, under the assumption of the public's weak rational expectation or the certainty setting, the government will keep the inflation at zero if and only if  $a(1 - \theta) < 2$ . This inequality is satisfied if the rate of the aggregate output gain from the unanticipated inflation,  $a$ , is small (less than 2) or the government puts more weight on stabilizing output than on stabilizing inflation ( $\theta > 1$ ). Furthermore, we investigate the robustness of the sequentially strong rational equilibrium behavior solution by showing that the imposed assumption is reasonable.

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# 1 Introduction

Time inconsistency is an interesting problem in macroeconomics in general, and monetary policy in particular. Although technologies, preferences, and information are the same at different time, the policymaker's optimal policy chosen at time  $t_1$  differs from the optimal policy for  $t_1$  chosen at  $t_0 < t_1$ . The study of time inconsistency is important. It not only provides positive theories that help us to understand the incentives faced by policymakers and provide the natural starting point for attempts to explain the actual behavior of policymakers and actual policy outcomes, but also require one to design policy-making institutions. Such a normative task can help one understand how institutional structures affect policy outcomes.

This problem was first noted by Kydland and Prescott (1977). Several solutions have been proposed to deal with this problem since then. Barro and Gordon (1983) were the first to build a game model to analyze "reputation" of monetary policy.<sup>1</sup> The second solution is based on the incentive contracting approach to monetary policy. Persson and Tabellini (1993), Walsh (1995) and Svensson (1997) developed models using this approach. The third solution is built on the legislative approach. The major academic contribution in this area was by Rogoff (1985).

Among these approaches, the "reputation" problem is key. If reputation consideration discourages the monetary authorities from attempting surprise inflation, then legal or contracting constraints on monetary authorities are unnecessary and may be harmful.

The main questions on reputation are when and how the government chooses inflation optimally to minimize welfare loss, and, whether the punishment can induce the government to choose zero inflation. The conclusions of Barro-Gorden models are: First, there exists a zero-inflation Nash equilibrium if the punishment for the government deviating from zero-inflation is large enough. However, this equilibrium is not sequentially rational over a finite time horizon. The only sequentially rational equilibrium is achieved if the government chooses discretionary inflation and the public expects it. Only over an infinite time horizon can one get a low-inflation equilibrium. Otherwise, the government would be sure in the last period to produce the discretionary outcome whatever the public's expectation were and, working backward, would be expected to do the same in the first period. Second, there are multiple Nash equilibria and there

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<sup>1</sup>Backus and Driffill (1983) extended the work of Barro and Gordon to a situation in which the public is uncertain about the preferences of the government. Persson and Tabellini (1990) gave an excellent summarization of these models. Al-Nowaihi and Levine (1994) discussed reputation equilibrium in the Barro-Gordon monetary policy game.

is no mechanism to choose between them.

Finite time horizon games are more reasonable than infinite ones in the real world. We can say that every government's lifetime is finite. Many experimental studies of games suggest that there are cooperation equilibrium when the players are told that the game will end. Consequently, how to induce cooperative behavior in a finitely repeated game is an interesting problem even for game theorists.

In this paper, we use the optimal stopping theory in the stochastic differential equations literature to study the time inconsistency problem in monetary policy with the continuous finite or infinite time horizon model. The optimal stopping theory can cover many dynamic economic applications under uncertainty. The optimal stopping theory, though relatively complete in its theoretical development, has not yet been widely applied in economics. By using the optimal stopping theory and introducing the notions of sequentially weak and strong rational equilibria, we give the conditions under which the time inconsistency problem may be solved with trigger reputation strategies within our setting not only for stochastic but also for nonstochastic settings even with a finite horizon. We provide the conditions for the existence of stationary sequentially strong rational equilibrium, and further completely characterize the existence of stationary sequentially weak rational equilibrium. We show that the government will keep the inflation at zero if and only if  $a(1 - \theta) < 2$ .

As a result, the government will keep inflation at zero if the rate of the output gain from the unanticipated inflation is small ( $a < 2$ ) or the government puts more weight on stabilizing output than on stabilizing inflation. It will act opportunistically if the rate of aggregate output gain from the unanticipated inflation is high or the government puts less weight on stabilizing output than on stabilizing inflation, and then be expected to behave (and will behave) accordingly in every succeeding period. Thus, the objective rate of the output gain from the unanticipated inflation and the subjective preferential policy on the relative importance between stabilizing output and stabilizing inflation can be used together to determine whether or not the government keeps inflation at zero.

The results obtained in the paper are sharply contrasted to the negative results from the certainty setting with a discrete time horizon. Our results on the existence of the stationary zero inflation policy as an equilibrium solution are also true for the nonstochastic continuous finite horizon settings, which demonstrate the advantage of our continuous time model compared to

the nonstochastic discrete time finite horizon model discussed in the literature. As we mentioned above, in the certainty setting with discrete time, a reputational equilibrium is possible only if the horizon is infinite. Thus, a striking advantage of using a continuous-time formulation is that it yields a solution to the time inconsistency problem whereas a discrete-time counterpart does not. Why does the much more complicated continuous-time formulation yield a positive result that the discrete-time formulation could not? Intuitively speaking, it is because, in continuous time, the government has an option to change a policy any intermediate periods while, in the discrete-time formulation, the government can change a policy only in each stopped subinterval, and thus the solution to the continuous-time formulations can be viewed as the sum of the solutions to the discrete-time formulations for many small stopped subintervals. Thus, the embedded option in continuous-time formulation may appear to explain why the continuous-time formulation can yield a solution to the time inconsistency while the discrete-time versions in the existing literature fails.<sup>2</sup>

We also investigate the robustness of the equilibrium behavior by showing that the imposed assumption is reasonable. As long as the inequality  $a(1 - \theta) < 2$  holds, we can expect a stationary zero-inflation outcome by the sequentially strong rational behavior so that the rational expectation reputation can discourage the monetary authority from attempting surprise. When  $a(1 - \theta) \geq 2$ , whether or not we can expect the monetary policy to have a tendency to become stable depends not only on the lifetime of the government, but also on the beliefs of the government and the public, *ceteris paribus*. If the time horizon is long enough, we may expect the monetary policy tends to stable beyond some point of time. Although the initial economy shocks may not implement a stationary sequentially rational equilibrium at the beginning, under the sequentially strong rational strategy behavior assumption, the reputation trigger equilibrium have a tendency to reach a new stationary equilibrium beyond some point in time. If the life time of the government is not long enough to reach such a point, we may be able to use an incentive contract or a legislative approach to reach it.

The remainder of the paper is organized as follows. Section 2 will set up the model and provides a solution for the optimal stopping problem faced by the government. In Section 3, we

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<sup>2</sup>Such an advantage of the continuous-time formulations can be also found in other fields such as the principal-agent literature. For instance, Holmstrom and Milgrom's (1987) continuous-time Brownian model not only generate the second-best solution, but their solution is remarkably simple. Schättler and Sung (1997) provided the above explanation for the principal-agent models.

study the equilibrium behavior. The robustness of this monetary game is discussed in Section 4. Section 5 gives the conclusion.

## 2 Model

### 2.1 The Setup

We consider a continuous time game theoretical model with two players: the government and the public. The government's strategy space is  $R^+ \times L[0, T]$ , from which the government is to choose an action  $(\tau, \{\pi_t\}_{t \in T})$ . Here  $\tau$  is the time that the government changes its monetary policy from the zero-inflation rule to a discretion rule;  $\pi_t$  is the inflation rate chosen by the government at time  $t$ ;  $T$  is the lifetime of the government which can be finite or infinite; and  $L[0, T]$  is the class of Lebesgue integral functions defined on  $[0, T]$ . The public's strategy space is  $L[0, T]$ , from which the public is to choose an action  $(\{\pi_t^e\}_{t \in T})$ . Here  $\pi_t^e$  is the expected inflation rate formed by the public at time  $t$ .

Suppose that, at the beginning, the government commits an inflation rate  $\pi_0 = 0$ , and the public believes it so that  $\pi_0^e = \pi_0 = 0$ . The government has the right to switch from the zero-inflation to a discretion rule  $\pi_t \neq 0$  at the time  $t$  between 0 and  $T$ . However, after he changes his policy, he loses his reputation.

The government's payoff function is described by a quadratic discounted expected loss function of the form:

$$\Lambda = E \int_0^T e^{-\rho t} \left[ \frac{1}{2} \theta (y_t - \bar{y}_t - k)^2 + \frac{1}{2} \pi_t^2 \right] dt \quad (1)$$

where  $\rho$  is the discount factor with  $0 < \rho < 1$ ,  $y_t$  is aggregate output,  $\bar{y}_t$  is the economy's natural rate of output, and  $\theta$  is a positive constant that represents the relative weight the government puts on output expansions relative to inflation stabilization. Here, the target inflation  $\pi$  is zero.<sup>3</sup> (1) is a typical marco welfare function that has played an important role in the literature, and means that the government desires to stabilize both output around  $\bar{y}_t + k$ , which exceeds the economy's equilibrium output of  $\bar{y}_t$  by a constant  $k$ , and inflation around zero.

The government's objective is to minimize this discounted expected loss function subject to the constraint imposed by a Lucas-type aggregate supply function, the so-called Phillips curve,

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<sup>3</sup>Without loss of generality, the target inflation rate is assumed to be zero. The results obtained in the paper will continue to be true if the monetary authority has a target inflation that differs from zero.

which describes the relationship between output and inflation in each period:

$$y_t - \bar{y}_t = a(\pi_t - \pi_t^e) + X_t, \quad (2)$$

where  $a$  is a positive constant that represents the effect of a money surprise on output, i.e., the rate of the output gain from the unanticipated inflation so that the larger is  $a$ , the greater is the central bank's incentive to inflate, and  $X_t$  is the shock at time  $t$  that is assumed to be an Ito diffusion process:

$$dX_t = \sigma dB_t, \quad X_0 = x,$$

which is a special case of the general Ito diffusion:

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t$$

with  $b(X_t) = 0$  and  $\sigma(X_t) = \sigma$ . Here,  $B_t$  is 1-dimensional Brownian Motion and  $\sigma$  is the diffusion coefficient with  $\sigma < \infty$ .

The public has complete information about the policymaker's objectives. It is assumed that the public forms his expectations rationally, and thus the assumption of rational expectation implicitly defines the loss function for the public as  $E[\pi_t - \pi_t^e]^2$ . The public's objective is to minimize this expected inflation error. Given the public's understanding of the government's decision problem, its choice of  $\pi^e$  is optimal.

We first examine the "one-shot" game. The single-period loss function  $\ell_t$  for the government is:

$$\begin{aligned} \ell_t(\pi_t, \pi_t^e) &= \frac{1}{2}\theta(y_t - \bar{y}_t - k)^2 + \frac{1}{2}\pi_t^2 \\ &= \frac{1}{2}\theta[a(\pi_t - \pi_t^e) + X_t - k]^2 + \frac{1}{2}\pi_t^2. \end{aligned} \quad (3)$$

The equilibrium concept in this game is noncooperative Nash. Then the government minimizes  $\ell_t$  by taking  $\pi_t^e$  as given, and thus we have the best response function for the policymaker:

$$\pi_t^D = \frac{a\theta}{1 + a^2\theta} (a\pi_t^e - X_t + k). \quad (4)$$

The public is assumed to understand the incentive facing the government so they use (4) in forming their expectations about inflation so that

$$\pi_t^e = E\pi_t^D = \frac{a\theta}{1 + a^2\theta} (a\pi_t^e - EX_t + k). \quad (5)$$

Solving (5) for  $\pi_t^e$ , we get the unique Nash equilibrium  $\pi_t^{e*} = E\pi_t^{D*} = a\theta(EX_t - k)$ . Thus, as long as  $EX_t \neq k$ , the policymaker has incentives to use the discretion rule although the loss at  $\pi_t^e = \pi_t = 0$  is lower than at  $\pi_t^{e*} = E\pi_t^{D*}$ .

Note that, if  $X_t = k$  a.s. for  $t \geq 0$ , the unique Nash equilibrium of the “one-shot” game for the public and the government, is  $\pi_t^{e*} = \pi_t^* = 0$  a.s., and thus, the time-inconsistency problem will not appear. To make the problem non-trivial, without loss of generality, we assume that  $X_t \neq k$  a.s. for  $t \geq 0$  in the rest of the paper.

A potential solution to the above time inconsistency problem is to force the government to bear some consequence penalties if it deviates from its announced policy of low inflation. One of such penalties that may take is a loss of reputation, and so, in this paper, we will adopt the reputation approach that incorporates notions of reputation into a repeated-game framework to avoid this time consistency problem. If the government deviates from the low-inflation solution, credibility is lost and the public expects high inflation in the future. That is, the public expects zero-inflation as long as government has fulfilled the inflation expectation in the past. However, if actual inflation exceeds what was expected, the public anticipates that the policymaker will apply discretion in the future. So the public forms their expectation according to the trigger strategy: Observing “good” behavior induces the expectation of continued good behavior and a single observation of “bad” behavior triggers a revision of expectations.

## 2.2 The Optimal Stopping Problem for Government

In order to solve the time inconsistency problem by using the reputation approach, we first incorporate the government’s loss minimization problem into a general optimal stopping time problem. During any time in  $[0, T]$ , the policymaker has the right to reveal his type (discretion or zero-inflation). Since he has the right but not the obligation to reveal his type, we can think it is an option for the policymaker. So the policymaker’s decision problem is to choose a best time  $\tau \in [0, T]$  to exercise this option.

The policymaker considers the following time-inhomogeneous optimal stopping problem: Find  $\tau^*$  such that

$$L^*(x) = \inf_{\tau} E^x \left[ \int_0^{\tau} f(t, X_t) dt + g(\tau, X_{\tau}) \right] = E^x \left[ \int_0^{\tau^*} f(t, X_t) dt + g(\tau^*, X_{\tau^*}) \right], \quad (6)$$

where

$$f(s, X_t) = \frac{1}{2}\theta e^{-\rho s}(X_t - k)^2 \quad (7)$$

is the instantaneous loss function for the policymaker when he uses the zero-inflation rate which is clearly Lipschitz continuous, and

$$g(s, X_\tau) = e^{-\rho s} E^{X_\tau} \left[ \int_s^T e^{-\rho(t-s)} \left[ \frac{\theta}{2} [a(\pi_t^D - \pi_t^e) + X_t - k]^2 + \frac{\pi_t^{D^2}}{2} \right] dt \right] \quad (8)$$

is the expected loss function for policymaker in which he begins to use the discretion rule at time  $s$ . Note that  $g(\cdot) \geq 0$  since the loss function  $\ell_t \geq 0$ . We assume that  $g(\cdot)$  is a bounded function, i.e.,  $g(\cdot) \leq M$  for some constant number  $M$ .

We assume that the public uses stationary strategy:  $\pi_t^e = \pi^e$  for  $t \geq \tau$ . To compute  $g(\tau, X_\tau)$ , putting (4) into (8), we have

$$\begin{aligned} g(\tau, X_\tau) &= e^{-\rho\tau} E^{X_\tau} \left[ \int_\tau^T e^{-\rho(t-\tau)} \left[ \frac{\theta}{2} [a(\pi_t^D - \pi^e) + X_t - k]^2 + \frac{1}{2} \pi_t^{D^2} \right] dt \right] \\ &= \frac{1}{2} \frac{\theta}{1 + a^2\theta} e^{-\rho\tau} E^{X_\tau} \left[ \int_\tau^T e^{-\rho(t-\tau)} (-X_t + k + a\pi^e)^2 dt \right] \\ &= \frac{\theta}{2(1 + a^2\theta)} e^{-\rho\tau} \cdot \left\{ E^{X_\tau} \left[ \int_\tau^T e^{-\rho(t-\tau)} X_t^2 dt \right] - 2(k + a\pi^e) E^{X_\tau} \left[ \int_\tau^T e^{-\rho(t-\tau)} X_t dt \right] \right. \\ &\quad \left. + (k + a\pi^e)^2 E^{X_\tau} \left[ \int_\tau^T e^{-\rho(t-\tau)} dt \right] \right\} \quad (9) \end{aligned}$$

We now calculate the conditional expectation for  $X_t^2$  and  $X_t$ . Let  $A$  be the characteristic operator of Ito diffusion  $dX_t = b(X_t)dt + \sigma(X_t)dB$  (with  $b = 0$ ). Then

$$\begin{aligned} Af &= \sum_i b_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j} (\sigma\sigma^T)_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} \\ &= \frac{1}{2} \sum_{i,j} (\sigma\sigma^T)_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} \end{aligned}$$

Then, by Dynkin formula (Øksendal 1998, p. 118), we have

$$E^{X_\tau} [X_t] = X_\tau + E^{X_\tau} \left[ \int_\tau^t AX_s ds \right] = X_\tau \quad (10)$$

$$E^{X_\tau} [X_t^2] = X_\tau^2 + E^{X_\tau} \left[ \int_\tau^t AX_s^2 ds \right] = X_\tau^2 + \sigma^2(t - \tau). \quad (11)$$

Substituting (10) and (11) into (9), we have

$$\begin{aligned} g(\tau, X_\tau) &= \frac{1}{2} \frac{\theta}{1 + a^2\theta} \left\{ \sigma^2 \left[ \frac{1}{\rho^2} (e^{-\rho\tau} - e^{-\rho T}) - \frac{1}{\rho} (T - \tau) e^{-\rho T} \right] \right. \\ &\quad \left. + (X_\tau - k - a\pi^e)^2 \frac{1}{\rho} (e^{-\rho\tau} - e^{-\rho T}) \right\}. \quad (12) \end{aligned}$$

Note that, if we define

$$f_1(s, X_t) = -f(s, X_t),$$

$$g_1(s, X_\tau) = -g(s, X_\tau) + M \geq 0,$$

then the loss minimization problem in (6) can be reduced to the following maximization problem:

Find  $\tau^*$  such that

$$\begin{aligned} G_0^*(x) &= \sup_{\tau \in [0, T]} E^x \left[ \int_0^\tau [-f(t, X_t)] dt - g(\tau, X_\tau) + M \right] \\ &= \sup_{\tau \in [0, T]} E^x \left[ \int_0^\tau f_1(t, X_t) dt + g_1(\tau, X_\tau) \right]. \end{aligned} \quad (13)$$

In the following, we will use the optimal stopping approach to solve the optimization problem (13).

### 2.3 Solve the Optimal Stopping Problem

In order to solve the government's optimization problem (13) by using a standard framework of the optimal stopping problem involving an integral (cf. Øksendal (1998, p. 213)), we make the following transformations: Let

$$W_\tau = \int_0^\tau f_1(t, X_t) dt + w, \quad w \in R$$

and define the Ito diffusion  $Z_t = Z_t^{(s, x, w)}$  in  $R^3$  by

$$Z_t = \begin{bmatrix} s + t \\ X_t \\ W_t \end{bmatrix}$$

for  $t \geq 0$ . Then

$$dZ_t = \begin{bmatrix} dt \\ dX_t \\ dW_t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -\frac{1}{2}\theta e^{-\rho t}(X_t - k)^2 \end{bmatrix} dt + \begin{bmatrix} 0 \\ \sigma \\ 0 \end{bmatrix} dB_t, \quad Z_0 = (s, x, w).$$

So  $Z_t$  is an Ito diffusion starting at  $z := Z_0 = (s, x, w)$ . Let  $R^z = R^{(s, x, w)}$  denote the probability law of  $\{Z_t\}$  and let  $E^z = E^{(s, x, w)}$  denote the expectation with respect to  $R^z$ . In terms of  $Z_t$  the problem (13) can be written

$$G_0^*(x) = G^*(0, x, 0) = \sup_{\tau} E^{(0, x, 0)}[W_\tau + g_1(\tau, X_\tau)] = \sup E^{(0, x, 0)}[G(Z_\tau)]$$

which is a special case of the problem

$$G^*(s, x, w) = \sup_{\tau} E^{(s, x, w)}[W_\tau + g_1(\tau, X_\tau)] = \sup E^{(s, x, w)}[G(Z_\tau)]$$

with

$$G(z) = G(s, x, w) := w + g_1(s, x).$$

Then, for

$$\begin{aligned} f_1(s, x) &= -\frac{1}{2}\theta e^{-\rho s}(x-k)^2 \\ g_1(s, x) &= -\frac{1}{2}\frac{\theta}{1+a^2\theta} \left\{ \sigma^2 \left[ \frac{1}{\rho^2} (e^{-\rho s} - e^{-\rho T}) - \frac{1}{\rho}(T-s)e^{-\rho T} \right] \right. \\ &\quad \left. + (x-k - a\pi^e)^2 \frac{1}{\rho} (e^{-\rho s} - e^{-\rho T}) \right\} + M \end{aligned}$$

and

$$G(s, x, w) = w + g_1(s, x),$$

the characteristic operator  $A_Z$  of  $Z_t$  is given by

$$\begin{aligned} A_Z G &= \frac{\partial G}{\partial s} + \frac{1}{2}\sigma^2 \frac{\partial^2 G}{\partial x^2} - \frac{1}{2}\theta e^{-\rho s}(x-k)^2 \frac{\partial G}{\partial w} \\ &= \frac{1}{2}\frac{\theta}{1+a^2\theta} (x-k - a\pi^e)^2 e^{-\rho s} - \frac{1}{2}\theta (x-k)^2 e^{-\rho s} \\ &= \frac{1}{2}\frac{\theta}{1+a^2\theta} [(x-k - a\pi^e)^2 - (1+a^2\theta)(x-k)^2] e^{-\rho s}. \end{aligned} \tag{14}$$

Let

$$U = \{(s, x, w) : G(s, x, w) < G^*(s, x, w)\}$$

and

$$V = \{(s, x, w) : AG(x) > 0\}.$$

Then, by (14) we have

$$\begin{aligned} V &= \{(s, x, w) : A_Z G(s, x, w) > 0\} \\ &= R \times \{x : (x-k - a\pi^e)^2 > (1+a^2\theta)(x-k)^2\} \times R. \end{aligned} \tag{15}$$

**Remark 1** Øksendal (1998, p. 205) shows that:  $V \subset U$ , which means that it is never optimal to stop the process before it exits from  $V$ . For each  $x \neq k$ , if we choose a suitable  $\pi^e(x)$  such that  $(x-k - a\pi^e(x))^2 > (1+a^2\theta)(x-k)^2$ , then we have  $U = V = R \times \{(-\infty, k) \cup (k, \infty)\} \times R$ . Therefore, any stopping time less  $T$  will not be optimal for all  $(s, x, w) \in V$ , and thus  $\tau^* = T$  is the optimal stopping time. We will use this fact to study the time inconsistency problem of the monetary policy game in the following sections.

**Remark 2** In fact, we can verify directly that  $\frac{dL(x)}{d\tau} < 0$  when  $\pi^e$  is bigger enough, where  $L(x)$  is defined by

$$L(x) = E^x \left[ \int_0^\tau f(t, X_t) dt + g(\tau, X_\tau) \right] = E^x \left[ \int_0^\tau f(t, X_t) dt + g(\tau, X_\tau) \right].$$

Thus,  $\tau^* = T$  is the optimal stopping time.

### 3 The Equilibrium Behavior of the Monetary Policy Game

In order to study the equilibrium behavior of the monetary policy game, we first give the following lemma that shows that the government will keep the zero-inflation policy when the public uses trigger strategies and reputation penalties imposed by the public are large enough.

**Lemma 1** *Let  $\tau = \inf\{s > 0 : \pi_s \neq 0\}$ . Then, for all  $x$  with  $x \neq k$ , any trigger strategy of the public,  $\{\pi_t^e(x)\}$ , which has the form of*

$$\pi_t^e = \begin{cases} 0 & \text{if } t = 0 \\ 0 & \text{if } 0 < t < \tau \\ \pi^e(x) \in \{h : (x - k - ah)^2 > (1 + a^2\theta)(x - k)^2\} & \text{if } t > s \text{ and } t \geq \tau \end{cases}, \quad (16)$$

*discourages the policymaker from attempting surprise inflation.*

Proof: For each  $x \in R$  with  $x \neq k$ , if we choose any  $\pi^e \in \{h : (x - k - ah)^2 > (1 + a^2\theta)(x - k)^2\}$ , we have

$$(x - k - a\pi^e(x))^2 > (1 + a^2\theta)(x - k)^2 \quad \text{for all } x \in R \text{ with } x \neq k.$$

Then,  $V$  in (15) becomes  $V = R \times \{(-\infty, k) \cup (k, \infty)\} \times R$ , and thus any stopping time less  $T$  is not optimal for the government. Hence,  $\tau^* = T$ . Thus, when the public applies this trigger strategy, it is never optimal for government to stop the zero-inflation policy. Q.E.D

Although there are (infinitely) many trigger strategies given in Lemma 1 that can discourage the policymaker from attempting surprise inflation, most of them are not optimal for the public in terms of minimizing the public's expected inflation error:  $\pi_t - \pi_t^e$ . To rule out the those non-optimal strategies, we have to impose some assumptions how the public form an expectation and what an equilibrium solution should be used to describe the public's self-interested behavior. Different assumptions on the public's behavior may result in different the optimal solutions. In the following, we introduce two types of sequentially rational equilibrium solution concepts.

Let  $\{\mathcal{F}_t\}$  be a filtration, i.e., a nondecreasing family  $\{\mathcal{F}_t : t \geq 0\}$  of sub- $\sigma$ -fields of  $\mathcal{F}$ :  $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$  for  $0 \leq s < t < \infty$ , which is assumed to be generated by the process itself, i.e.,  $\mathcal{F}_t := \sigma(X_s : 0 \leq s \leq t)$ . Then,  $\mathcal{F}_t$  can be regarded the set of accumulated information up to time  $t$ .

Suppose the government knows the distribution of the shock,  $X_t$ , exactly, that is,

$$d\tilde{P}^G = dP,$$

where  $\tilde{P}^G$  is the belief of the government for the movement of the shock,  $P$  is the measure of the shock.

We suppose that the public does not know the distribution of the shock, but it's belief  $\tilde{P}^P$  is absolutely continuous with respect to  $P$ <sup>4</sup>, which means that if an event does not occur in probability, then the public will believe that this event will not happen.

Then, by Randon-Nikodym Theorem (Lipster & Ahiryaev 2001, p. 13), there exist Randon-Nikodym derivatives,  $M(t)$ , such that

$$d\tilde{P}^P = M(t)dP, (a.s.),$$

and  $M(t)$  is a martingale. This means that, whenever new information becomes available, the belief of the public is adjusted. We can interpreter  $M(t)$  is the information structure of the society, it is a measurement of how the public knows the real shock.

We suppose that  $M(t)$  is  $P$ -square-integrable and  $X_t$  is  $\tilde{P}^P$ -integrable. We also suppose that  $\langle X_t, M(t) \rangle = 0$ <sup>5</sup>, heuristically, this assumption can be interpreted as: the history of the shock can't help the public to predict the movement of the future shock.<sup>6</sup>

We denote by  $\tilde{E}$  the expectation operator with respect to  $\tilde{P}^P$ .

A strategy  $(\tau, \{\pi_t, \pi_t^e\})$  is said to be a *sequentially strong rational equilibrium strategy* for the dynamic model defined above if

(1) the belief of the public for the movement of the shocks  $X_t$ ,  $\tilde{P}^P$ , satisfies Bayes'

rule:

$$\tilde{E}[X_t | \mathcal{F}_s] = \frac{1}{M(s)} E[X_t M(t) | \mathcal{F}_s] \quad (17)$$

---

<sup>4</sup>  $\tilde{P}^P(A) = 0$  for each  $A \in \mathcal{F}_t$ , such that  $P(A) = 0$ .

<sup>5</sup>  $\langle X, Y \rangle$  is cross-variation, which is defined by  $\langle X_t, Y_t \rangle = \lim_{\|\Pi\| \rightarrow 0} \sum_{1 \leq k \leq m} (X_{t(k)} - X_{t(k-1)})(Y_{t(k)} - Y_{t(k-1)})$ , where  $X_t$  and  $Y_t$  are square-integrable, and  $\Pi = [t_0, t_1, \dots, t_m]$  is a partition of  $[0, t]$ .

<sup>6</sup> Note that, if one assumes that the public knows the distributions of the shocks,  $X_t$ , exactly, then  $M(t) = 1$ . This is a usual assumption made in the literature.

for all  $s < t$ ;

(2) The expectation of the public is strong rational:  $\pi_t^e = E^{X_s} \pi_t^D := \tilde{E}[\pi_t^D | \mathcal{F}_s]$  for all  $s < t$ ;

(3) it optimizes the objectives of the public and the government.

A strategy  $(\tau, \{\pi_t, \pi_t^e\})$  is said to be a *sequentially weak rational equilibrium strategy* for the dynamic model defined above if

(1) the belief of the public for the movement of the shocks  $X_t, \tilde{P}^P$ , satisfies Bayes' rule:

$$\tilde{E}[X_t | \mathcal{F}_0] = \frac{1}{M(0)} E[X_t M(t) | \mathcal{F}_0]; \quad (18)$$

(2) The expectation of the public is weak rational:  $\pi_t^e = E^x \pi_t^D := \tilde{E}[\pi_t^D | \mathcal{F}_0]$  for all  $s < t$ ;

(3) it optimizes the objectives of the public and the government.

The difference between sequentially strong rational equilibrium and sequentially weak rational equilibrium is that the sequentially weak rational equilibrium uses the information only at time 0 to form the public's belief and expectation on the government's policy while the sequentially strong rational equilibrium uses accumulated information up to the present to form the public's belief and expectation on the government's policy. Thus, the sequentially weak rational equilibrium, in general, is a weaker equilibrium solution concept to describe the public's behavior. This implies that every sequentially strong rational equilibrium is clearly a sequentially weak rational equilibrium, but the reverse may not be true. However, when the shocks,  $\{X_t\}$ , are nonstochastic, these two equilibrium solutions are equivalent.

Now we use these two types of sequentially rational equilibria to study the time consistency problem in monetary policy. Propositions 1 and 2 below show the existence of such equilibria.

**Proposition 1** *Suppose the shocks  $\{X_t\}$  satisfy the inequality:*

$$[(x - k) + a\theta(X_t - k)]^2 > (1 + a^2\theta)(x - k)^2 \text{ for all } t \in [0, T] \text{ and } x \in R \text{ with } x \neq k. \quad (19)$$

Let  $(\tau, \{\pi_s\})$  be the strategy of the government, where  $\tau$  is the first time that the government changes its policy from zero-inflation to discretion rule, i.e.,  $\tau = \inf\{s > 0 : \pi_s \neq 0\}$ . Let the

strategy of the public  $\{(\pi_t^e)\}$  be given by

$$\pi_t^e = \begin{cases} 0 & \text{if } t = 0 \\ 0 & \text{if } 0 < t < \tau \\ a\theta(k - X_\tau) & \text{if } t \geq \tau \end{cases} . \quad (20)$$

Then,  $(\tau^*, \{\pi_t^*, \pi_t^{e*}\})$  with  $\tau^* = T$ ,  $\pi_t^* = 0$  and  $\pi_t^{e*} = 0$  for all  $t \geq 0$  is a sequentially strong rational equilibrium strategy for the policymaker and the public.

Proof: To prove  $(\tau, \{\pi_t, \pi_t^e\})$  defined above results in a sequentially strong rational equilibrium,  $\tau^* = T$ ,  $\pi_t^* = 0$  and  $\pi_t^{e*} = 0$  for all  $t \geq 0$ , we need to show that (1) it satisfies Bayes' rule, (2) the strong rational expectation condition holds:  $\pi_t^e = E^{X_\tau} \pi_t^D := \tilde{E}[\pi_t^D | \mathcal{F}_\tau]$ , (3)  $\pi_t^e \in \{h : (x - k - ah)^2 > (1 + a^2\theta)(x - k)^2\}$ , and (4)  $(\tau^*, \{\pi_t^*, \pi_t^{e*}\})$  optimizes the objectives of the public and the government.

We first claim that the public updates its belief by Bayes' Rule. Indeed, since  $M(t)$  is a martingale and, for  $s < t$ ,  $X_t$  is a  $\tilde{P}^P$ -integrable random variable, then, by Lemma of Shreve & Kruzhilin (1999, p. 438), the Bayes' Rule holds:

$$\tilde{E}[X_t | \mathcal{F}_s] = \frac{1}{M(s)} E[X_t M(t) | \mathcal{F}_s].$$

To show  $\pi_t^e = E^{X_\tau} \pi_t^D$ , first note that  $X_t$  and  $M(t)$  are square-integrable martingale, using the fact that  $X_t M(t) - \langle X_t, M(t) \rangle$  is a martingale (Karatzas & Shreve(1991, p. 31)) and the assumption  $\angle X_t, M(t) = 0$ , We can get that  $X_t M(t)$  is a martingale, by Bayes' Rule:

$$\tilde{E}[X_t | \mathcal{F}_\tau] = \frac{1}{M(\tau)} E[X_t M(t) | \mathcal{F}_\tau] = \frac{1}{M(\tau)} X_\tau M(\tau) = X_\tau.$$

which means  $\{X_t\}$  is also a martingale under  $\tilde{P}^P$ . Since the policymaker's best response function is given by

$$\pi_t^D = \frac{a\theta}{1 + a^2\theta} (a\pi_t^e - X_t + k),$$

$\{X_t\}$  is a martingale under  $\tilde{P}^P$ , and  $\pi_t^e = a\theta(k - X_\tau)$  is complete information at time  $t$ , we have

$$\begin{aligned} E^{X_\tau} \pi_t^D &= E^{X_\tau} \frac{a\theta}{1 + a^2\theta} (a\pi_t^e - X_t + k) \\ &= \frac{a\theta}{1 + a^2\theta} (a\pi_t^e - E^{X_\tau} X_t + k) \\ &= \frac{a\theta}{1 + a^2\theta} (a\pi_t^e - X_\tau + k). \end{aligned} \quad (21)$$

Substituting  $\pi_t^e = a\theta(k - X_\tau)$  into (21), we have  $E^{X_\tau} \pi_t^D = \frac{a\theta}{1 + a^2\theta} [a^2\theta(k - X_\tau) - X_\tau + k] = a\theta(k - X_\tau) = \pi_t^e$ .

Now, if condition (19) is satisfied, we have  $(x - k - a\pi_t^e)^2 > (1 + a^2\theta)(x - k)^2$  and thus  $\pi_t^e \in \{h : (x - k - ah)^2 > (1 + a^2\theta)(x - k)^2\}$  for all  $x \in R$  with  $x \neq k$ . Then, by Lemma 1, and the optimal stopping time is  $\tau^* = T$ . Therefore, we must have  $\pi_t^* = 0$  for all  $t \in [0, T]$ .

Since the public only cares about his inflation prediction errors, so  $\pi_t^e = \theta(k - X_t)$  minimizes the public's expected loss when the policy change occurs at time  $t$  in this game. Hence, if both the policymaker and public believe that future shocks will grow enough to make the inequality (19) hold, the threat of the public is creditable. Hence, we must have  $\pi_t^{e*} = 0$  for all  $t \in [0, T]$  since  $\tau^* = T$ . Thus, we have shown that the trigger strategies  $(\tau, \{\pi_t, \pi_t^e\})$  result in a sequentially strong rational equilibrium, which is  $\tau^* = T$ ,  $\pi_t^* = 0$ , and  $\pi_t^{e*} = 0$  for all  $t \geq 0$ . Q.E.D.

Thus, Proposition 1 implies that, as long as all disturbance shocks  $X_t$  are bigger enough, the public can use a trigger strategy to induce a stationary zero-inflation sequentially strong rational equilibrium. Of course, the assumption that  $[(x - k) + a\theta(X_t - k)]^2 > (1 + a^2\theta)(x - k)^2$  for all  $t \in [0, T]$  and  $x \in R$  with  $x \neq k$  seems very strong. Proposition 3 in the next section shows that this is a reasonable assumption. As long as this inequality holds for the initial shock  $x$ , the public and the government will have a strong belief that it will be true for all  $t \in (0, T]$  and  $x \in R$ .

The sequentially strong rational equilibrium has imposed a strong assumption on the public's self-interested behavior. If the public's self-interested behavior is described by the sequentially weak rational equilibrium, we can have the following proposition which completely characterizes the existence of the stationary zero inflation strategy as an equilibrium outcome.

**Proposition 2** *Let  $(\tau, \{\pi_s\})$  be the strategy of the government, where  $\tau = \inf\{s > 0 : \pi_s \neq 0\}$ .*

*Let the strategy of the public  $\{\pi_t^e\}$  be given by*

$$\pi_t^e = \begin{cases} 0 & \text{if } t = 0 \\ 0 & \text{if } 0 < t < \tau \\ a\theta(k - x) & \text{if } t \geq \tau \end{cases} . \quad (22)$$

*Then, the stationary zero-inflation policy, i.e.,  $\tau^*$ ,  $\pi_t^* = 0$  and  $\pi_t^{e*} = 0$  for all  $t \geq 0$ , is a sequentially weak rational equilibrium strategy for the policymaker and the public if and only if  $a(1 - \theta) < 2$ .*

Proof: We first note that  $(1 + a\theta)^2 > 1 + a^2\theta$  if and only if  $a(1 - \theta) < 2$ . Substituting  $\pi_t^e = a\theta(k - x)$  for  $t \geq \tau$  into  $(x - k - a\pi_t^e)^2$ , we have

$$(x - k - a\pi_t^e)^2 = (1 + a\theta)^2(x - k)^2$$

$$> (1 + a^2\theta)(x - k)^2$$

for all  $x \in R \setminus \{k\}$  if and only if  $a(1 - \theta) < 2$ . Then, we have

$$U = V = \begin{cases} R \times \{(-\infty, k) \cup (k, \infty)\} & \text{if } a(1 - \theta) < 2 \\ R \times \{\emptyset\} \times R & \text{otherwise} \end{cases}. \quad (23)$$

and thus the optimal stopping time is given by

$$\tau^* = \begin{cases} T & \text{if } a(1 - \theta) < 2 \\ 0 & \text{otherwise} \end{cases}. \quad (24)$$

Hence,  $\tau^* = T$ ,  $\pi_t^* = 0$  and  $\pi_t^{r*} = 0$  for all  $t \in [0, T]$  if and only if  $a(1 - \theta) < 2$ . The proofs of the other parts are the same as those in Proposition 1. Therefore, the trigger strategies  $(\tau, \{\pi_t, \pi_t^e\})$  result in a stationary zero-inflation sequentially weak rational equilibrium, which is given by  $\tau^*$ ,  $\pi_t^* = 0$  and  $\pi_t^{e*} = 0$  for all  $t \geq 0$ , if and only if  $a(1 - \theta) < 2$ . Q.E.D.

Thus, as long as  $a(1 - \theta) < 2$ , the public can use a trigger strategy to induce a stationary zero-inflation equilibrium outcome that is sequentially weak rational. Figure 1 shows the range of the parameters  $a$  and  $\theta$ , which guarantees the existence of a stationary sequentially weak rational equilibrium. Note that, if the government's relative weight parameter  $\theta \geq 1$ , i.e., if the government thinks that stabilizing output is at least as important as stabilizing inflation, or if the economic environment parameter  $a \leq 2$  so that the rate of the output gain from the unanticipated inflation is not too high, this inequality always holds and stationary sequentially weak rational equilibrium can be guaranteed. When the inequality  $a(1 - \theta) \geq 2$ , the sequentially weak rational trigger equilibrium, which is given by  $\pi_0^{e*} = \pi_0 = 0$  and  $\pi_t^{e*} = \pi_t^* = \theta(k - x)$  for  $0 < t \leq T$ , is not stationary.

[Insert Figure 1 around here]

As such, the government will act opportunistically if the rate of the output gain from the unanticipated inflation is high ( $a > 2$ ) or the government puts less weight on stabilizing output than on stabilizing inflation ( $\theta < 1$ ), and then be expected to behave (and will behave) that way in every succeeding periods. It will keep the inflation at zero if the rate of the output gain from the unanticipated inflation is small or the government puts more weight on stabilizing output than on stabilizing inflation. Thus, the objective rate of the output gain from the unanticipated inflation, combining together with and the subjective preferential policy on the relative importance between stabilizing output and stabilizing inflation, can determine whether

or not the government keeps inflation at zero. In the literature, some papers assume that the economic environment parameter  $a$  is normalized to one. However, from our results, the choice of  $a$  will affect whether or not a monetary policy is time consistency, and thus it cannot be arbitrarily normalized to be one.

When shocks  $X_t$  becomes nonstochastic, i.e.,  $X_t = X_0 = x$ , the sequentially strong rational equilibrium and sequentially weak rational equilibrium are the same. Thus, from Propositions 1 and 2, we have the following corollary that shows that the existence of a stationary zero inflation equilibrium is completely characterized by the inequality  $a(1 - \theta) < 2$ .

**Corollary 1** *Let  $(\tau, \{\pi_s\})$  be the strategy of the government, where  $\tau = \inf\{s > 0 : \pi_s \neq 0\}$ .*

*Let the strategy of the public  $\{\pi_t^e\}$  be given by*

$$\pi_t^e = \begin{cases} 0 & \text{if } t = 0 \\ 0 & \text{if } 0 < t < \tau \\ a\theta(k - x) & \text{if } t \geq \tau \end{cases} . \quad (25)$$

*Then, the stationary zero-inflation policy, i.e.,  $\tau^*$ ,  $\pi_t^* = 0$  and  $\pi_t^{e*} = 0$  for all  $t \geq 0$ , is a sequentially (strong) rational equilibrium strategy for the policymaker and the public if and only if  $a(1 - \theta) < 2$ .*

This possibility result on the existence of the stationary zero inflation policy as an equilibrium shows the advantage of our nonstochastic continuous finite horizon setting compared to the nonstochastic discrete time finite horizon settings discussed in the literature. It is well known that, in the certainty setting with discrete time, a reputational equilibrium is possible only if the horizon is infinite. Otherwise, the government would be sure in the last period to produce the discretionary outcome whatever the public's expectation were and, working backward, would be expected to do the same in the first period. Thus, our results are sharply contrasted to the negative results from the certainty setting with discrete time horizon.

## 4 Robustness of Equilibrium Solutions

In this section we study the robustness of sequentially strong rational equilibrium. In order to get the sequentially strong rational equilibrium in Proposition 1, we imposed the assumption that  $B = \{X_t : [(x - k) + a\theta(X_t - k)]^2 > (1 + a^2\theta)(x - k)^2\}$  for all  $0 \leq t \leq T$  and  $x \in R$  with  $x \neq k$ . It might appear that the result in Proposition 1 is sensitive to this assumption. Is this

assumption reasonable? The following proposition shows that the result is quite robust in the sense that, as long as the initial starting point  $x$  is in  $B$ , the expected first exit time from  $B$  will be infinite.

**Proposition 3** *Let  $B = \{X_t : [(x - k) + a\theta(X_t - k)]^2 > (1 + a^2\theta)(x - k)^2 \text{ for } t \geq 0\}$ , and let  $\eta = \inf\{t > 0 : X_t \notin B\}$  be the first time  $X_t$  exits from  $B$ . Suppose that  $x \in B$ , i.e.,  $a(1 - \theta) < 2$ . Then, we have*

$$E^x[\eta] = \infty$$

for all  $x \in R$  with  $x \neq k$ .

Proof: Solving  $[(x - k) + a\theta(X_t - k)]^2 > (1 + a^2\theta)(x - k)^2$  for  $X_t$ , we have

$$X_t > \frac{1}{a\theta} \left[ (1 + a\theta)k - x + \sqrt{1 + a^2\theta}|x - k| \right]$$

or

$$X_t < \frac{1}{a\theta} \left[ (1 + a\theta)k - x - \sqrt{1 + a^2\theta}|x - k| \right].$$

Let  $C = \frac{1}{a\theta} \left[ (1 + a\theta)k - x + \sqrt{1 + a^2\theta}|x - k| \right]$  and  $D = \frac{1}{a\theta} \left[ (1 + a\theta)k - x - \sqrt{1 + a^2\theta}|x - k| \right]$ .

Since  $X_0 = x \in B$  for all  $x \in R$ , there are two cases to be considered: (1)  $x > C$  and (2)  $x < D$ .

Case 1.  $x > C$ . Let  $\eta_c = \inf\{t > 0 : X_t \leq C\}$ , and let  $\eta_n$  be the first exit time from the interval

$$\{X_t : C \leq X_t \leq n\}$$

for all integers  $n$  with  $n > C$ . We first show that  $P^x(X_{\eta_n} = C) = \frac{n-x}{n-C}$  and  $P^x(X_{\eta_n} = n) = \frac{x-C}{n-C}$ .

Consider function  $h \in C_0^2(R)$  defined by  $h(x) = x$  for  $C \leq x \leq n$  ( $C_0^2(R)$  means the functions in  $C^2(R)$  with compact support in  $R$ ). By Dynkin's formula,

$$E^x [h(X_{\eta_n})] = h(x) + E^x \left[ \int_0^{\eta_n} Ah(X_s) ds \right] = h(x) = x, \quad (26)$$

we have

$$CP^x(X_{\eta_n} = C) + nP^x(X_{\eta_n} = n) = x.$$

Thus,

$$P^x(X_{\eta_n} = C) = \frac{n-x}{n-C}$$

and

$$P^x(X_{\eta_n} = n) = 1 - P^x(X_{\eta_n} = C) = \frac{x-C}{n-C}.$$

Now Consider  $h \in C_0^2(R)$  such that  $h(x) = x^2$  for  $C \leq x \leq n$ . Applying Dynkin's formula again, we have

$$E^x [h(X_{\eta_n})] = h(x) + E^x \left[ \int_0^{\eta_n} Ah(X_s) ds \right] = x^2 + \sigma^2 E^x [\eta_n], \quad (27)$$

and thus

$$\sigma^2 E^x [\eta_n] = C^2 P^x(X_{\eta_n} = C) + n^2 P^x(X_{\eta_n} = n) - x^2.$$

Hence, we have

$$E^x [\eta_n] = \frac{1}{\sigma^2} \left[ C^2 \frac{n-x}{n-C} + n^2 \frac{x-C}{n-C} - x^2 \right].$$

Letting  $n \rightarrow \infty$ , we conclude that  $P^x(X_{\eta_n} = n) = \frac{x-C}{n-C} \rightarrow 0$  and  $\eta_c = \lim \eta_n < \infty$  a.s. Therefore, we have

$$E^x[\eta_c] = \lim_{n \rightarrow \infty} E^x[\eta_n] = \infty.$$

Case 2.  $X_0 = x < D$ . Define  $\eta_D = \inf\{t > 0; X_t \geq D\}$ . Let  $\eta_n$  be the first exit time from the interval

$$\{X_t : -n \leq X_t \leq D\}$$

for all integers  $n$  with  $-n < D$ . By the same method, we can prove that

$$E^x [\eta_n] = \frac{1}{\sigma^2} \left[ D^2 \frac{n+x}{n+D} + n^2 \frac{D-x}{n+D} - x^2 \right].$$

Letting  $n \rightarrow \infty$ , we conclude that  $P^x(X_{\eta_n} = n) = \frac{D-x}{n+D} \rightarrow 0$  and  $\eta_D = \lim \eta_n < \infty$  a.s., and thus

$$E^x[\eta_D] = \lim_{n \rightarrow \infty} E^x[\eta_n] = \infty.$$

Thus, in either case, we have  $E^x[\eta] = \infty$ . Q.E.D.

Proposition 3 thus implies that, because the expected exit time from  $B$  is infinite since the expectation  $E^x[\eta] = \infty$  for all  $x \in R$  with  $x \neq k$ , the policymaker will have the belief that the future shocks will stay in  $B$  forever, and consequently they will likely make decisions and behave according to this belief. As a result, the sequentially strong rational equilibrium will likely appear in the game when the public has the same belief as the government. So, in this sense, we can regard the class  $B$  as an absorbing class for  $X_t$  as long as  $x \in B$ . Note that, for  $x \neq k$ ,  $x \in B$  if and only if  $a(1-\theta) < 2$ .<sup>7</sup> Therefore,  $(1+a\theta)^2(x-k)^2 > (1+a^2\theta)(x-k)^2$  if and only if  $a(1-\theta) < 2$ . Thus, as long as  $a(1-\theta) < 2$ , the sequentially strong rational equilibrium solution is stable.

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<sup>7</sup>Indeed,  $a(1-\theta) < 2$  is equivalent to  $(1+a\theta)^2 > 1+a^2\theta$ .

What happens if the initial shock  $x$  is not in  $B$  (i.e., if  $2 \geq a(1 - \theta)$ )? We have following proposition:

**Proposition 4** Define  $\tau = \inf\{t > 0 : Z_t \in B\}$ . Then for  $x \notin B$ , i.e.,  $a(1 - \theta) \geq 2$ , we have

$$E^x[\tau] = \frac{a(1 - \theta) - 2}{\sigma^2 a \theta} (k - x)^2$$

for all  $D \leq x \leq C$ .

Proof: Since  $x \notin B$ , we have  $D \leq x \leq C$ . Define  $\tau_C = \inf\{t > 0 : X_t \geq C\}$  and  $\tau_D = \inf\{t > 0 : X_t \leq D\}$ . Then  $\tau = \tau_C \wedge \tau_D := \min\{\tau_C, \tau_D\}$ . We first show that  $P^x(X_\tau = C) = \frac{x-D}{C-D}$  and  $P^x(X_\tau = D) = \frac{C-x}{C-D}$ . Consider  $h \in C_0^2(\mathbb{R})$  such that  $h(x) = x$  for  $D \leq x \leq C$ . By Dynkin's formula,

$$E^x [h(X_{\tau_C \wedge \tau_D})] = h(x) + E^x \left[ \int_0^{\tau_C \wedge \tau_D} Ah(X_s) ds \right] = h(x) = x, \quad (28)$$

we have

$$CP^x(X_\tau = C) + DP^x(X_\tau = D) = x.$$

Thus,

$$P^x(X_\tau = C) = \frac{x - D}{C - D}.$$

and thus

$$P^x(X_\tau = D) = 1 - P^x(X_\tau = C) = \frac{C - x}{C - D}.$$

Now consider  $h \in C_0^2(\mathbb{R})$  such that  $h(x) = x^2$  for  $D \leq x \leq C$ . By Dynkin's formula:

$$E^x [h(X_{\tau_C \wedge \tau_D})] = h(x) + E^x \left[ \int_0^{\tau_C \wedge \tau_D} Ah(X_s) ds \right] = h(x) + \sigma^2 E^x [\tau_C \wedge \tau_D], \quad (29)$$

we have

$$\sigma^2 E^x [\tau_C \wedge \tau_D] = C^2 P^x(X_\tau = C) + D^2 P^x(X_\tau = D) - x^2$$

and thus

$$\begin{aligned} E^x [\tau_C \wedge \tau_D] &= \frac{1}{\sigma^2} \left[ C^2 \frac{x - D}{C - D} + D^2 \frac{C - x}{C - D} - x^2 \right] \\ &= \frac{1}{\sigma^2} [(C + D)x - CD - x^2] \\ &= \frac{2x}{\sigma^2 a \theta} [(1 + a\theta)k - x] - \frac{1}{\sigma^2 a^2 \theta^2} [(1 + a\theta)k - x]^2 + \frac{1 + a^2 \theta}{\sigma a^2 \theta^2} (x - k)^2 - \frac{1}{\sigma} x^2 \\ &= \frac{1}{\sigma^2 a^2 \theta^2} \{ [(1 + a\theta)k - x][2xa\theta - (1 + a\theta)k + x] - a^2 \theta^2 x^2 + (1 + a^2 \theta)(k - x)^2 \} \\ &= \frac{1}{\sigma^2 a^2 \theta^2} \{ -(1 + a\theta)^2 k^2 - (1 + 2a\theta)x^2 + (1 + a\theta)kx + (1 + a\theta)(1 + 2a\theta)xk \} \end{aligned}$$

$$\begin{aligned}
& -a^2\theta^2x^2 + (1 + a^2\theta)(x - k)^2\} \\
= & \frac{1}{\sigma^2a^2\theta^2} \{-(1 + a\theta)^2k^2 - (1 + a\theta)^2x^2 + 2(1 + a\theta)^2kx + (1 + a^2\theta)(x - k)^2\} \\
= & \frac{1}{\sigma^2a^2\theta^2} \{-(1 + a\theta)^2(k - x)^2 + (1 + a^2\theta)(x - k)^2\} \\
= & \frac{a(1 - \theta) - 2}{\sigma^2a\theta}(k - x)^2 \geq 0. \tag{30}
\end{aligned}$$

by noting that  $a(1 - \theta) \geq 2$ . Q.E.D.

Notice that, from (30), one can see that, the bigger shock (measured by  $\sigma$ ), the faster the convergence rate. In particular, if  $\sigma^2 \rightarrow 0$ ,  $E^x[\tau_c \wedge \tau_D] \rightarrow \infty$ . This means that, when the shocks  $\{X_t\}$  degenerate to a non-stochastic process and the public has the same belief as the government, the government and the public will believe that  $X_t \notin B$  for all  $t \in [0, T]$ , and thus a stationary sequentially strong rational equilibrium does not exist. This is actually the result we have already obtained in Corollary 1. On the other hand, if  $\sigma^2 \rightarrow \infty$ , then  $E^x[\tau_c \wedge \tau_D] \rightarrow 0$ . This means that, when the shocks  $\{X_t\}$  become very large and the public has the same belief as the government, the public and the government may believe that  $X_t$  will be in  $B$  for  $t \in (0, T]$ . As such, the policymaker and the public will likely have the beliefs that the shocks  $X_t$  will be in  $B$  right after the initial shock  $x$ , and consequently, the stationary sequentially strong rational equilibrium will likely appear in the time horizon  $(0, T]$ .

When  $0 < \sigma < \infty$ , from Proposition 4, the expected time of entering  $B$ ,  $E^x[\tau] = E^x[\tau_c \wedge \tau_D]$  is a finite number. Suppose the public has the same belief as the government. There are two cases to be considered: (1)  $E^x[\tau] \geq T$ . In this case, the government and the public likely believe that  $X_t \notin B$  for all  $t \in [0, T]$ , and thus a stationary sequentially strong rational equilibrium will unlikely exist. (2)  $E^x[\tau] < T$ . In this case, we should not expect the zero-inflation stationary monetary policy for the time period between  $[0, E^x[\tau]]$  since  $X_t \notin B$  for all  $t \in [0, E^x[\tau]]$ . However, once  $X_t$  enters  $B$  at the first time  $E^x[\tau]$ , we can regard  $X_\tau$  as a new starting point. Then, by Proposition 3, the policymaker and the public will believe  $X_t$  will stay in  $B$  for all  $t \in [E^x[\tau], T]$ , and thus we can expect to have a non-zero inflation stationary monetary policy on  $[E^x[\tau], T]$ . This implies that, although we do not have a time consistency policy on the whole time horizon  $[0, T]$  when  $x \notin B$ , we could have a time consistency monetary policy beyond the point  $E^x[\tau]$ . In other words, one will have an instationary policy period if the initial shock  $x \notin B$ , however, after a certain point  $\tau$ , the monetary policy may become stationary. Thus, the time inconsistency can happen at most once.

Summarizing the above discussion, we can draw the following conclusions:

- (i) When  $a(1 - \theta) < 2$ , any initial shock  $x$  with  $x \neq k$  is in the class  $B$ , one can expect all future shocks  $X_t$  are in  $B$  and thus can expect a stationary zero-inflation outcome by the sequentially strong rational behavior.
- (ii) When  $a(1 - \theta) \geq 2$ , any initial shock  $x$  with  $x \neq k$  is not in the class  $B$ , whether or not we can expect the monetary policy to have a tendency to become stable depends on  $T$ , the lifetime of the government. If the expected first entry time to  $B$ ,  $E^x[\tau]$  is greater than the lifetime of the government, we do not expect to have a stationary monetary policy and thus we have the time inconsistency problem. If the first entering time into  $B$ ,  $E^x[\tau]$  is less than the lifetime of the government, we may expect a stationary monetary policy beyond the point  $E^x[\tau]$ , and the policy becomes stationary. Thus, the monetary policy can jump at most once.

Thus, for this continuous time dynamic stochastic game, the sequentially strong rational equilibrium behavior can be well predicted for any initial shock.

## 5 Conclusion

In this paper, we examine the equilibrium behavior of the time inconsistency problem in a continuous time stochastic world. We introduce the notions of sequentially weak and strong rational equilibria, and show that the time inconsistency problem may be solved with trigger reputation strategies not only for stochastic but also for nonstochastic settings even with finite horizon. We provide the conditions for the existence of stationary sequentially strong rational equilibrium, and also completely characterize the existence of stationary sequentially weak rational equilibrium. We show that, when  $x \neq k$ , the government will keep the inflation at zero if and only if  $a(1 - \theta) < 2$ . Thus, the reputation can discourage the monetary authority from attempting surprise inflation as long as this inequality holds. Furthermore, we investigate the robustness of the sequentially strong rational equilibrium behavior solution by showing that the imposed assumption is reasonable and the sequentially rational equilibrium is very stable.

## 6 References

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