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Double implementation of Lindahl allocations by a pure mechanism

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Abstract. This paper considers the issue of designing mechanisms whose Nash allocations and strong Nash allocations coincide with Lindahl allocations for public goods economies when coalition patterns, preferences, and endowments are unknown to the designer. It will be noted that the mechanism presented here is feasible and continuous, and the implementation result is obtained without defining an artificial preference profile on prices announced by individuals. In addition, unlike most existing Nash-implementing mechanisms which need to distinguish the case of two agents from that of three or more agents, this paper provides a unified mechanism which is irrespective of the number of agents.

1 Introduction

The incentive issue is a basic problem that any social (especially economic) organization needs to consider when participants have private information, and may use such information strategically to advance their own interests. A basic principle of mechanism design then must require an organization to provide individuals with appropriate incentives so that individuals' interests are compatible with the goals of the organization. Implementation theory, which regards mechanisms as unknown, deals with precisely this problem by

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designing a game form (rules of the game) such that a prespecified welfare criterion is guaranteed to be achieved by the game across a large domain of possible environments. That is, constructing a mechanism such that the set of equilibrium outcomes of the mechanism coincides with the set of socially desirable alternatives for all environments under consideration.

In the general equilibrium approach to the efficient provision of public goods, the Lindahl equilibrium has been commonly used in the literature. As is well known, the Lindahl mechanism, however, is not incentive-compatible, in the sense that it has a free-rider problem. Since Hurwicz (1972) formalized a general model to deal with the incentive problem of economic organizations, many incentive-compatible mechanisms have been proposed to implement the Lindahl correspondence for various economic environments, including those in Hurwicz (1979), Hurwicz et al. (1995), Walker (1981), Nakamura (1989), Tian (1989, 1990, 1993), Li, Nakamura, and Tian (1995), and Tian and Li (1995) for complete information and Tian (1996a) for incomplete information.

However, all of the foregoing mechanisms only considered implementation of Lindahl allocations by using Nash equilibrium or Bayesian Nash equilibrium as a solution concept to describe individuals' self-interested behavior. (Bayesian) Nash equilibrium is a strictly noncooperative notion and only concerned with single individual deviations at which no one's position can be improved by unilateral deviation from a prescribed strategy profile. No cooperation among agents is allowed. As a result, although a Nash equilibrium may be easy to reach, it may not be stable in the sense that there may exist a group of agents whose positions can be improved by forming a coalition. Thus it is natural to adopt the strong Nash equilibrium which allows all possible cooperation (coalitions) among agents. Thus, to have a solution concept combining the properties of Nash and strong Nash equilibria, it is desirable to construct a mechanism which doubly implements a social choice rule by Nash and strong Nash equilibria so that its equilibrium outcomes are not only easy to reach, but also hard to leave. Also, by double implementation, the solution concept can cover the situation where agents in some coalitions will cooperate and in some other coalitions will not, and thus the designer does not need to know which coalitions are permissible and consequently it allows the possibility for agents to manipulate coalition patterns.

While the domain of Nash implementable social choice rules (correspondences) is relatively large, the domain of strong Nash implementable social choice rules is much smaller, as is the domain of double implementable social choice rules in Nash and strong Nash equilibria. Maskin (1979) proved that any social choice rule which satisfies no veto power cannot in general be implemented in strong Nash equilibrium when the domain of preferences is unrestricted. However, when the domain of economic environments is restricted, the results can be positive. Maskin showed that the social choice correspondence, which selects all Pareto-efficient and individual allocations, are doubly implementable in Nash and strong Nash equilibria. Suh (1997) further provided a necessary and sufficient condition for a social choice correspondence to be doubly implementable in Nash and strong Nash equilibrium for a class of economic environments. By applying his characterization result, Suh (1997) investigated double implementation of Lindahl allocations in Nash and strong Nash equilibria. However, due to the general nature of the social choice rules under consideration, the implementing mechanisms turn out to be quite complex. Characterization results show what is possible for the implementation of a social choice rule, but not what is realistic. Thus, like most characterization results in the literature, Suh's mechanism is not natural in the sense that it is not continuous; small variations in an agent's strategy choice may lead to large jumps in the resulting allocations. Further, it has a message space of infinite dimension.

Recently, Peleg (1996b) gave a feasible and continuous mechanism with a finite dimensional message space, which attempts to doubly implement Lindahl allocations. The main drawback of his mechanism is that the mechanism is not a pure mechanism in the sense that a preference relation is artificially introduced for individuals to rank announced prices. In other words, his implementation result is obtained based not on the original preferences defined on the allocation space, but on the re-defined preferences on the outcome/message space which consists of allocations and price determinations (see Peleg (1996b, p. 318)). In Peleg's approach, an individual is not only a player, but also an *inside* auctioneer. Unlike the usual auctioneer defined in the literature, the auctioneers defined in Peleg (1996b) not only announce and adjust prices, but are also as assigned preferences on the level of prices announced. In an incentive mechanism design, the preferences of agents should be given, not assigned, since it is private information to the designer. The designer cannot vary an agent's preferences since they are in fact determined by the agent himself. Thus, it leaves a question as to there is a pure natural mechanism which is feasible and continuous, and, further, doubly implements the Lindahl correspondence in Nash and strong Nash equilibria.

A similar situation prevailed with regard to double implementation of the (constrained) Walrasian correspondence in Nash and strong Nash equilibria until Tian (1996b) presented a pure continuous and feasible mechanism which doubly implements the constrained Walrasian correspondence in Nash and strong Nash equilibria.

This paper will affirmatively answer this question by giving a pure feasible and continuous mechanism which doubly implements the Lindahl correspondence in Nash and strong Nash equilibria. Our implementation result is obtained without changing individuals' preferences and thus improves the mechanism proposed in Peleg (1996b) so as to avoid introducing artificial preference relations for prices announced by individuals.[1] In addition, our mechanism works not only for three or more agents, but also for a two-agent world. While most of the mechanisms mentioned above need to distinguish

[1] We also use a weaker assumption that preferences are increasing only in private goods while Peleg (1996b) assumed that preferences are increasing in both private and public goods.

the case of two agents from that of three or more agents, this paper gives a unified mechanism which is irrespective of the number of agents. Further, our implementation result holds on a large domain of economic environments, including some non-neoclassical economic environments. No continuity assumption on preferences is needed, and, further, preferences may be nontotal or nontransitive. Finally, our mechanism is simple and natural. Not only does it have a finite dimensional message space, but also a type of "market game" which is similar to the Lindahl rule: the strategies of the mechanism are "prices" and "quantities", and agents' consumption is chosen from their budget sets.

The plan of this paper is as follows. Section 2 sets forth a public goods model and gives some notation and definitions. Section 3 presents a mechanism which has the desirable properties mentioned above. Section 4 shows that this mechanism doubly implements the Lindahl correspondence in Nash and strong Nash equilibria. Finally, concluding remarks are offered in Section 5.

2 The model

2.1 Economic environments

In a public goods economy, there are n agents ($n \geq 2$) who consume L private goods and K public goods, x being private and y being public. Denote by $N = \{1, 2, \ldots, n\}$ the set of agents. Each agent's characteristic is denoted by $e_i = (\mathring{w}_i, P_i)$, where $\mathring{w}_i \in \mathbb{R}^L_{++}$ is the initial endowment of private goods and P_i is the strict (irreflexive) preference relation defined on $\mathbb{R}^{L+K}_+$ which may be nontotal or nontransitive. We assume that there is no initial endowment of public goods, but that public goods can be produced from private goods. Let $\mathscr{Y}$ be the production possibility set. A generic element of $\mathscr{Y}$ is (r, y), where $r \in -\mathbb{R}^L_+$ is the vector of private goods inputs and $y \in \mathbb{R}^K_+$ is the vector of public goods outputs. An economy is the full vector $e = (e_1, \ldots, e_n, \mathscr{Y})$. The following assumptions are made on e:

Assumption 1. *P_i is convex*[2], *strictly monotonically increasing in private goods, and nondecreasing in public goods.*

Assumption 2 (Indispensability of private goods). $(x_i, y) P_i (x'_i, y')$ *for all* $i \in N$, $x_i \in \mathbb{R}^L_{++}$, $x'_i \in \partial\mathbb{R}^L_+$, *and* $y, y' \in \mathbb{R}^K_+$, *where* $\partial\mathbb{R}^L_+$ *is the boundary of* $\mathbb{R}^L_+$.

Assumption 3. *The production possibility set $\mathscr{Y}$ is a closed convex cone;* $0 \in \mathscr{Y}$; $(-\mathbb{R}^L_+, 0) \subseteq \mathscr{Y}$ *(free disposal); and for any* $y \in \mathbb{R}^K_+$, *there is an* $r \in -\mathbb{R}^L_+$ *such that* $(r, y) \in \mathscr{Y}$.

Remark 1. Observe that under Assumption 3, if $(r, y) \in \mathscr{Y}$, then $(r', y) \in \mathscr{Y}$ for any $r' \leq r$ and $(r, y') \in \mathscr{Y}$ for any $y' \leq y$ (cf. Debreu 1959, p. 42).

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[2] P_i is convex if for bundles a, b, c with $0 < \lambda \leq 1$ and $c = \lambda a + (1 - \lambda) b$, the relation $a P_i b$ implies $c P_i b$.

Remark 2. Assumption 2 cannot be dispensed with. Tian (1988) showed that (constrained) Lindahl allocations violate Maskin's (1977) monotonicity condition even under the strict monotonicity and convexity of utility functions, and thus cannot be Nash-implemented by a feasible mechanism. Consequently, it cannot be dispensed with for double implementation in Nash and strong Nash equilibria.

2.2 *Lindahl allocations and core allocations*

An allocation $(x, y) = (x_1, \dots, x_n, y)$ is *feasible* if $(x, y) \in \mathbb{R}^{nL+K}_+$ and $(\sum_{i=1}^n x_i - \sum_{i=1}^n \mathring{w}_i, y) \in \mathscr{Y}$.

An allocation (x^*, y^*) is a *Lindahl allocation* for an economy e if it is feasible and there is a price vector $p^* \in \mathbb{R}^L_+$ and personalized price vectors $q_i^* \in \mathbb{R}^K_+$, one for each i, such that

1) $p^* \cdot x_i^* + q_i^* \cdot y^* = p^* \cdot \mathring{w}_i$ for all $i \in N$;
2) $(x_i, y)P_i(x_i^*, y^*)$ implies $p^* \cdot x_i + q_i^* \cdot y > p^* \cdot \mathring{w}_i$ for all $i \in N$;
3) $\hat{q}^* \cdot y^* + p^* \cdot r^* \geq \hat{q}^* \cdot y + p^* \cdot r$ for all $(r, y) \in \mathscr{Y}$,

where $r^* = \sum_{i=1}^n x_i^* - \sum_{i=1}^n \mathring{w}_i$, and $\hat{q}^* = \sum_{i=1}^n q_i^*$ which should be regarded as the (market) price vector of public goods. Note that conditions 1 and 3 in the preceding definition is equivalent to $\hat{q}^* \cdot y + p^* \cdot r \leq 0$ for all $(r, y) \in \mathscr{Y}$, which is the familiar zero-profit maximizing condition under constant returns. Denote by $L(e)$ the set of all such allocations. Let E be the class of public goods economies which satisfies Assumptions 1–3, and on which $L(e) \neq \emptyset$.

Let $\mathscr{Y}^*$ be the dual cone of $\mathscr{Y}$ which is defined by

$$\mathscr{Y}^* = \{(p, \hat{q}) \in \mathbb{R}^{L+K} : p \cdot r + \hat{q} \cdot y \leq 0 \ \forall (r, y) \in \mathscr{Y}\}. \tag{1}$$

Denote by $\mathscr{Y}^*_+ = \mathscr{Y}^* \cap [\mathbb{R}^L_{++} \times \mathbb{R}^K_+]$ the nonnegative dual cone of $\mathscr{Y}$. Since $L(e) \neq \emptyset$, $\mathscr{Y}^*_+ \neq \emptyset$ by Assumptions 1 and 3. Thus, any element in $\mathscr{Y}^*_+$ gives an efficiency price system of private and public goods, which means the zero-profit (maximizing) condition holds, i.e.,

$$\hat{q} \cdot y + p \cdot r \leq 0 \tag{2}$$

for all $(r, y) \in \mathscr{Y}$.

A *coalition* C is a non-empty subset of N.

A feasible allocation $(x, y) \in \mathbb{R}^{nL+K}_+$ can be improved upon by $C \subset N$ if there exists an allocation (x', y') such that

(i) $(\sum_{i \in C}(x_i - \mathring{w}_i), y) \in \mathscr{Y}$, and
(ii) $(x_i', y')P_i(x_i, y)$ for all $i \in C$.

A feasible allocation (x, y) is in the core of e if there does not exist a coalition C that can improve upon (x, y).

An allocation (x, y) is *Pareto-optimal* with respect to the strict preference profile $P = (P_1, \dots, P_n)$ if it cannot be improved upon by N.

An allocation (x, y) is *individually rational* with respect to the strict preference profile $P = (P_1, \dots, P_n)$ if it cannot be improved upon by every single individual i.

Note that every Lindahl allocation defined on $e \in E$ must be in the core, and thus it is Pareto-optimal and individually rational (cf. Foley 1970).

3 Mechanism

In the following we will present a feasible and continuous mechanism which doubly implements the Lindahl correspondence in Nash and strong Nash equilibria.

Let M_i denote the i-th message domain. Its elements are written as m_i and called messages. Let $M = \prod_{i=1}^n M_i$ denote the message space. The message spaces of agents are defined as follows.

For each $i \in N$, the message domain of agent i is of the form

$$M_i = (0, \mathring{w}_i] \times \overline{\mathscr{Y}}_+^* \times \mathbb{R}^{nL} \times \mathbb{R}^K, \tag{3}$$

Here $\overline{\mathscr{Y}}_+^* = \{(p, q_1, \ldots, q_n) \in \mathbb{R}_+^{L+nK} : (p, \sum_{j=1}^n q_j) \in \mathscr{Y}_+^*\}$.

A generic element of M is $m_i = (w_i, p_i, q_{i1}, \ldots, q_{in}, x_{i1}, \ldots, x_{in}, y_i)$, whose components have the following interpretations. The component w_i denotes a profession of agent i's endowment, the inequality $0 < w_i \leq \mathring{w}_i$ means that the agent cannot overstate his own endowment; on the other hand, the endowment can be understated, but the claimed endowment w_i must be positive. Note that, although the true endowment is the upper bound of the reported endowment, the designer does not need to know this upper bound. This is because whenever an agent claims an endowment of a certain amount, the designer can ask him to *exhibit* it (one may, for instance, imagine that the rules of the game require that the agent 'put on the table' the reported amount w_i). The component p_i is an efficiency price vector of private goods proposed by agent i and is used as a price vector of agent $i-1$ where $i-1$ is read to be n when $i=1$. The component $\bar{q}_i \equiv (q_{i1}, \ldots, q_{in})$ is an efficiency personalized price vector profile, so that the sum of personalized price vectors is the efficiency price vector of public goods. The component x_{ij} is a proposed contribution that agent i is willing to make to agent j (a negative x_{ij} means agent i wants to get $-x_{ij}$ amount of goods from agent j) and the component y_i denotes the proposed contribution to the production of public goods by agent i (a negative y_i means the agent wants to receive a subsidy from society).

For each $i \in N$, define a price vector function for private goods $p_i : M \to \mathbb{R}_+^L$ by

$$p_i(m) = p_{i+1}. \tag{4}$$

and a price vector function for public goods $\hat{q}_i : M \to \mathbb{R}_+^K$ by

$$\hat{q}_i(m) = \sum_{j=1}^n q_{i+1,j}, \tag{5}$$

where $n+1$ is to be read as 1.

Define the personalized price vector for the i-th consumer by

$$q_i(m) = q_{i+1,i}. \tag{6}$$

Note that even though $p_i(m)$, $\hat{q}_i(m)$, and $q_i(m)$ are functions of only the price-components p_{i+1} and $q_{i+1,i}$, announced by agent $i+1$ for agent i, we can write it as a function of m without loss of generality.

Define a correspondence $B_y : M \to 2^{\mathbb{R}_+^K}$ by

$$B_y(m) = \Bigg\{ y \in \mathbb{R}_+^K : \left(-\sum_{i \in N} w_i, y \right) \in \mathscr{Y}, \frac{p_i(m) \cdot w_i}{1 + \|p_i - p_{i+1}\| + \|\bar{q}_i - \bar{q}_{i+1}\|} - q_i(m) \cdot y \geq 0 \ \forall\, i \in N \Bigg\}, \tag{7}$$

where $\|\cdot\|$ is the Euclidian norm. $B_y(m)$ is clearly nonempty, compact, and convex for all $m \in M$. We will show the following lemma in the Appendix.

Lemma 1. *$B_y(\cdot)$ is continuous on M.*

Define the outcome function for public goods $Y : M \to B$ by

$$Y(m) = \left\{ y : \min_{y \in B_y(m)} \|y - \tilde{y}\| \right\}, \tag{8}$$

which is the closest point to $\tilde{y}$. Here $\tilde{y} = \sum_{i=1}^n y_i$. Then $Y(m)$ is single-valued and continuous on M.[3]

We then define a feasible correspondence $B_x : M \to 2^{\mathbb{R}_+^{nL}}$ by

$$B_x(m) = \Bigg\{ x \in R_+^{nL} : \left(\sum_{j \in N} x_j - \sum_{j \in N} w_j, Y(m) \right) \in \mathscr{Y} \quad p_i(m) \cdot x_i + q_i(m) \cdot Y(m) \leq \frac{p_i(m) \cdot w_i}{1 + \|p_i - p_{i+1}\| + \|\bar{q}_i - \bar{q}_{i+1}\|} \ \forall i \Bigg\} \tag{9}$$

which is a correspondence with non-empty, compact, and convex values. We will prove the following lemma in the Appendix.

Lemma 2. *$B_x(\cdot)$ is continuous on M.*

Now define the outcome function for private goods consumption $X(m) : M \to B_x$ by

$$X(m) = \left\{ x : \min_{x \in B_x(m)} \|x - \tilde{x}\| \right\}, \tag{10}$$

which is the closest point to $\tilde{x}$. Here $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)$ with $\tilde{x}_j = \sum_{i=1}^n x_{ij}$

[3] This is because $Y(m)$ is an upper semi-continuous correspondence by Berge's Maximum Theorem (see Debreu 1959, p. 19) and single-valued (see Mas-Colell 1985, p. 28).

for $j = 1, \ldots, n$. Then $X(m) = (X_1(m), \ldots, X_n(m))$ is single-valued and continuous on M.

Thus the outcome function is continuous and also feasible on M since $(X(m), Y(m)) \in \mathbb{R}_+^{nL+K}$ and $(\sum_{i=1}^n X_i(m) - \sum_{i=1}^n \mathring{w}_i, Y(m)) \in \mathscr{Y}$ for all $m \in M$.[4]

Denote by $h : M \to \mathbb{R}_+^{nL+K}$ the outcome function, or more explicitly, $h_i(m) = (X_i(m), Y(m))$. Then the mechanism consists of $\langle M, h \rangle$ which is defined on E.

A message $m^* = (m_1^*, \ldots, m_n^*) \in M$ is said to be a *Nash equilibrium* of the mechanism $\langle M, h \rangle$ for an economy e if for each $i \in N$ and $m_i \in M_i$, it is not true that

$$h_i(m_i, m_{-i}^*) P_i h_i(m^*), \tag{11}$$

where $(m_i, m_{-i}^*) = (m_1^*, \ldots, m_{i-1}^*, m_i, m_{i+1}^*, \ldots, m_n^*)$. $h(m^*)$ is then called a *Nash (equilibrium) allocation* of the mechanism for the economy e. Denote by $V_{M,h}(e)$ the set of all such Nash equilibria and by $N_{M,h}(e)$ the set of all such Nash (equilibrium) allocations.

The mechanism $\langle M, h \rangle$ is said to *Nash-implement* the Lindahl correspondence L on E, if, for all $e \in E$, $N_{M,h}(e) = L(e)$.

A message $m^* = (m_1^*, \ldots, m_n^*) \in M$ is said to be a *strong Nash equilibrium* of the mechanism $\langle M, h \rangle$ for an economy e if there does not exist any coalition C and $m_C \in \prod_{i \in C} M_i$ such that for all $i \in C$,

$$h_i(m_C, m_{-C}^*) P_i h_i(m^*). \tag{12}$$

$h(m^*)$ is then called a *strong Nash (equilibrium) allocation* of the mechanism for the economy e. Denote by $SV_{M,h}(e)$ the set of all such strong Nash equilibria and by $SN_{M,h}(e)$ the set of all such strong Nash (equilibrium) allocations.

The mechanism $\langle M, h \rangle$ is said to doubly *implement* the Lindahl correspondence L on E, if, for all $e \in E$, $SN_{M,h}(e) = N_{M,h}(e) = L(e)$.

Remark 3. The notion of strong Nash equilibrium adopted in the paper has been used in the double implementation literature such as those in Peleg (1996a,b) and Suh (1996, 1997). It may be remarked that, even when continuity and strict monotonicity conditions on preferences are imposed, Proposition 3 below will no longer be true if the following stronger condition of strong Nash equilibrium is adopted: A message profile $m^* = (m_1^*, \ldots, m_n^*) \in M$ is a *strong Nash equilibrium* if there does not exist any coalition C and a combination of messages to the coalition $m_C \in \prod_{i \in C} M_i$ such that (1) it is not true that $h_i(m^*) P_i h_i(m_C, m_{-C}^*)$ for all $i \in C$, and (2) $h_i(m_C, m_{-C}^*) P_i h_i(m^*)$ for some $i \in C$. This is because the mechanism does not allow taking small

[4] Although the outcome function is feasible, it is not balanced (merely weakly balanced) in the sense that the resulting allocation may not be on the frontier of the possibility set for some messages. One can see this fact from formula (9).

amounts of private goods from one agent and distributing them among other agents to make everyone better off, or it may violate someone's budget constraint when the agent's budget set cannot be changed by varying a combination of messages to the coalition m_C.

Remark 4. The above mechanism is irrespective of the number of agents. It is thus a unified mechanism which works for two-agent economies as well as for economies with three or more agents. For two-agent economies with one private good and one public good, a feasible and continuous mechanism which Nash implements the Lindahl correspondence was given only by Nakamura (1989). Here we give an even simpler feasible and continuous mechanism which implements the Lindahl correspondence not only in Nash equilibrium, but also in strong Nash equilibrium.

Remark 5. By the construction of the mechanism, one can see that the mechanism constructed in the paper only yields weakly balanced allocations since the feasibility condition may not hold with equality for all messages, although in equilibrium it necessarily holds with equality. However, since the above mechanism is a unified mechanism which deals with both cases of two-agent economies and economies with three or more agents, by a result given by Kwan and Nakamura (1987), it is impossible to find a mechanism which Nash implements the Lindahl correspondence with a balanced (not merely weakly balanced) feasible and continuous mechanism that works for two-agent economic environments.[5] Consequently, it is impossible to have a mechanism which doubly implements the Lindahl correspondence in Nash and strong Nash equilibria by a balanced feasible and continuous mechanism that works for two-agent economic environments.

4 Double implementation

The remainder of this paper is devoted to the proof of equivalence among Nash allocations, strong Nash allocations, and Lindahl allocations. Proposition 1 below proves that every Nash allocation is a Lindahl allocation. Proposition 2 below proves that every Lindahl allocation is a Nash allocation. Proposition 3 below proves that every Nash equilibrium is a strong Nash equilibrium. To show these results, we first prove the following lemmas.

Lemma 3. *Under Assumptions 1–3, if* $(X(m^*), Y(m^*)) \in N_{M,h}(e)$, *then* $X_i(m^*) \in \mathbb{R}^L_{++}$ *for all* $i \in N$.

Proof: Suppose, by way of contradiction, that $X_i(m^*) \in \partial\mathbb{R}^L_+$ for some $i \in N$. Since $w_i^* \in \mathbb{R}^L_{++}$, there is some $x_i \in \mathbb{R}^L_{++}$ such that $p_i(m^*) \cdot x_i \leq$

[5] For economies with three or more agents, however, it is possible to find a mechanism which Nash implements the Lindahl correspondence by a balanced feasible and continuous mechanism. Tian (1990, 1993), Li et al. (1995), and Tian and Li (1995) gave this type of mechanisms.

$\frac{p_i(m^*)\cdot w_i^*}{1+\|p_i^*-p_{i+1}^*\|+\|\bar{q}_i^*-\bar{q}_{i+1}^*\|}$, and $(x_i,0)P_i(X_i(m^*),Y(m^*))$ by Assumption 2. Now suppose agent i chooses $y_i=-\sum_{j\neq i}^n y_j^*$, $x_{ii}=x_i-\sum_{k\neq i}x_{k,i}^*$, $x_{ij}=-\sum_{k\neq i}x_{k,j}^*$ for $j\neq i$, and keeps other components of the message unchanged. Since $0\in\mathscr{Y}$, we have $0\in B_y(m_i,m_{-i}^*)$. Also, $(0,\dots,0,x_i,0,\dots,0)\in B_x(m_i,m_{-i}^*)$. Then, we have $(X_i(m_i,m_{-i}^*),Y(m_i,m_{-i}^*))=(x_i,0)$, and thus $(X_i(m_i,m_{-i}^*),Y(m_i,m_{-i}^*))P_i(X_i(m^*),Y(m^*))$. This contradicts $(X(m^*),Y(m^*))\in N_{M,h}(e)$ and thus we must have $X_i(m^*)\in\mathbb{R}_{++}^L$ for all $i\in N$. Q.E.D.

Lemma 4. *Under Assumptions 1–3, if m^* is a Nash equilibrium, then $p_1^*=p_2^*=\cdots=p_n^*$, $\bar{q}_1^*=\bar{q}_2^*=\cdots=\bar{q}_n^*$. Consequently, $p_1(m^*)=p_2(m^*)=\cdots=p_n(m^*):=p(m^*)\in\mathbb{R}_{++}^L$, $\hat{q}_1(m^*)=\hat{q}_2(m^*)=\cdots=\hat{q}_n(m^*):=\hat{q}(m^*)$, and $\sum_{i\in N}q_i(m^*)=\hat{q}(m^*)$.*

Proof: Suppose, by way of contradiction, that $p_i^*\neq p_{i+1}^*$ and/or $\bar{q}_i^*\neq\bar{q}_{i+1}^*$ for some $i\in N$. Then $p_i(m^*)\cdot X_i(m^*)+q_i(m^*)\cdot Y(m^*)\leq \frac{1}{1+\|p_i^*-p_{i+1}^*\|+\|\bar{q}_i^*-\bar{q}_{i+1}^*\|}p_i(m^*)\cdot w_i^* < \frac{1}{1+\|p_i-p_{i+1}^*\|+\|\bar{q}_i-\bar{q}_{i+1}^*\|}\cdot p_i(m^*)\cdot w_i^*$ for all p_i and $\bar{q}_i$ such that $\|p_i-p_{i+1}^*\|+\|\bar{q}_i-\bar{q}_{i+1}^*\|<\|p_i^*-p_{i+1}^*\|+\|\bar{q}_i^*-\bar{q}_{i+1}^*\|$. Thus, there is $x_i>X_i(m^*)$ such that $p_i(m^*)\cdot x_i+q_i(m^*)\cdot Y(m^*)\leq\frac{1}{1+\|p_i-p_{i+1}^*\|+\|\bar{q}_i-\bar{q}_{i+1}^*\|}p_i(m^*)\cdot w_i^*$, and $(x_i,Y(m^*))P_i(X_i(m^*),Y(m^*))$ by monotonicity of preferences. Since $X(m^*)\in\mathbb{R}_{++}^{nL}$ by Lemma 3, we can actually make $x_i<\sum_{j=1}^n X_j(m^*)$, and thus we have $(x_i-\sum_{j\in N}w_j^*,Y(m^*))\in\mathscr{Y}$ by Remark 1. Also, by Lemma 3, $q_j(m^*)\cdot Y(m^*)<\frac{1}{1+\|p_j^*-p_{j+1}^*\|+\|\bar{q}_j^*-\bar{q}_{j+1}^*\|}p_j(m^*)\cdot w_j^*$ for all $j\in N$. Thus, if agent i chooses p_i and $\bar{q}_i$ such that they are sufficiently close to p_i^* and $\bar{q}_i^*$ and satisfy $\|p_i-p_{i+1}^*\|+\|\bar{q}_i-\bar{q}_{i+1}^*\|<\|p_i^*-p_{i+1}^*\|+\|\bar{q}_i^*-\bar{q}_{i+1}^*\|$, $x_{ii}=x_i-\sum_{k\neq i}x_{k,i}^*$, $x_{ij}=-\sum_{k\neq i}x_{k,j}^*$ for $j\neq i$, and keeps other components of the message unchanged, then $Y(m^*)\in B_y(m_i,m_{-i}^*)$ by noting that $q_j(m_i,m_{-i}^*)\cdot Y(m^*)<\frac{1}{1+\|p_j^*-p_{j+1}^*\|+\|\bar{q}_j^*-\bar{q}_{j+1}^*\|}p_j(m^*)\cdot w_j^*$ for all $j\in N$, $p_j(\cdot)$ and $q_j(\cdot)$ are continuous. Also, $(0,\dots,0,x_i,0,\dots,0)\in B_x(m_i,m_{-i}^*)$. Therefore, $(X_i(m_i,m_{-i}^*),Y(m_i,m_{-i}^*))=(x_i,Y(m^*))$. Hence, $(X_i(m_i,m_{-i}^*),Y(m_i,m_{-i}^*))P_i(X_i(m^*),Y(m^*))$. This contradicts $(X(m^*),Y(m^*))\in N_{M,h}(e)$. Thus we must have $p_1^*=p_2^*=\cdots=p_n^*$ and $\bar{q}_1^*=\bar{q}_2^*=\cdots=\bar{q}_n^*$. Consequently, $p_1(m^*)=p_2(m^*)=\cdots=p_n(m^*)=p(m^*)$, $\hat{q}_1(m^*)=\hat{q}_2(m^*)=\cdots=\hat{q}_n(m^*)=\hat{q}(m^*)$, and $\sum_{i\in N}q_i(m^*)=\hat{q}(m^*)$. Q.E.D.

Lemma 5. *Under Assumptions 1–3, if m^* is a Nash equilibrium, then $w_i^*=\mathring{w}_i$ for all $i\in N$.*

Proof: Suppose, by way of contradiction, that $w_i^*\neq\mathring{w}_i$ for some $i\in N$. Then $p_i(m^*)\cdot X_i(m^*)+q_i(m^*)\cdot Y(m^*)\leq p_i(m^*)\cdot w_i^*<p_i(m^*)\cdot\mathring{w}_i$, and thus there is $x_i>X_i(m^*)$ such that $p_i(m^*)\cdot x_i+q_i(m^*)\cdot Y(m^*)\leq p_i(m^*)\cdot\mathring{w}_i$,

$(x_i - \sum_{j \in N} w_j^*, Y(m^*)) \in \mathscr{Y}$, and $(x_i, Y(m^*))P_i(X_i(m^*), Y(m^*))$ by monotonicity of preferences. Thus if agent i chooses $w_i = \mathring{w}_i$, $x_{ii} = x_i - \sum_{k \neq i} x_{k,i}^*$, $x_{ij} = -\sum_{k \neq i} x_{k,j}^*$ for $j \neq i$, and keeps other components of the message unchanged, then $Y(m^*) \in B_y(m_i, m_{-i}^*)$ and $(0, \ldots, 0, x_i, 0, \ldots, 0) \in B_x(m_i, m_{-i}^*)$, and thus $(X_i(m_i, m_{-i}^*), Y(m_i, m_{-i}^*)) = (x_i, Y(m^*))$. Therefore, $(X_i(m_i, m_{-i}^*), Y(m_i, m_{-i}^*))P_i(X_i(m^*), Y(m^*))$, a contradiction. Q.E.D.

Lemma 6. *If* $(X(m^*), Y(m^*)) \in N_{M,h}(e)$, *then* $p(m^*) \cdot X_i(m^*) + q_i(m^*) \cdot Y(m^*) = p(m^*) \cdot \mathring{w}_i$.

Proof: Suppose, by way of contradiction, that $p(m^*) \cdot X_i(m^*) + q_i(m^*) \cdot Y(m^*) < p(m^*) \cdot \mathring{w}_i$. Then there is $x_i > X_i(m^*)$ such that $p(m^*) \cdot x_i + q_i(m^*) \cdot Y(m^*) \leq p(m^*) \cdot \mathring{w}_i$, $(x_i - \sum_{j \in N} w_j^*, Y(m^*)) \in \mathscr{Y}$, and $(x_i, Y(m^*))P_i(X_i(m^*), Y(m^*))$ by monotonicity of preferences. Thus if agent i chooses $x_{ii} = x_i - \sum_{k \neq i} x_{k,i}^*$, $x_{ij} = -\sum_{k \neq i} x_{k,j}^*$ for $j \neq i$, and keeps other components of the message unchanged, then $Y(m^*) \in B_y(m_i, m_{-i}^*)$ and $(0, \ldots, 0, x_i, 0, \ldots, 0) \in B_x(m_i, m_{-i}^*)$, and thus $(X_i(m_i, m_{-i}^*), Y(m_i, m_{-i}^*)) = (x_i, Y(m^*))$. Therefore, $(X_i(m_i, m_{-i}^*), Y(m_i, m_{-i}^*))P_i(X_i(m^*), Y(m^*))$. This contradicts $(X(m^*), Y(m^*)) \in N_{M,h}(e)$. Q.E.D.

Proposition 1. *Under Assumptions 1–3, if* m^* *is a Nash equilibrium of the mechanism defined above, then the Nash allocation* $(X(m^*), Y(m^*))$ *is a Lindahl allocation with the price system* $(p(m^*), q_1(m^*), \ldots, q_n(m^*))$ *as the Lindahl price vector, i.e.,* $N_{M,h}(e) \subseteq L(e)$ *for all* $e \in E$.

Proof: Let m^* be a Nash equilibrium. Now we prove that $(X(m^*), Y(m^*))$ is a Lindahl allocation with $(p(m^*), q_1(m^*), \ldots, q_n(m^*))$ as the price vector. First note that, by Lemmas 3–6, the mechanism is feasible, $\sum_{i=1}^n q_i(m^*) = \hat{q}(m^*)$, $p(m^*) \cdot X_i(m^*) + q_i(m^*) \cdot Y(m^*) = p(m^*) \cdot \mathring{w}_i$ for all $i \in N$. Also, since $(p^*(m^*), \hat{q}(m^*)) \in \mathscr{Y}^*$ is an efficiency price vector, we have $\hat{q}(m^*) \cdot y + p(m^*) \cdot r \leq 0 = \hat{q}(m^*) \cdot Y(m^*) + p(m^*) \cdot (\sum_{i=1}^n X_i(m^*) - \sum_{i=1}^n \mathring{w}_i)$ for all $(r, y) \in \mathscr{Y}$, which means $Y(m^*)$ is a profit maximizing level of output. We only need to show that each individual is maximizing his/her preferences. Suppose, by way of contradiction, that there is some $(x_i, y) \in \mathbb{R}_+^{L+K}$ such that $(x_i, y)P_i(X_i(m^*), Y(m^*))$ and $p(m^*) \cdot x_i + q_i(m^*) \cdot y \leq p(m^*) \cdot \mathring{w}_i$. Because of the monotonicity of preferences, it will be enough to confine ourselves to the case of $p(m^*) \cdot x_i + q_i(m^*) \cdot y = p(m^*) \cdot \mathring{w}_i$. Also, by Assumption 3, for such a $y \in \mathbb{R}_+^K$, there is $r \in -\mathbb{R}_+^L$ such that $(r, y) \in \mathscr{Y}$. Let

$$x_{\lambda i} = \lambda x_i + (1 - \lambda) X_i(m^*)$$

$$y_\lambda = \lambda y + (1 - \lambda) Y(m^*).$$

$$r_\lambda = \lambda r + (1 - \lambda) r(m^*).$$

Here $r(m^*) = \sum_{j=1}^n X_j(m^*) - \sum_{j=1}^n \mathring{w}_j$. Then, by convexity of preferences and the production possibility set, we have $(x_{\lambda i}, y_\lambda)P_i(X_i(m^*), Y(m^*))$ and $(r_\lambda, y_\lambda) \in \mathscr{Y}$ for any $0 < \lambda < 1$. Also, $(x_{\lambda i}, y_\lambda) \in \mathbb{R}_+^{L+K}$ and $p(m^*) \cdot x_{\lambda i} + q_i(m^*) \cdot y_\lambda = p(m^*) \cdot w_i$.

Since $X(m^*) \in \mathbb{R}^{nL}_{++}$, we must have $p(m^*) \cdot w_j - q_j(m^*) \cdot Y(m^*) > 0$ for all $j \in N$ and $r(m^*) = \sum_{j=1}^n X_j(m^*) - w^* > X_i(m^*) - w^*$, where $w^* = \sum_{i=1}^n w_i^*$. Then, we have $p(m_i, m^*) \cdot w_j - q_j(m_i, m^*) \cdot y_\lambda > 0$ for all $j \in N$ and $x_{\lambda i} - w^* < r_\lambda$ as λ is a sufficiently small positive number. Thus, by Remark 1, $(x_{i\lambda} - w^*, y_\lambda) \in \mathscr{Y}$ and $(-w^*, y_\lambda) \in \mathscr{Y}$ since $-w^* < x_{i\lambda} - w^* < r_\lambda$.

Now suppose that player i chooses $y_i = y_\lambda - \sum_{j \neq i}^n y_j^*$, $x_{ii} = x_i - \sum_{k \neq i} x_{k,i}^*$, and keeps x_{ij}^* $(j \neq i)$, w_i^* and p_i^* unchanged. Then $w_j^* - q_j(m_i, m^*) \cdot y_\lambda > 0$ for all $j \in N$, $(-\sum_{j \in N} w_j^*, y_\lambda) \in \mathscr{Y}$, and $(x_{\lambda i} - \sum_{j \in N} w_j^*, y_\lambda) \in \mathscr{Y}$ as λ is sufficiently small. Thus, $y_\lambda \in B_y(m_i, m_{-i}^*)$ and $(0, \ldots, 0, x_{\lambda i}, 0, \ldots, 0) \in B_x(m_i, m_{-i}^*)$. Then $Y(m_i, m_{-i}^*) = y_\lambda$ and $X_i(m_i, m_{-i}^*) = x_{\lambda i}$. From $(x_{\lambda i}, y_\lambda) P_i(X_i(m^*), Y(m^*))$, we have $(X_i(m_i, m_{-i}^*), Y(m_i, m_{-i}^*)) P_i(X_i(m^*), Y(m^*))$. This contradicts the hypothesis that $(X(m^*), Y(m^*)) \in N_{M,h}(e)$. Q.E.D.

Proposition 2. *Under Assumptions 1–3, if (x^*, y^*) is a Lindahl allocation with the price system $(p^*, q_1^*, \ldots, q_n^*)$, then there is a Nash equilibrium m^* for the mechanism defined above such that $X_i(m^*) = x_i^*$, $p(m^*) = p^*$, and $q_i(m^*) = q_i^*$, for all $i \in N$, $Y(m^*) = y^*$, i.e., $L(e) \subseteq N_{M,h}(e)$ for all $e \in E$.*

Proof: We first note that $p^* \in \mathbb{R}^L_{++}$ and $x^* \in \mathbb{R}^{nL}_{++}$ by Assumptions 1–2. We need to show that there is a message m^* such that (x^*, y^*) is a Nash allocation. For each $i \in N$, define $m_i^* = (w_i^*, p_i^*, q_{i1}^*, \ldots, q_{in}^*, x_{i1}^*, \ldots, x_{in}^*, y_i^*)$ by $w_i^* = \mathring{w}_i$, $p_i^* = p^*$, $q_{ij}^* = q_j^*$ for $j = 1, \ldots, n$, $(x_{i1}^*, \ldots, x_{in}^*) = (0, \ldots, 0, x_i^*, 0, \ldots, 0)$, and $y_i^* = y^*/n$. Then, it can be easily verified that $Y(m^*) = y^*$, $p(m^*) = p^*$, and $q_i(m^*) = q_i^*$, $X_i(m^*) = x_i^*$, for all $i \in N$. Notice that $p(m_i, m_{-i}^*) = p(m^*)$ and $q_i(m_i, m_{-i}^*) = q_i(m^*)$ for all $m_i \in M_i$, $(X(m_i, m_{-i}^*), Y(m_i, m_{-i}^*)) \in \mathbb{R}^{nL+K}_{+}$ and $p(m^*) \cdot X_i(m_i, m_{-i}^*) + q_i(m^*) \cdot Y(m_i, m_{-i}^*) = p(m^*) \cdot \mathring{w}_i$ for all $i \in N$ and $m_i \in M_i$. Therefore, for all $m_i \in M_i$, it is not true that

$$(X_i(m_i, m_{-i}^*), Y(m_i, m_{-i}^*)) P_i(X_i(m^*), Y(m^*)),$$

or it contradicts the fact that $(X_i(m^*), Y(m^*))$ is a Lindahl allocation. Q.E.D.

Proposition 3. *Under Assumptions 1–3, every Nash equilibrium m^* of the mechanism defined above is a strong Nash equilibrium, that is $N_{M,h}(e) \subseteq SN_{M,h}(e)$ for all $e \in E$.*

Proof: Let m^* be a Nash equilibrium. By Proposition 1, we know that $(X(m^*), Y(m^*))$ is a Lindahl allocation with $(p(m^*), q_1(m^*), \ldots, q_n(m^*))$ as the Lindahl price system. Then $(X(m^*), Y(m^*))$ is Pareto optimal and thus the coalition N cannot be improved upon by any $m \in M$. Now for any coalition C with $\emptyset \neq C \neq N$, choose $i \in C$ such that $i + 1 \notin C$. Then no strategy played by C can change the budget set of i since $p_i(m)$ and $q_i(m)$ are determined by $p_{i+1,i}$ and $q_{i+1,i}$, respectively. Furthermore, because $(X(m^*), Y(m^*)) \in L(e)$, it is P_i-maximal in the budget set of i, and thus C cannot improve upon $(X(m^*), Y(m^*))$. Q.E.D.

Since every strong Nash equilibrium is clearly a Nash equilibrium, by combining Propositions 1–3, we have the following theorem.

Theorem 1. *For public goods economies E, there exists a feasible and continuous mechanism which doubly implements the Lindahl correspondence in Nash and strong Nash equilibria. That is,* $N_{M,h}(e) = SN_{M,h}(e) = L(e)$ *for all* $e \in E$.

5 Concluding remarks

In this paper, we have presented a simple mechanism which doubly implements the Lindahl correspondence in Nash and strong Nash equilibria when coalition patterns, preferences, and endowments are unknown to the planner. This implementation result is obtained without defining an artificial preference profile on prices announced by individuals. The important reasons for preferring double implementation over Nash implementation or strong Nash implementation are: (1) the double implementation covers the case where agents in some coalitions may cooperate and in some other coalitions may not, when such information is unknown to the designer, and (2) this combining solution concept which characterizes agents' strategic behavior may give a state that is easy to reach and hard to leave. An advantage of our mechanism is that it works not only for three or more agents, but also for a two-agent world. While all the mechanisms in the existing literature, which Nash implement Lindahl allocations, need to distinguish the case of two agents from that of three or more agents, this paper gives a unified mechanism which is irrespective of the number of agents. Furthermore, we allow preferences of agents to be nontotal-nontransitive and discontinuous. In addition, this mechanism is well-behaved and natural in the sense that it is feasible and continuous, and has a message space of finite dimension. Of course, a disadvantage of the mechanism is that it is merely weakly balanced, but not balanced in the sense that the resulting allocation may not be on the frontier of the possibility set for non-equilibrium messages. Whether or not there exists a balanced feasible and continuous mechanism which doubly implements the Lindahl correspondence in Nash and strong Nash equilibrium for economies with three or more agents remains an open question. Another disadvantage of the mechanism is that it requires that production technologies be known to the designer. Tian (1999) gave a mechanism which doubly implements Lindahl allocations for constant return economies with one private good and K public goods when production functions are unknown to the designer.

Appendix

Proof of Lemma 1: $B_y(m)$ is clearly upper hemi-continuous for all $m \in M$. We only need to show that $B_y(m)$ is also lower hemi-continuous at every $m \in M$. Let $m \in M$, $y \in B_y(m)$, and let $\{m_t\}$ be a sequence such that $m_t \to m$, where $m_t = (m_1^t, \ldots, m_n^t)$ and $m_i^t = (w_i^t, p_i^t, q_{i1}^t, \ldots, q_{in}^t, x_{i1}^t, \ldots, x_{in}^t, y_i^t)$. We want to prove that there is a sequence $\{y_t\}$ such that $y_t \to y$, and, for all t, $y_t \in B_y(m_t)$, i.e., $y_t \in \mathbb{R}_+^K$, $(-\sum_{i\in N} w_i^t, y_t) \in \mathcal{Y}$, and $q_i(m_t)\cdot y_t \leq \dfrac{p_i(m_t)\cdot w_i^t}{1+\|p_i^t - p_{i+1}^t\| + \|\bar{q}_i^t - \bar{q}_{i+1}^t\|}$ for all $i \in N$.

Let $N' = \left\{ i \in N : q_i(m)\cdot y = \dfrac{p_i(m)\cdot w_i}{1+\|p_i - p_{i+1}\| + \|\bar{q}_i - \bar{q}_{i+1}\|}\right\}$. Two cases will be considered.

Case 1. $N' = \emptyset$, i.e., $q_i(m)\cdot y < \dfrac{p_i(m)\cdot w_i}{1+\|p_i - p_{i+1}\| + \|\bar{q}_i - \bar{q}_{i+1}\|}$ for all $i \in N$. Then, for all t larger than a certain integer t', we have $q_i(m_t)\cdot y < \dfrac{p_i(m_t)\cdot w_i^t}{1+\|p_i^t - p_{i+1}^t\| + \|\bar{q}_i^t - \bar{q}_{i+1}^t\|}$. Let $T(m) = \{z \in \mathbb{R}_+^K : (-\sum_{i\in N} w_i, z) \in \mathcal{Y}\}$. Let $\bar{y}_t$ be the closest point to y in $[0, y] \cap T(m_t)$. Since $0 \in T(m)$ and $T(m)$ is closed and convex, $\bar{y}_t$ is well-defined and unique. Notice that, since $\bar{y}_t \in [0, y]$, $\bar{y}_t = \alpha_t y$ for some $\alpha_t \in [0, 1]$. We claim that $\bar{y}_t \to y$, i.e., $\alpha_t \to 1$. Suppose, by way of contradiction, that there is a subsequence $\{y_{t_k}\}$ such that $y_{t_k} \to y_0$ and $y_0 \neq y$, i.e., $y_0 = \alpha_0 y$ for some $\alpha_0 < 1$. Then, there is k' such that, for every $k > k'$, the production plan $(-\sum_{i\in N} w_i^{t_k}, y_{t_k})$ must be on the frontier of $\mathcal{Y}$ (i.e., it is technologically efficient), and thus $\hat{q}\cdot Y(m_{t_k}) - p\cdot \sum_{i\in N} w_i^{t_k} = 0$ for an efficiency price vector $(p, q_1, \ldots, q_1) \in \mathcal{Y}_+^*$. Thus, $\hat{q}\cdot y_0 - p\cdot\sum_{i\in N} w_i = 0 \geq \hat{q}\cdot y - p\cdot \sum_{i\in N} w_i$, which implies $\alpha_0 \hat{q}\cdot y \geq \hat{q}\cdot y$ and thus $\alpha_0 \geq 1$. This contradicts the fact that $\alpha_0 < 1$. So we must have $\bar{y}_t \to y$. Now, let $y_t = \bar{y}_t$ for all $t > t'$ and $y_t = 0$ for $t \leq t'$. Then, $y_t \to y$, and, for all t, $y_t \in B_y(m_t)$, i.e., $y_t \in \mathbb{R}_+^K$, $(-\sum_{i\in N} w_i^t, y_t) \in \mathcal{Y}$, and $q_i(m_t)\cdot y_t \leq \dfrac{p_i(m_t)\cdot w_i^t}{1+\|p_i^t - p_{i+1}^t\| + \|\bar{q}_i^t - \bar{q}_{i+1}^t\|}$ for all $i \in N$ by noting that $y_t \leq y$. Thus, the sequence $\{y_t\}$ has all the desired properties.

Case 2. $N' \neq \emptyset$, i.e., $q_i(m)\cdot y = \dfrac{p_i(m)\cdot w_i}{1+\|p_i - p_{i+1}\| + \|\bar{q}_i - \bar{q}_{i+1}\|}$ for all $i \in N'$.

Let $\omega_i = \dfrac{p_i(m)\cdot w_i}{1+\|p_i - p_{i+1}\| + \|\bar{q}_i - \bar{q}_{i+1}\|}$, $\omega_{it} = \dfrac{p_i(m_t)\cdot w_i^t}{1+\|p_i^t - p_{i+1}^t\| + \|\bar{q}_i^t - \bar{q}_{i+1}^t\|}$, and let $a_{it} = \dfrac{\omega_{it}}{q_i(m_t)\cdot y} y$ for $i \in N'$. For each t, let $N'(t) = \{i \in N' : q_i(m_t)\cdot y \geq \omega_{it}\}$, let $a_t = \min_{i\in N'(t)}\{a_{it}\}$, and define $\hat{y}_t$ as follows:

$$\hat{y}_t = \begin{cases} a_t & \text{if } N'(t) \neq \emptyset \\ y & \text{otherwise} \end{cases}.$$

Then $\hat{y}_t \leq y$ by noting that $\dfrac{\omega_{ik}}{q_i(m_t)\cdot y} \leq 1$ and $a_{it} = \dfrac{\omega_{it}}{q_i(m_t)\cdot y} y \leq y$ for $i \in N'(t)$. Also, since $\dfrac{\omega_{it}}{q_i(m_t)\cdot y} \to \dfrac{\omega_i}{q_i(m)\cdot y} = 1$ for all $i \in N'$, we have $a_{it} \to y$ for all $i \in N'$ and thus $\hat{y}_t \to y$. Now we claim that $\hat{y}_t$ also satisfies all individuals' budget sets. Indeed, for any $i \in N'(t)$, we have

$$q_i(m_t) \cdot \hat{y}_t \leq q_i(m_t) \cdot a_{it} = \omega_{it}.$$

For any $i \in N' \backslash N'(t)$, we have

$$q_i(m_t) \cdot \hat{y}_t = q_i(m_t) \cdot y \leq \omega_{it}$$

by noting that $\hat{y}_t = y$ and $q_i(m_t) \cdot y \leq \omega_{it}$ for $i \in N' \backslash N'(t)$. For all $i \in N \backslash N'$, since $q_i(m) \cdot y < \omega_{it}$, we have $q_i(m_t) \cdot \hat{y}_t \leq q_i(m_t) \cdot y < \omega_{it}$ for all t larger than a certain integer t'. Finally, let $y_t = \min(\bar{y}_t, \hat{y}_t)$. Then $y_t \to y$ since $\bar{y}_t \to y$ and $\hat{y}_t \to y$. Also, $y_t \geq 0$, $(-\sum_{i \in N} w_i^t, y_t) \in \mathcal{Y}$ by Remark 1 (because $y_t \leq \bar{y}_t$ and $(-\sum_{i \in N} w_i^t, \bar{y}_t) \in \mathcal{Y}$), and $q_i(m_t) \cdot y_t \leq \dfrac{p_i(m_t) \cdot w_i^t}{1 + \|p_i^t - p_{i+1}^t\| + \|\bar{q}_i^t - \bar{q}_{i+1}^t\|}$ for all $i \in N$ by noting that $y_t \leq \hat{y}_t$. Thus, $y_t \in B_y(m_t)$ for all $t > t'$. Therefore, the sequence $\{y_t\}$ has all the desired properties. So $B_y(m)$ is lower hemi-continuous at every $m \in M$. Q.E.D.

Proof of Lemma 2: $B_x(m)$ is clearly upper hemi-continuous for all $m \in M$. We only need to show that $B_x(m)$ is also lower hemi-continuous at every $m \in M$. Let $m \in M$, $x = (x_1, \ldots, x_n) \in B_x(m)$, and let $\{m_t\}$ be a sequence such that $m_t \to m$, where $m_t = (m_1^t, \ldots, m_n^t)$ and $m_i^t = (w_i^t, p_i^t, q_{i1}^t, \ldots, q_{in}^t, x_{i1}^t, \ldots, x_{in}^t, y_i^t)$. We want to prove that there is a sequence $\{x_t\}$ such that $x_t \to x$, and, for all t, $x_t \in B_x(m_t)$, i.e., $x_t = (x_1^t, \ldots, x_n^t) \in \mathbb{R}_+^{nL}$, $(\sum_{i \in N} x_i^t - \sum_{i \in N} w_i^t, Y(m_t)) \in \mathcal{Y}$, and $p_i(m_t) \cdot x_i^t + q_i(m_t) \cdot Y(m_t) \leq \dfrac{p_i(m_t) \cdot w_i^t}{1 + \|p_i^t - p_{i+1}^t\| + \|\bar{q}_i^t - \bar{q}_{i+1}^t\|}$ for all $i \in N$. We first prove that there is a sequence $\{\hat{x}_t\}$ such that $\hat{x}_t \to x$, and, for all t, $\hat{x}_t \in \mathbb{R}_+^{nL}$ and $p_i(m_t) \cdot \hat{x}_i^t + q_i(m_t) \cdot Y(m_t) \leq \dfrac{p_i(m_t) \cdot w_i^t}{1 + \|p_i^t - p_{i+1}^t\| + \|\bar{q}_i^t - \bar{q}_{i+1}^t\|}$ for all $i \in N$. For each $i \in N$, two cases will be considered.

Case 1. $p_i(m) \cdot x_i + q_i(m) \cdot Y(m) < \dfrac{p_i(m) \cdot w_i}{1 + \|p_i - p_{i+1}\| + \|\bar{q}_i - \bar{q}_{i+1}\|}$. Hence, for all t larger than a certain integer t', we have $p_i(m_t) \cdot x_i + q_i(m_t) \cdot Y(m_t) < \dfrac{p_i(m_t) \cdot w_i^t}{1 + \|p_i^t - p_{i+1}^t\| + \|\bar{q}_i^t - \bar{q}_{i+1}^t\|}$ by noting that $Y(\cdot)$, $q_i(\cdot)$, and $p_i(\cdot)$ are all continuous. Let $\hat{x}_i^t = x_i$ for all $t > t'$ and $\hat{x}_i^t = 0$ for $t \leq t'$. Then, we have $p_i(m_t) \cdot \hat{x}_i^t + q_i(m_t) \cdot Y(m_t) < \dfrac{p_i(m_t) \cdot w_i^t}{1 + \|p_i^t - p_{i+1}^t\| + \|\bar{q}_i^t - \bar{q}_{i+1}^t\|}$.

Case 2. $p_i(m) \cdot x_i + q_i(m) \cdot Y(m) = \dfrac{p_i(m) \cdot w_i}{1 + \|p_i - p_{i+1}\| + \|\bar{q}_i - \bar{q}_{i+1}\|}$. If $x_i = 0$, we can simply let $\hat{x}_i^t = 0$ for all t. We assume $x_i \neq 0$. Let $\omega_i = \dfrac{p_i(m) \cdot w_i}{1 + \|p_i - p_{i+1}\| + \|\bar{q}_i - \bar{q}_{i+1}\|}$ and $\omega_{it} = \dfrac{p_i(m_t) \cdot w_i^t}{1 + \|p_i^t - p_{i+1}^t\| + \|\bar{q}_i^t - \bar{q}_{i+1}^t\|}$. Define $\hat{x}_t$ as follows:

$$\hat{x}_{it} = \begin{cases} \frac{\omega_{it} - q_i(m_t) \cdot Y(m_t)}{p_i(m_t) \cdot x_i} x_i & \text{if } \frac{\omega_{it} - q_i(m_t) \cdot Y(m_t)}{p_i(m_t) \cdot x_i} \leq 1 \\ x_i & \text{otherwise} \end{cases}.$$

Then $\hat{x}_i^t \leq x_i$, and $p_i(m_t) \cdot \hat{x}_i^t + q_i(m_t) \cdot Y(m_t) \leq \frac{p_i(m_t) \cdot w_i^t}{1 + \|p_i^t - p_{i+1}^t\| + \|\bar{q}_i^t - \bar{q}_{i+1}^t\|}$. Also, since $\frac{\omega_{it} - q_i(m_t) \cdot Y(m_t)}{p_i(m_t) \cdot x_i} \to \frac{\omega_i - q_i(m) \cdot Y(m)}{p_i(m) \cdot x_i} = 1$, we have $\hat{x}_i^t \to x_i$. Thus, in both cases, there is a sequence $\{\hat{x}_t\}$ such that $\hat{x}_t \to x$, and, for all t, $\hat{x}_t \in \mathbb{R}_+^{nL}$ and $p_i(m_t) \cdot \hat{x}_i^t + q_i(m_t) \cdot Y(m_t) \leq \frac{p_i(m_t) \cdot w_i^t}{1 + \|p_i^t - p_{i+1}^t\| + \|\bar{q}_i^t - \bar{q}_{i+1}^t\|}$ for all $i \in N$.

We now show that there is a sequence $\{\bar{x}_t\}$ such that $\bar{x}_t \to x$, and, for all t, $\bar{x}_t \in \mathbb{R}_+^{nL}$ and $(\sum_{i \in N} \bar{x}_i^t - \sum_{i \in N} w_i^t, Y(m_t)) \in \mathscr{Y}$. Again, we need to consider only the case $x \neq 0$.

Let $G(m) = \{x' \in \mathbb{R}_+^{nL} : (\sum_{i \in N} x_i' - \sum_{i \in N} w_i, Y(m)) \in \mathscr{Y}\}$. Let $\bar{x}_t$ be the closest point to x in $[0, x] \cap G(m_t)$. Since $0 \in G(m)$ and $G(m)$ is closed and convex, $\bar{x}_t$ is well-defined and unique. Notice that, since $\bar{x}_t \in [0, x]$, $\bar{x}_t = \beta_t x$ for some $\beta_t \in [0, 1]$. We claim that $\bar{x}_t \to x$, i.e., $\beta_t \to 1$. Suppose, by way of contradiction, that there is a subsequence $\{x_{t_k}\}$ such that $x_{t_k} \to x_0$ and $x_0 \neq x$, i.e., $x_0 = \beta_0 x$ for some $\beta_0 < 1$. Then, there is k' such that, for every $k > k'$, the production plan $(\sum_{i \in N}(x_i^{t_k} - w_i^{t_k}), Y(m_{t_k}))$ must be on the boundary of $\mathscr{Y}$, and thus $\hat{q} \cdot Y(m_{t_k}) - p \cdot \sum_{i \in N}(w_i^{t_k} - x_i^{t_k}) = 0$ for an efficiency price vector $(p, q_1, \ldots, q_1) \in \mathscr{Y}_+^*$. Therefore, $\hat{q} \cdot Y(m) - p \cdot \sum_{i \in N}(w_i - x_{0i}) = 0 \geq \hat{q} \cdot Y(m) - p \cdot \sum_{i \in N}(w_i - x_i)$, which implies $\beta_0 p_i \cdot \sum_{i \in N} x_i \geq p \cdot \sum_{i \in N} x_i$ and thus $\beta_0 \geq 1$. This contradicts the fact that $\beta_0 < 1$. So we must have $\bar{x}_t \to x$.

Finally, let $x_t = \min(\bar{x}_t, \hat{x}_t)$ with $x_i^t = \min(\bar{x}_i^t, \hat{x}_i^t)$ for $i = 1, \ldots, n$. Then $x_t \to x$ since $\bar{x}_t \to x$ and $\hat{x}_t \to x$. Also, for every t, $x_t \geq 0$, $(\sum_{i \in N} x_i^t - \sum_{i \in N} w_i^t, Y(m_t)) \in \mathscr{Y}$ by Remark 1 (because $x_t \leq \bar{x}_t$ and $(\sum_{i \in N} \bar{x}_i^t - \sum_{i \in N} w_i^t, Y(m_t)) \in \mathscr{Y}$), and $p_i(m_t) \cdot x_i^t + q_i(m_t) \cdot Y(m_t) \leq \frac{p_i(m_t) \cdot w_i^t}{1 + \|p_i^t - p_{i+1}^t\| + \|\bar{q}_i^t - \bar{q}_{i+1}^t\|}$ for all $i \in N$ by noting that $x_i^t \leq \hat{x}_i^t$. Thus, $x_t \in B_x(m_t)$ for all $t > t'$. Therefore, the sequence $\{x_t\}$ has all the desired properties. So $B_x(m)$ is lower hemi-continuous at every $m \in M$. Q.E.D.

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