On the Existence of Strong Nash Equilibria

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Abstract

This paper investigates the existence of strong Nash equilibria (SNE) in continuous and concave games. We show that the coalition consistence property introduced in the paper, together with the concavity and continuity of payoffs, permits the existence of strong Nash equilibria in games with compact and convex strategy spaces. The coalition consistency property is a general condition that cannot be dispensed with for the existence of strong Nash equilibrium. It is satisfied in many economic games and can be checked with a similar way as finding weak Pareto efficient outcomes. We also characterize the existence of SNE by providing necessary and sufficient conditions. Moreover, we suggest an algorithm for efficiently computing strong Nash equilibrium. The results are illustrated with applications to economies with multilateral environmental externalities and to the simple oligopoly static model.

Keywords: Noncooperative game, strong Nash equilibrium, coalition, weak Pareto-efficiency.

1 Introduction

The concept of Nash equilibrium introduced by Nash [1950] is probably the most important behavioral solution concept in game theory. Nevertheless, Nash equilibrium has some serious shortcomings. First, Nash equilibrium is a strictly noncooperative notion and is only concerned with

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unilateral deviations from which no one can be improved. No cooperation among agents is allowed. As such, although a Nash equilibrium may be easy to reach, it may not be stable in the sense that there may exist a group of agents that can be improved by forming a coalition. Thus it is natural to have an equilibrium concept that allows possible cooperation or coalitions among agents.

Secondly, many games have multiple equilibria, and players may not be clear about which one to focus on.[1] This leads to a selection problem. Many refinements, which can be used to separate reasonable equilibria from unreasonable ones, have been proposed, such as the perfect equilibrium (Selten [1975]), the proper equilibrium (Myerson [1978]), the sequential equilibrium (Kreps and Wilson [1982]), and more recently the strong Berge equilibrium (Larbani and Nessah [2001]). All these equilibria are related to one another in varying degrees. However, in these solution concepts, there are still opportunities for joint deviations that are mutually beneficial for some group of players.

Thirdly, in designing an incentive mechanism, we may want equilibrium outcomes are not only easy to reach, but also hard to leave. Thus, one may desire to construct a mechanism which doubly implements a social choice rule by Nash and strong Nash equilibria. By double implementation, it can cover the situation where agents in some coalitions will cooperate and in some other coalitions will not, and thus the designer does not need to know which coalitions are permissible. Consequently, it allows the possibility for agents to manipulate coalition patterns. This solution concept needs to combine the properties of Nash equilibrium and strong Nash equilbimany and such mechanisms have been proposed in literature such as those in Tian [1999, 2000, 2003], Suh [1996, 1997, 2001, 2003], and Shin and Suh [1996].

These shortcomings on Nash equilibrium may motive us to adopt the solution concept of strong Nash equilibrium (SNE), which was introduced by Aumann [1959], to ensure more stable outcomes than the Nash equilibrium. A strong Nash equilibrium is defined as a strategic profile for which no subset of players has a joint deviation that strictly benefits all of them, while all other players are expected to maintain their equilibrium strategies. Since the deviating coalition can be a single player or the whole set of players, it implies that a SNE is also a Nash equilibrium and weakly Pareto efficient (in the sense that there is no other profile strictly preferred by all players). A SNE is then not only immune to unilateral deviations, but also to deviations by coalitions. Thus, it is also a refinement of Nash equilibrium.

The SNE has been used to study different noncooperative games as coalition formation (Hart and Kurz [1983,1984], Bernheim *et al.* [1987], Chander and Tulkens [1997], Le Breton and We-

[1] For a detailed discussion on the shortcomings of Nash equilibrium, see Maskin [2009].

ber [2005]), congestion games (Hotzman and Law-Yone [1997], Voorneveld *et al.* [1999]), voting models (Keiding and Peleg [2001], Brams and Sanver [2006] and Moulin [1982]), network formation (Matsubayachi and Yamakawa [2006]), production externality games (Moulin and Shenker [1992], Moulin [1994]), and many other economic situations: Abreu and Sen [1991], Hirai *et al.* [2006], Konishi *et al.* [1997a,1997b,1997c], Ma [2002], Milchtaich [1996], Yoshihara [1999], Nishihara [1999], Perry and Reny [1994], Ray [2001], Savvateev [2003], Slikker [2001], Voorneveld and Grahn [2002], Yi [1999] and Young [1998]. These series of examples reveal the explanatory power of such an equilibrium concept.

However, the existence of strong Nash equilibrium is a largely unsolved problem. There does not exist a general theorem that establishes clear existence conditions for the SNE. Ichiishi [1981] introduced the notion of social coalitional equilibrium and proved its existence under a set of assumptions of a society[2]. The concept of social coalitional equilibrium extends the notion of social equilibrium introduced by Debreu [1952], to prevent deviations by coalitions. It can also be specialized to strong Nash equilibrium. Then, the sufficient conditions for the existence of social coalitional equilibria are also sufficient for the existence of strong Nash equilibria. However, the assumptions imposed in Ichiishi [1981] are difficult to verify. [3]

There are also several other studies on the existence of strong Nash equilibria in various specific environments: Guesnerie and Oddou [1981], Greenberg and Weber [1986] in models with local public goods, Greenberg and Weber [1986,1993] in voting models, Demange and Henriet [1991], Demange [1994] in industrial organization and location models, and Wako [1994] in a market with indivisible goods, Ichiishi [1993] in games with non-transferable utility functions, Konishi et al. [1997a,1997c] in finite games, Konishi *et al.* [1997b] in games without spillovers, Hotzman and Law-Yone [1997] and Rozenfeld and Tennenholtz [2006] in congestion games. Yet, there is no general theorem on the existence of strong Nash equilibrium.

In this paper we fill this gap by proposing some existence results on SNE in general games. We show that the coalition consistence property introduced in the paper, together with the concavity and continuity of payoffs, permits the existence of strong Nash equilibria in games with compact and convex strategy spaces. The coalition consistency property is a general condition that cannot

[2]Given a finite set of agents N, a society is a list of specified data $(\{X^j\}_{j\in N}, \{S^C\}_{C\in\mathfrak{P}}, \{u^j_C\}_{j\in C\in\mathfrak{P}}, \mathfrak{F})$. X^j is the strategy set of jth agent, $S^C : X \to X^C$, $S^C(x)$ is the set of all feasible strategy bundles for a coalition C, $u^j_C : GraphS^C \to \mathbb{R}$ is the preference relation of the jth agent in C and $\mathfrak{F}$ is a nonempty collection of coalition structures.

[3]For instance, it is hard to verify Assumption 4: For every $x \in X$ and for every $v \in \mathbb{R}^n$, if there exists a balanced collection $\mathfrak{B}$ such that for each $C \in \mathfrak{B}$ there exists $y_C \in S_C(x)$ for which $v_j \leq u^j_C(x, y_C)$ for every $j \in C$, then there exist $P \in \mathfrak{F}$ and $z_D \in S_D(x)$ for every $D \in P$ such that $v_j \leq u^j_D(x, z_D)$.

be dispensed with for the existence of strong Nash equilibrium, which requires the existence of a strategy profile x such that for every coalition S, x_S is an element of the weighted best-reply correspondence forming from the coalition S. The coalition consistency property is satisfied in many economic games, and relatively easy to check, say, by using the same methods for finding the maximum of utilitarian social welfare function for every coalition and then checking if there exists a suitable weight such that every component of such coalitions is the same as those obtained from single individual deviations.

We also characterize the existence of SNE by providing necessary and sufficient conditions. Moreover, we suggest an algorithm that can be used to efficiently compute strong Nash equilibrium. The results are illustrated with applications to economies with multilateral environmental externalities and to the simple oligopoly static model.

The remainder of the paper is organized as follows. Section 2 presents the notions, definitions, and some properties. Section 3 establishes sufficient conditions for the existence of a strong Nash equilibrium and provides a method for its computation. Section 4 is dedicated to applications of the main new results to economies with multilateral environmental externalities and the simple oligopoly static model. Section 5 concludes.

2 Preliminaries

Consider the following noncooperative game in normal form:

$$G = \langle X_i,\ u_i \rangle_{i\in I} \tag{2.1}$$

where $I = \{1, ..., n\}$ is the finite set of players, X_i is the set of strategies of player i which is a subset of a locally convex Hausdorff vector space, and u_i is player i's payoff function from the set of strategy profiles $X = \prod_{i\in I} X_i$ to $\mathbb{R}$. Denote by $u = (u_1, u_2, ..., u_n)$ the profile of utility functions.

Let $\Im$ denote the set of all coalitions (*i.e.*, nonempty subsets of I). For each coalition $S \in \Im$, denote by $-S = \{i \in I : i \notin S\}$ the remaining of coalition S. If S is reduced to a singleton $\{i\}$, we denote simply by $-i$ all other players rather than player i. We also denote by $X_S = \prod_{i\in S} X_i$ the set of strategies of players in coalition S. If $\{K_j\}_{j\in\{1,...,s\}\subset\mathbb{N}}$ is a partition of I, any strategy profile $x = (x_1, ..., x_n) \in X$ then can be written as $x = (x_{K_1}, x_{K_2}, .., x_{K_s})$ with $x_{K_i} \in X_{K_i}$.

We say that a game $G = (X_i,\ u_i)_{i\in I}$ is compact, convex, concave, and continuous, respectively if, for all $i \in I$, X_i is compact and convex, and u_i is concave and continuous on X, respectively.

We say that a strategy profile $x^* \in X$ is a *Nash equilibrium* of a game G if,

$$u_i(y_i, x^*_{-i}) \leq u_i(x^*)\ \forall i \in I,\ \forall y_i \in X_i.$$

Definition 2.1 *(Aumann [1959])* A strategy profile $\overline{x} \in X$ is said to be *strong Nash equilibrium* (SNE) of a game G , if $\forall S \in \Im$, there does not exist any $y_S \in X_S$ such that

$$u_i(y_S, \overline{x}_{-S}) > u_i(\overline{x}), \ \ \forall \, i \in S. \tag{2.2}$$

Definition 2.2 A strategy profile $\overline{x} \in X$ of a game G is said to be *weakly Pareto efficient* if there does not exist any $y \in X$ such that $u_i(y) > u_i(\overline{x})$ for all $i \in I$.

A strategy profile is a strong Nash equilibrium if no coalition (including the grand coalition, *i.e.*, all players collectively) can profitably deviate from the prescribed profile. This definition immediately implies that any strong equilibrium is both weakly Pareto efficient and a Nash equilibrium. This equilibrium is stable with regard to the deviation of any coalition.

Definition 2.3 *(The Weakly α-Core)* A strategy profile $\overline{x} \in X$ is in the weakly α-core of a game G, if $\forall S \in \Im$ and $\forall x_S \in X_S$, there exists a $y_{-S} \in X_{-S}$ such that

$$u_i(x_S, y_{-S}) \leq u_i(\overline{x}) \ \ \text{for at least some } i \in S.$$

A strategy profile $\overline{x}$ is in the weakly α-core means that for any coalition S and any deviation x_S of $\overline{x}_S$, the coalition of the remaining players $(-S)$ can find a strategy y_{-S} such as in the new strategy (x_S, y_{-S}), the payoffs of at least one player in coalition S cannot be better than those in the strategy $\overline{x}$ (for all the players of the coalition S at the same time).

Definition 2.4 *(The Weakly β-Core)* A strategy profile $\overline{x} \in X$ is in the weakly β-core of a game G, if $\forall S \in \Im$, there exists a $y_{-S} \in X_{-S}$ such that for every $x_S \in X_S$,

$$u_i(x_S, y_{-S}) \leq u_i(\overline{x}) \ \ \text{for at least some } i \in S.$$

A strategy profile $\overline{x}$ is in the weakly β-core means that for any coalition S, the coalition of players $-S$ possesses a strategy y_{-S} which prevents all deviations of the coalition S of the strategy $\overline{x}$. Thus stability property of an outcome in the weakly β-core is stronger than that of the weak α-core: a deviating coalition S can be countered by the complement coalition $-S$ even if the players of S keep secret their joint strategy X_S.

Definition 2.5 *(The k-Equilibrium)* A strategy profile $\overline{x} \in X$ is said to be k-equilibrium ($k \in \{1, 2, ..., \#I\}$) of a game G, if for all coalitions S with $\#S = k$, there does not exist any $y_S \in X_S$ such that

$$u_i(y_S, \overline{x}_{-S}) > u_i(\overline{x}), \ \ \forall \, i \in S.$$

No k-players' coalition can make all these players win at the same time by deviating from the strategy $\overline{x}$.

We deduce the following properties:

1. SNE is a Nash equilibrium. It is sufficient to consider $S = \{i\}$ in Definition 2.1.

2. SNE is weakly Pareto optimal. It is sufficient to consider $S = I$ in Definition 2.1.

3. SNE is an element in the weakly α-core set. It is sufficient to consider $y_{-S} = \overline{x}_{-S}, \forall S \in \Im$ in Definition 2.3.

4. SNE is an element in the weakly β-core set. It is sufficient to consider $y_{-S} = \overline{x}_{-S}, \forall S \in \Im$ in Definition 2.4.

5. SNE is also a k-equilibrium, $\forall k \in \{1, 2, ..., n\}$. It is sufficient to consider $\forall S \in \Im$ with $\#S = k$ in Definition 2.1.

The following lemma characterizes the strong Nash equilibrium of the game G.

LEMMA 2.1 *The strategy profile $\overline{x} \in X$ is a strong Nash equilibrium of the game $G = \langle X_i,\ u_i \rangle_{i \in I}$ if and only if for each $S \in \Im$, the strategy $\overline{x}_S \in X_S$ is weakly Pareto efficient for the sub-game $\langle X_j, u_j(., \overline{x}_{-S}) \rangle_{j \in S}$.*

PROOF. It is a straightforward consequence of Definition 2.1. ■

3 Existence Results

In this section we investigate the existence of strong Nash equilibria in general games. We first provide some sufficient conditions for the existence of strong Nash equilibria (SNE). To do so, we use the following g-fixed point Theorem given by Nessah and Chu [2004].

Denote by cl (A) the closure of a set A and by ∂A its boundary. Letting Y_0 be a nonempty convex subset of a convex subset Y of a vector space and $y \in Y_0$, we denote by $Z_{Y_0}(y)$ the following set: $Z_{Y_0}(y) = \left[\text{cl} \left(\bigcup_{h>0} [Y_0 - \{y\}] / h \right) + \{y\} \right] \cap Y$. Note that cl $\left(\bigcup_{h>0} [Y_0 - \{y\}] / h \right)$ is called tangent cone to Y_0 at the point y.

LEMMA 3.1 *(Nessah and Chu [2004]) Let X be a nonempty compact set in a metric space E, and Y a nonempty convex and compact set in a locally convex Hausdorff space F. Let $g : X \to Y$ be a continuous function and $C : X \to 2^Y$ an upper hemicontinuous correspondence with nonempty closed and convex values. Suppose that the following conditions are met:*

(a) *$g(X)$ is convex in Y;*

(b) *for each $g(x) \in \partial g(X)$, $C(x) \cap Z_{g(X)}(g(x)) \neq \emptyset$.*

Then, there exists $\overline{x} \in X$ such that $g(\overline{x}) \in C(\overline{x})$.

Let

$$\Delta_S = \{\lambda_S = (\lambda_1, ..., \lambda_{\#S}) \in \mathbb{R}_+^{\#S} : \sum_{j \in S} \lambda_j = 1\}$$

be the unit simplex of $\mathbb{R}^{\#S}$ ($S \in \Im$), and let

$$\Delta = \{\lambda = (\lambda_S, S \in \Im) : \lambda_S \in \Delta_S\}, \quad \widehat{X} = \prod_{S \in \Im} X_S.$$

For each coalition S, define the S-weighted best-reply correspondence $C_S : X_{-S} \times \Delta_S \to 2^{X_S}$ by

$$C_S(x_{-S}, \lambda_S) = \{z_S \in X_S : \sup_{y_S \in X_S} \sum_{i \in S} \lambda_{i,S} u_i(y_S, x_{-S}) \leq \sum_{i \in S} \lambda_{i,S} u_i(z_S, x_{-S})\},$$

and then the $\Im$-weighted best-reply correspondence $C : X \times \Delta \to 2^{\widehat{X}}$ by

$$x \mapsto C(x, \lambda) = \{\widehat{z} = (z_S, S \in \Im) \in \widehat{X} : z_S \in C_S(x_{-S}, \lambda_S)\}.$$

Define the function $\phi : X \to \widehat{X}$ by

$$\phi(x) = ((x_S) : \ S \in \Im).$$

We then have the following lemma.

LEMMA 3.2 *Suppose that for all $i \in I$, X_i is convex and compact. Then we have:*

(a) *The function ϕ is continuous on X.*

(b) *The set $\phi(X)$ is convex and compact.*

PROOF. The continuity of function ϕ is a consequence of its definition and the construction of the set $\widehat{X}$. The compactness of the set $\phi(X)$ is a consequence of the Weierstrass Theorem. The convexity of $\phi(X)$ is a consequence of the linearity of ϕ, which is easily verified. ■

To show the existence of strong Nash equilibrium, we assume the $\Im$-weighted best-reply correspondence $C(x, \lambda)$ satisfies following coalition consistence property:

DEFINITION 3.1 (COALITION CONSISTENCE PROPERTY) A game $G = (X_i,\ u_i)_{i\in I}$ is said to satisfy the coalition consistence property if there exists $\lambda \in \Delta$ such that for each $x \in X$, there exists $z \in X$ such that

$$z_S \in C_S(x_{-S}, \lambda_S),\ \forall S \in \Im. \tag{3.1}$$

REMARK 3.1 The coalition consistence property is relatively easy to check, much easier than Condition 4 given in Ichiishi [1981]. Indeed, by the definition of the $\Im$-weighted best-reply correspondence $C(x, \lambda)$, $z_S \in C_S(x_{-S}, \lambda_S),\ \forall S \in \Im$ implies that z_S is the maximum of utilitarian social welfare function, *i.e.*, the weighted average of payoff functions, of individuals in S for every $S \in \Im$, and consequently, is weakly Pareto efficient to the sub-game $\langle X_j, u_j(., x_{-S})\rangle_{j\in S}$, $\forall S \in \Im$.[4] Thus, by the Second Fundamental Theorem of Welfare Economics, all such z_S can be found by varying wights $\lambda_{i,S}$ and the methods for finding weakly Pareto efficient outcomes can be used to find such z_S that are the candidates of strong Nash equilibria. Then, to check if the coalition consistence property is satisfied is reduced to check if there exists a suitable weight $\lambda \in \Delta$ such that every component $z_{i,S}$ of z_S is equal to $z_{\{i\}}$ that is obtained for singleton coalition $S = \{i\}$. If so, z is a consistent coalition, i.e., $z \in X$ and $z_S \in C_S(x_{-S}, \lambda_S),\ \forall S \in \Im$, which means the coalition consistence property is satisfied.

We then establish the following existence theorem on strong Nash equilibria.

THEOREM 3.1 *Suppose the game $G = (X_i,\ u_i)_{i\in I}$ is compact, continuous, concave, and satisfies the coalition consistence property. Then, it possesses a strong Nash equilibrium.*

PROOF. We prove step by step that the functions ϕ and C defined by $\phi(x) = ((x_S) : S \in \Im)$ and $C(x, \lambda) = \{\widehat{z} = (z_S : S \in \Im) \in \widehat{X} : z_S \in C_S(x_{-S}, \lambda_S)\}$, respectively, satisfy the conditions of Lemma 3.1:

1) $\forall x \in X$ and $\forall \lambda \in \Delta$, $C(x, \lambda) \neq \emptyset$. Indeed, for any $x \in X$, the function $y_S \mapsto \sum_{j\in S} \lambda_{j,S} u_j(y_S, x_{-S})$, $S \in \Im$ is continuous on the compact X_S and by the Weierstrass Theorem, there exists $\overline{z}_S \in X_S$ such that

$$\max_{y_S\in X_S} \sum_{j\in S} \lambda_{j,S} u_j(y_S, x_{-S}) = \sum_{j\in S} \lambda_{j,S} u_j(\overline{z}_S, x_{-S}),\ \textit{i.e.}\ \overline{z}_S \in C_S(x_{-S}, \lambda_S).$$

Hence $\widehat{z} = (\overline{z}_S, S \in \Im) \in C(x, \lambda)$ and consequently $C(x, \lambda) \neq \emptyset$.

[4] Thus, to guarantee that the first order conditions for the social maximization be also sufficient, we need to assume that payoff functions of players are concave.

2) $\forall x \in X$ and $\forall \lambda \in \Delta$, $C(x,\lambda)$ is convex in $\widehat{X}$. Indeed, let $x \in X$, $\lambda \in \Delta$, $\overline{z}$ and $\overline{\overline{z}}$ be two elements of $C(x,\lambda)$ and $\theta \in [0,1]$. We want to prove that $\theta\overline{z}+(1-\theta)\overline{\overline{z}} \in C(x,\lambda)$. Since $\overline{z}_S$ and $\overline{\overline{z}}_S$ are both the maximum of $\sum\limits_{j\in S}\lambda_{j,S}u_j(y_S,x_{-S})$, we must have: $\sum\limits_{j\in S}\lambda_{j,S}u_j(\overline{z}_S,x_{-S}) = \sum\limits_{j\in S}\lambda_{j,S}u_j(\overline{\overline{z}}_S,x_{-S})$ and thus, by the concavity of function u_i, we have

$$\max_{y_S\in X_S}\sum_{j\in S}\lambda_{j,S}u_j(y_S,x_{-S}) \leq \sum_{j\in S}\lambda_{j,S}u_j(\overline{z}_S,x_{-S}) = \sum_{j\in S}\lambda_{j,S}u_j(\overline{\overline{z}}_S,x_{-S}) \quad (3.2)$$

$$\leq \sum_{j\in S}\lambda_{j,S}u_j(\theta\overline{z}_S+(1-\theta)\overline{\overline{z}}_S,x_{-S}),\ \theta\in[0,1]. \quad (3.3)$$

Therefore, $\theta\overline{z}+(1-\theta)\overline{\overline{z}} \in C(x,\lambda)$.

3) C is upper hemicontinuous over X. Note that X is compact, and thus $\widehat{X}$ is compact (Tychonoff Theorem). Thus, to prove that C is upper hemicontinuous on X, it suffices to prove that $Graph(C) \subset X\times\widehat{X}$ is closed.

To see this, let $(x,\widehat{z}) \in \mathrm{cl}\,(Graph(C))$. Then there exists a sequence $\{(x^p,\widehat{z}^p)\}_{p\geq 1}$ in $Graph(C)$ that converges to $(x,\widehat{z})$.

Hence, we have $\forall p \geq 1$, $\widehat{z}^p \in C(x^p,\lambda)$, *i.e.*,

$$\max_{y_S\in X_S}\sum_{j\in S}\lambda_{j,S}u_j(y_S,x^p_{-S}) \leq \sum_{j\in S}\lambda_{j,S}u_j(z^p_S,x^p_{-S}),\ \forall S\in\Im.$$

Then, by the continuity of functions u_i, as $p\to\infty$, we have

$$\max_{y_S\in X_S}\sum_{j\in S}\lambda_{j,S}u_j(y_S,x_{-S}) \leq \sum_{j\in S}\lambda_{j,S}u_j(z_S,x_{-S}),\ \forall S\in\Im,$$

i.e., $\widehat{z} \in C(x,\lambda)$, hence $(x,\widehat{z}) \in Graph(C)$ which means that $Graph(C)$ is closed in $X\times\widehat{X}$. Thus the function C is upper hemicontinuous on X.

4) For each $\phi(x) \in \partial\phi(X)$, $C(x,\lambda)\cap Z_{\phi(X)}(\phi(x) \neq \emptyset$ where $Z_{\phi(X)}(\phi(x)) = \left[\mathrm{cl}\,(\bigcup\limits_{h>0}\frac{\phi(X)-\{\phi(x)\}}{h})+\{\phi(x)\}\right]\cap\widehat{X} = \left[\mathrm{cl}\,(\bigcup\limits_{h>0}\{h[\phi(u)-\phi(x)],\ u\in X\})+\{\phi(x)\}\right]\cap\widehat{X}$.

Indeed, by the coalition consistence property, there exists $\lambda\in\Delta$ such that for each $x\in X$ with $\phi(x)\in\partial\phi(X)$, there exists $z\in X$ such that

$$z_S \in C_S(x_{-S},\lambda_S),\ \forall S\in\Im.$$

For each $a\geq 1$, let $y_S = \frac{1}{a}z_S + \frac{a-1}{a}x_s$. Since $\frac{1}{a}>0$, $\frac{a-1}{a}\geq 0$ and $\frac{1}{a}+\frac{a-1}{a}=1$, we have $y_S\in X_S$ by the convexity of X, and $ay_s+(1-a)x_S = z_S \in C_S(x_{-S})$ for all S.

Thus, $\phi(ay + (1-a)x) = a\phi(y) + (1-a)\phi(x) \in C(x, \lambda)$ (because ϕ is linear). Since $a[\phi(y) - \phi(x)] \in \frac{\phi(X) - \{\phi(x)\}}{1/a} \subset \text{cl}\,(\bigcup_{h>0} \frac{\phi(X)-\{\phi(x)\}}{h})$, then $a\phi(y) + (1-a)\phi(x) = a[\phi(y) - \phi(x)] + \phi(x) \in Z_{\phi(X)}(\phi(x))$. Therefore, $a\phi(y) + (1-a)\phi(x) \in C(x, \lambda) \cap Z_{\phi(X)}(\phi(x))$, *i.e.* $C(x, \lambda) \cap Z_{\phi(X)}(\phi(x)) \neq \emptyset$.

From 1)-4), we conclude that the correspondence C satisfies all the conditions of Lemma 3.1. Consequently, $\exists \overline{x} \in X$ such that $\phi(\overline{x}) \in C(\overline{x}, \lambda)$, *i.e.*, $\forall S \in \Im$, $\overline{x}_S \in C_S(\overline{x}_{-S}, \lambda_S)$. Therefore, $\forall S \in \Im$, $\forall y_S \in X_S$, we have:

$$\sum_{j \in S} \lambda_{j,S} u_j(y_S, \overline{x}_{-S}) \leq \sum_{j \in S} \lambda_{j,S} u_j(\overline{x}_S, \overline{x}_{-S}) = \sum_{j \in S} \lambda_{j,S} u_j(\overline{x}). \tag{3.4}$$

Now we prove that $\forall S \in \Im$, $\overline{x}_S$ is weakly Pareto efficient to the sub-game $\langle X_j, u_j(., \overline{x}_{-S}) \rangle_{j \in S}$.
Suppose that $\exists S_0 \in \Im$ such that $\overline{x}_{S_0}$ is not weakly Pareto efficient to the sub-game $\langle X_j, u_j(., \overline{x}_{-S_0}) \rangle_{j \in S_0}$. Then, there exists $\widetilde{y}_{S_0} \in X_{S_0}$ such that:

$$\forall j \in S_0, \ u_j(\widetilde{y}_{S_0}, \overline{x}_{-S_0}) > u_j(\overline{x}). \tag{3.5}$$

System (3.5), together with $\lambda \in \Delta$ implies that $\sum_{j \in S_0} \lambda_{j,S} u_j(\widetilde{y}_{S_0}, \overline{x}_{-S_0}) > \sum_{j \in S_0} \lambda_{j,S} u_j(\overline{x})$. This contradicts inequality (3.4) for $S = S_0$ and $y_S = \widetilde{y}_{S_0}$. Hence $\overline{x}_S$ is weakly Pareto efficient to the sub-game $\langle X_j, u_j(., \overline{x}_{-S}) \rangle_{j \in S}$, $\forall S \in \Im$, and consequently, by Lemma 2.1 it is a strong Nash equilibrium. The proof is completed. ∎

REMARK 3.2 From the above theorem, one can see that the existence of strong Nash equilibrium requires much stronger conditions than the existence of Nash equilibrium. First, note that, in order to apply for a fixed-point theorem, we need to impose concavity of payoffs to ensure the $\Im$-weighted best-reply correspondence to be convex-valued since quasi-concavity of payoffs may not guarantee the (quasi-)concavity of a weighted average of payoff functions. Also, the coalition consistence property cannot be dispensed with. The following example shows that a game may not possess a strong Nash equilibrium even if it is compact, continuous and concave.

EXAMPLE 3.1 Consider a game with $n = 2$, $I = \{1, 2\}$, $X_1 = X_2 = [0, 1]$ and

$$u_1(x) = -x_1 + 2x_2,$$
$$u_2(x) = 2x_1 - x_2.$$

It can be easy seen that the game is compact, continuous and concave. Moreover, it processes a unique Nash equilibrium that is $(0, 0)$. However, there is no strong Nash equilibrium. Indeed, it is sufficient to show that $(0,0)$ is not weakly Pareto efficient. To see this, let $y = (1,1)$. Then $u_1(1,1) = 1 > u_1(0,0) = 0$ and $u_2(1,1) = 1 > u_2(0,0) = 0$. Thus the strategy $(0,0)$ is not a strong Nash equilibrium, and consequently there is no strong Nash equilibrium since $(0,0)$ it is a unique Nash equilibrium.

To illustrate how to use Theorem 3.1 to find strong Nash equilibrium, we consider the following examples.

EXAMPLE 3.2 Consider a game with $n = 2$, $I = \{1,2\}$, $X_1 = [1/3, 2]$, $X_2 = [3/4, 2]$, and

$$u_1(x) = -x_1^2 + x_2 + 1,$$
$$u_2(x) = x_1 - x_2^2 + 1.$$

Since X is compact and convex and payoff functions are continuous and concave on X, we only need to show that the coalition consistence property is also satisfied so that we know there exists a strong Nash equilibrium by Theorem 3.1. To see this, let $x = (x_1, x_2)$ and $\Im = \{\{1\},\{2\},\{1,2\}\}$. Then there exists $\lambda = (1, 1, (0.6, 0.4))$ such that:

1) for $S = \{1\}$ and $\lambda_S = 1$, we have $\max\limits_{y_1 \in X_1} u_1(y_1, x_2) = \max\limits_{y_1 \in X_1}(-y_1^2 + x_2 + 1) = -(1/3)^2 + x_2 + 1$.

2) for $S = \{2\}$ and $\lambda_S = 1$, we have $\max\limits_{y_2 \in X_2} u_2(x_1, y_2) = \max\limits_{y_2 \in X_2}(x_1 - y_2^2 + 1) = x_1 - (3/4)^2 + 1$.

3) for $S = \{1,2\}$ and $\lambda_S = (0.6, 0.4)$, we have $\max\limits_{(y_1,y_2) \in X}[0.6u_1(y_1,y_2) + 0.4u_2(y_1,y_2)] = \max\limits_{(y_1,y_2) \in X}[-0.6y_1^2 + 0.4y_1 - 0.4y_2^2 + 0.6y_2 + 1] = [-0.6(1/3)^2 + 0.4(1/3) - 0.4(3/4)^2 + 0.6(3/4) + 1]$.

Thus, for all $x \in X$, there exist $z = (1/3, 3/4) \in X$ such that

$$z_S \in C_S(x_{-S}, \lambda_S),\ \forall S \in \Im.$$

Therefore, the coalition consistence property is satisfied, and thus by Theorem 3.1, the game has a strong Nash equilibrium.

EXAMPLE 3.3 Let $I_0 = \{1, 2, ..., n-1\}$ be the set of agents. The set of all coalitions of I_0 is denoted by $\aleph$. There are m commodities. For each agent i, his strategy space is X^i, a subset of $\mathbb{R}^m \times \mathbb{R}^m \times E^i$ where E^i is a vector space over $\mathbb{R}^m$, and $u_i : \prod\limits_{h \in I_0} X^h \to \mathbb{R}$ is the expected utility function of the i-th agent. A generic element $x^i \in X^i$ is denoted by (x_1^i, x_2^i, x_3^i) with $x_1^i, x_2^i \in \mathbb{R}^m$ and $x_3^i \in E^i$. The total excess demand for the marketed commodities is $\sum\limits_{i \in I_0}(x_1^i + x_2^i)$. [5]

$\overline{x} \in \prod\limits_{h \in I_0} X^h$ is an equilibrium for this market economy $\mathcal{E} = (X_i,\ u_i)_{i \in I_0}$ if

[5] For more details, see Ichiishi [1981].

(i) $\overline{x}$ is a strong Nash equilibrium of $\mathcal{E}$;

(ii) $\sum\limits_{i\in I_0}(\overline{x}_1^i+\overline{x}_2^i)\leq 0.$

Let P be the market price domain $\{p\in\mathbb{R}_+^m : \sum\limits_{h=1}^{m}p_h=1\}$. Also, let $I=I_0\cup\{n\}$, $X^n=P$, $X=\prod\limits_{i\in I}X^i$ and $u_n(x,p)=p.\sum\limits_{i=1}^{n-1}(x_1^i+x_2^i)$, where $x\in\prod\limits_{i\in I_0}X^i$.

The market economy $\mathcal{E}$ is said to satisfy the weak form of Walras' law if

$$\text{For every } (x,p)\in(\prod_{i\in I_0}X^i)\times P,\ p.\sum_{i=1}^{n-1}(x_1^i+x_2^i)\leq 0.$$

COROLLARY 3.1 *Suppose that the market economy game $\mathcal{E}=(X_i,\ u_i)_{i\in I_0}$ is convex, compact, continuous, concave and satisfies the weak form of Walras's law. If the game $G'=(X_i,\ u_i)_{i\in I}$ satisfies the coalition consistence property, then $\mathcal{E}$ has an equilibrium.*

In the following, we characterize the existence of strong Nash equilibria by providing a necessary and sufficient condition. To do so, define a function $\digamma : X\times\Delta\times\widehat{X}\longrightarrow\mathbb{R}$ by

$$\digamma(x,\lambda,\widehat{y})=\sum_{S\in\Im}\sum_{i\in S}\lambda_i\{u_i(y_S,x_{-S})-u_i(x)\},$$

where $\widehat{X}=\prod\limits_{S\in\Im}X_S$.

Note that, by the definition of $\digamma$, we have

$$\forall x\in X,\ \forall\lambda\in\Delta,\ \max_{\widehat{y}\in\widehat{X}}\digamma(x,\lambda,\widehat{y})\geq 0. \tag{3.6}$$

Indeed, for $x\in X$ and $\lambda\in\Delta$, letting $\widehat{y}=\phi(x)=(x_S,\ S\in\Im)$,[6] we have $\digamma(x,\lambda,\widehat{y})=0$, and consequently, $\max\limits_{\widehat{y}\in\widehat{X}}\digamma(x,\lambda,\widehat{y})\geq 0$ for $\forall(x,\lambda)\in X\times\Delta$.

Let

$$\alpha=\inf_{\lambda\in\Delta}\inf_{x\in X}\sup_{\widehat{y}\in\widehat{X}}\digamma(x,\lambda,\widehat{y}).$$

We will use the following result.

LEMMA 3.3 *(Moulin and Fogelman-Soulié [1979], p. 162) Suppose that X is convex in a vectorial space and the functions u_i, $i\in I$, are concave on X. Then, $\overline{x}\in X$ is a weakly Pareto efficient strategy profile of the game $G=\langle X_i,\ u_i\rangle_{i\in I}$ if and only if there exists $\lambda\in\Delta_I$ such that $\sup\limits_{y\in X}\sum\limits_{i\in I}\lambda_{i,I}u_i(y)=\sum\limits_{i\in I}\lambda_{i,I}u_i(\overline{x})$.*

[6] The function ϕ is defined on page 6.

We then have the following theorem.

THEOREM 3.2 *(Necessity Theorem) Suppose that, $\forall i \in I$, X_i is a nonempty convex subset of a vectorial space and u_i is concave on X. If the game $G = \langle X_i,\ u_i \rangle_{i \in I}$ has a strong Nash equilibrium, then $\alpha = 0$.*

PROOF. Let $\overline{x} \in X$ be a strong Nash equilibrium of the game $G = \langle X_i,\ u_i \rangle_{i \in I}$. According to Lemma 2.1, $\overline{x}_S$ is weakly Pareto efficient to the sub-game $\langle X_j, u_j(., \overline{x}_{-S}) \rangle_{j \in S}$, $\forall S \in \Im$. Since X_i is nonempty and convex, and u_i is concave on X for all $i \in I$, then by Lemma 3.3, there exists $\overline{\lambda}_S \in \Delta_S$ such as $\sup\limits_{y_S \in X_S} \sum\limits_{i \in S} \overline{\lambda}_{i,S} \{u_i(y_S, \overline{x}_{-S}) - u_i(\overline{x})\} = 0$, $\forall S \in \Im$. This equality implies:

$$\sup_{\widehat{y} \in \widehat{X}} \digamma(\overline{x}, \overline{\lambda}, \widehat{y}) = 0.$$

Thus, we have:

$$\alpha = \inf_{x \in X} \inf_{\lambda \in \Delta} \sup_{\widehat{y} \in \widehat{X}} \digamma(x, \lambda, \widehat{y}) \leq \sup_{\widehat{y} \in \widehat{X}} \digamma(\overline{x}, \overline{\lambda}, \widehat{y}) = 0. \tag{3.7}$$

Inequalities (3.6) and (3.7) imply $\alpha = 0$. This proves the necessity. ■

THEOREM 3.3 *(Sufficiency Theorem) Suppose that $\forall i \in I$, X_i is a nonempty, compact subset of a topological Hausdorff space, and u_i is continuous on X. If $\alpha = 0$, then the game $G = \langle X_i,\ u_i \rangle_{i \in I}$ possesses a strong Nash equilibrium.*

PROOF. By the assumptions of Theorem 3.3, for all $x \in X$ and $\lambda \in \Delta$, the maximum of the function $\digamma(x, \lambda, .)$ over $\widehat{X}$ and $\min\limits_{x \in X} \min\limits_{\lambda \in \Delta} \max\limits_{\widehat{y} \in \widehat{X}} \digamma(x, \lambda, \widehat{y})$ exist.

Suppose that $\alpha = 0$. Since the functions $x \mapsto \digamma(x, \lambda, \widehat{y})$ and $\lambda \mapsto \digamma(x, \lambda, \widehat{y})$ are continuous over compact X and Δ, respectively, then the Weierstrass Theorem implies there exist $\overline{x} \in X$ and $\overline{\lambda} \in \Delta$ such that $\alpha = \max\limits_{\widehat{y} \in \widehat{X}} \digamma(\overline{x}, \overline{\lambda}, \widehat{y}) = 0$, and this equality implies $\forall \widehat{y} \in \widehat{X}$, $\digamma(\overline{x}, \overline{\lambda}, \widehat{y}) = \sum\limits_{S \in \Im} \sum\limits_{i \in S} \overline{\lambda}_{i,S} \{u_i(y_S, \overline{x}_{-S}) - u_i(\overline{x})\} \leq 0$.

For any arbitrarily fixed $S \in \Im$, we have $\forall \widehat{y} \in \widehat{X}$,

$$\digamma(\overline{x}, \overline{\lambda}, \widehat{y}) = \sum_{i \in S} \overline{\lambda}_{i,S} \{u_i(y_S, \overline{x}_{-S}) - u_i(\overline{x})\} + \sum_{K \in \Im, K \neq S} \sum_{i \in K} \overline{\lambda}_{i,S} \{u_i(y_K, \overline{x}_{-K}) - u_i(\overline{x})\} \leq 0.$$

For $\widehat{y} \in \widehat{X}$ such that y_S is arbitrarily chosen in X_S and $y_K = \overline{x}_K, \forall K \neq S$, we have $\sum\limits_{K \in \Im, K \neq S} \sum\limits_{i \in K} \overline{\lambda}_{i,S} \{u_i(y_K, \overline{x}_{-K}) - u_i(\overline{x})\} = 0$. Then, from the last inequality, we deduce that $\forall y_S \in X_S$, $\sum\limits_{i \in S} \overline{\lambda}_{i,S} u_i(y_S, \overline{x}_{-S}) \leq \sum\limits_{i \in S} \overline{\lambda}_{i,S} u_i(\overline{x})$. Since S is arbitrarily chosen in $\Im$, then

$$\forall y_S \in X_S,\ \sum_{i \in S} \overline{\lambda}_{i,S} u_i(y_S, \overline{x}_{-S}) \leq \sum_{i \in S} \overline{\lambda}_{i,S} u_i(\overline{x}),\ \forall S \in \Im. \tag{3.8}$$

Now we prove that $\forall S \in \Im$, $\overline{x}_S$ is weakly Pareto efficient for the sub-game $\langle X_j, u_j(., \overline{x}_{-S})\rangle_{j\in S}$.

Suppose that $\exists S_0 \in \Im$ such that $\overline{x}_{S_0}$ is not weakly Pareto efficient for the sub-game $\langle X_j, u_j(., \overline{x}_{-S_0})\rangle_{j\in S_0}$. Then, there exists $\widetilde{y}_{S_0} \in X_{S_0}$ such that:

$$\forall j \in S_0,\ u_j(\widetilde{y}_{S_0}, \overline{x}_{-S_0}) > u_j(\overline{x}). \tag{3.9}$$

System (3.9) implies that $\sum\limits_{j\in S_0} \overline{\lambda}_{j,S_0} u_j(\widetilde{y}_{S_0}, \overline{x}_{-S_0}) > \sum\limits_{j\in S_0} \overline{\lambda}_{j,S_0} u_j(\overline{x})$ ($\overline{\lambda}_{j,S_0} \geq 0$ and $\sum\limits_{j\in S_0} \overline{\lambda}_{j,S_0} = 1$). This contradicts inequality (3.8) for $S = S_0$ and $y_S = \widetilde{y}_{S_0}$. Hence, $\overline{x}_S$ is weakly Pareto efficient for the sub-game $\langle X_j, u_j(., \overline{x}_{-S})\rangle_{j\in S}$, $\forall S \in \Im$. Consequently, by Lemma 2.1, $\overline{x}_S$ is a strong Nash equilibrium. ■

Theorems 3.2 and 3.3 actually provides a method of finding a SNE of game (2.1) under certain conditions (see Algorithm 1).

Algorithm 1 . Procedure for the determination of a SNE

Require: Suppose that all the conditions of Theorems 3.2 and 3.3 are satisfied.

Require: Calculate the value $\alpha = \min\limits_{x\in X} \min\limits_{\lambda\in\Delta} \max\limits_{\widehat{y}\in\widehat{X}} \digamma(x, \lambda, \widehat{y})$.

if $\alpha > 0$, **then**

the game $G = (X_i,\ u_i)_{i\in I}$ has no SNE.

else

any strategy profile $\overline{x} \in X$ such that $\min\limits_{\lambda\in\Delta} \max\limits_{\widehat{y}\in\widehat{X}} \digamma(\overline{x}, \lambda, \widehat{y}) = 0$ is a SNE of the game $G = \langle X_i,\ u_i\rangle_{i\in I}$.

end if

The following example illustrates the application of Algorithm 1.

EXAMPLE 3.4 Let us consider the following example. Assume that in game (2.1) $n = 2$, $I = \{1, 2\}$, $X_1 = X_2 = [-1, 1]$, $x = (x_1, x_2)$ and

$$u_1(x) = 3x_1 - x_2^2 + 4x_2,$$
$$u_2(x) = -x_1^2 + x_1 - 2x_2.$$

It is obvious to see that the functions u_i are concave over the convex X, $i = 1, 2$.

In this example, we have $\widehat{X} = X_1 \times X_2 \times (X_1 \times X_2)$, we put $\widehat{y} = (a, b, (c, d)) \in X_1 \times X_2 \times (X_1 \times X_2)$ and $x = (u, v)$.

We have $\alpha = \min\limits_{(x,\lambda)\in X\times\Delta} \max\limits_{\widehat{y}\in\widehat{X}} \digamma(x, \widehat{y}) = \min\limits_{\lambda\in[0,1]} \min\limits_{u,v\in[-1,1]} \max\limits_{a,b,c,d\in[-1,1]} \{[u_1(a, v) - u_1(u, v)] + [u_2(u, b) - u_2(u, v)] + [\lambda(u_1(c, d) - u_1(u, v)) + (1 - \lambda)(u_2(c, d) - u_2(u, v))]\} =$

$\min_{u,v\in[-1,1]} \min_{\lambda\in[0,1]} \max_{a,b,c,d\in[-1,1]} \{[3a-2b]+[-(1-\lambda)c^2+(1+2\lambda)c]+[-\lambda d^2+2(3\lambda-1)d]+[(1-\lambda)u^2-2(2+\lambda)u]+[\lambda v^2+(4-6\lambda)v]\}$.

Let us consider the following function.

$$h : [0,1] \to \mathbb{R}$$

defined by $\lambda \mapsto h(\lambda) = \min_{u,v\in[-1,1]} \max_{a,b,c,d\in[-1,1]} \{[3a-2b]+[-(1-\lambda)c^2+(1+2\lambda)c]+[-\lambda d^2+2(3\lambda-1)d]+[(1-\lambda)u^2-2(2+\lambda)u]+[\lambda v^2+(4-6\lambda)v]\}$.

We recall that $\alpha = \min_{\lambda\in[0,1]} h(\lambda)$.

The minimum and maximum of function F are attained in respectively: $\widetilde{a} = \widetilde{u} = 1, \widetilde{b} = -1$,

$$\widetilde{c} = \begin{cases} \frac{1+2\lambda}{2(1-\lambda)}, \text{ if } 0 \le \lambda \le 1/4 \\ 1, \text{ if } 1/4 \le \lambda \le 1 \end{cases}, \widetilde{d} = \begin{cases} -1, \text{ if } 0 \le \lambda \le 1/4 \\ \frac{3\lambda-1}{\lambda}, \text{ if } 1/4 \le \lambda \le 1/2 \\ 1, \text{ if } 1/2 \le \lambda \le 1 \end{cases}, \text{ and}$$

$$\widetilde{v} = \begin{cases} -1, \text{ if } 0 \le \lambda \le 1/2 \\ \frac{3\lambda-1}{\lambda}, \text{ if } 1/2 \le \lambda \le 1/2. \end{cases}$$

We obtain then:

$$h(\lambda) = \begin{cases} \frac{16\lambda^2-8\lambda+1}{4(1-\lambda)}, & \text{if } 0 \le \lambda \le 1/4 \\ \frac{16\lambda^2-8\lambda+1}{\lambda}, & \text{if } 1/4 \le \lambda \le 1/2 \\ \frac{-4\lambda^2+12\lambda-4}{\lambda}, & \text{if } 1/2 \le \lambda \le 1. \end{cases}$$

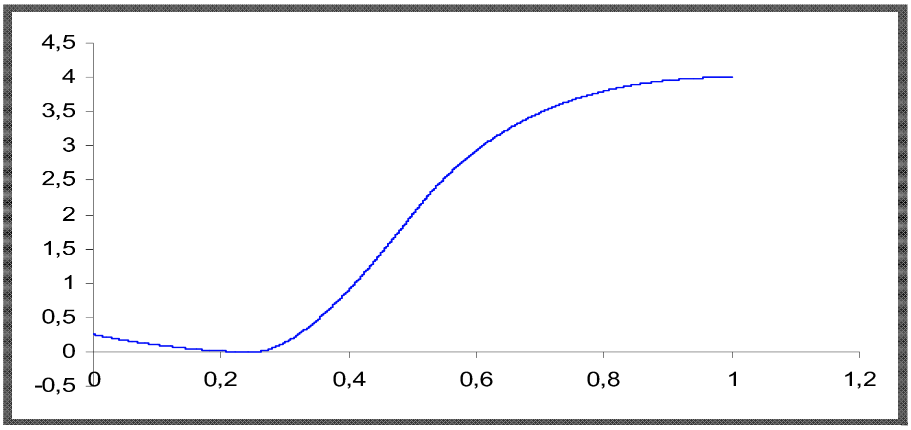

Figure 1: The graph of function h

We see that $\alpha = \min_{\lambda\in[0,1]} h(\lambda) = h(1/4) = 0$ (Figure 1). According to Algorithm 1, the considered game in this example has a strong Nash equilibrium which is $\overline{x} = (\widetilde{u}, \widetilde{v}) = (1, -1)$.

4 Applications

In this section we show how our main existence results are applied to some important economic games. We provide two applications: one is in an economy with multilateral environmental externalities that is intensively studied by Chander and Tulkens [1997], and the other is in a simple oligopoly game.

4.1 Economy with Multilateral Environmental Externalities

Consider an economy with multilateral externalities and n agents, indexed by $i \in I = \{1, ..., n\}$. A consumption good $y_i \geq 0$ is produced from an input $e_i \geq 0$. The technology is described by a production function $y_i = g_i(e_i)$, and each agent's preference is presented by a quasilinear utility function $u_i(y_i, z) = y_i - v_i(z)$ where $v_i(z)$ is i's disutility function of the externality given by $z = \sum\limits_{h \in I} e_h$.

Define an n-person noncooperative game $G = \langle X_i, u_i \rangle_{i \in I}$ as follows. Let

$$X_i = \{e_i \in \mathbb{R} :\ 0 \leq e_i \leq e_i^0\}$$

be the strategy set of each player i, and X_S the space of joint strategies of players in $S \in \Im$. Let X denote the space of joint strategies of all players, *i.e.*, $X = X_I$. For a strategy profile $[(e_1, ..., e_n)] \in X$, we choose $u_i(y_i, z) = y_i - v_i(z)$ with $z = \sum\limits_{i \in I} e_i$ as the payoff for player i. Let $u = (u_1, ..., u_n)$.

By Lemma 2.1, we know that $\overline{e} \in X$ is a strong Nash equilibrium of the game $G = \langle X_i, u_i \rangle_{i \in I}$ if and only if $\overline{e}_S \in X_S$ is weakly Pareto efficient for the subgame $G_S(\overline{e}) = \langle X_j, u_j(., \overline{e}_{-S}) \rangle_{j \in S}$. By Lemma 3.3, $\overline{e}_S \in X_S$ is weakly Pareto efficient for the subgame $G_S(\overline{e})$ if and only if there exists $\lambda_S \in \Delta_S$ such that

$$\sup_{d_S \in X_S} \sum_{i \in S} \lambda_{i,S}[g_i(d_i) - v_i(d_S + \overline{e}_{-S})] = \sum_{i \in S} \lambda_{i,S}[g_i(\overline{e}_i) - v_i(\overline{e})]$$

where $d_S + \overline{e}_{-S} = \sum\limits_{j \in S} d_j + \sum\limits_{j \in -S} \overline{e}_j$ and $\overline{e} = \sum\limits_{j \in I} \overline{e}_j$.

To characterize weak Pareto efficiency for the subgame $G_S(e)$, we get the first order conditions

$$\lambda_{j,S} g_j'(d_j) = \sum_{h \in S} \lambda_{h,S} v_h'(\sum_{i \in S} d_i + \sum_{i \in -S} e_i),\ j \in S,\ \lambda_S \in \Delta_S. \tag{4.1}$$

Consider two coalitions $S_1, S_2 \in \Im$ and player j such that $j \in S_1 \cap S_2$. Then, (4.1) implies:

$$\begin{cases} (1)\ \lambda_{j,S_1} g_j'(d_j^1) = \sum\limits_{h \in S_1} \lambda_{h,S_1} v_h'(\sum\limits_{i \in S_1} d_i^1 + \sum\limits_{i \in -S_1} e_i),\ \lambda_{S_1} \in \Delta_{S_1}. \\ (2)\ \lambda_{j,S_2} g_j'(d_j^2) = \sum\limits_{h \in S_2} \lambda_{h,S_2} v_h'(\sum\limits_{i \in S_2} d_i^2 + \sum\limits_{i \in -S_2} e_i),\ \lambda_{S_2} \in \Delta_{S_2}. \end{cases} \tag{4.2}$$

For $e \in X$ to be a strong Nash equilibrium, it is necessary that $d_j^1 = d_j^2 = ... = d_j^k = e_j$, for each $j \in S_1 \cap S_2 \cap ... \cap S_k$.

While we can use Theorems 3.2 and 3.3 to provide necessary and sufficient conditions for the existence of strong Nash equilibrium for this problem, here we provide sufficient conditions for the existence of strong Nash equilibrium by applying Theorem 3.1. To do so, we make the following assumptions.

Assumption 1. $g_i(e_i) - v_i(z)$ is concave and differentiable over an interval $\prod_{i \in I}[0, e_i^0]$.

Assumption 2. There exist $\lambda \in \Delta$ and $e \in X$ such that

$$\lambda_{j,S} g_j^{'}(e_j) = \sum_{h \in S} \lambda_{h,S} v_h^{'}(\sum_{i \in I} e_i), \ \forall j \in S, \ \forall S \in \Im. \tag{4.3}$$

Then, by Theorem 3.1, we have the following result.

PROPOSITION 4.1 *Suppose Assumptions 1 and 2 are satisfied.* [7] *Then, the game $G = \langle X_i, u_i \rangle_{i \in I}$ possesses a strong Nash equilibrium.*

EXAMPLE 4.1 Consider the game $G = \langle X_i, u_i \rangle_{i \in I}$ with $I = \{1, 2, ..., n\}$, $e = (e_1, ..., e_n)$, $z = \sum_{i=1}^{n} e_i$, and

$$g_i(e_i) = a_i e_i^2 - b_i e_i + c_i, \, u_i(y_i, z) = y_i - v_i(z),$$
$$v_i(z) = az^2 - bz + c \text{ with } a_i, b_i, a, b > 0, \, c \geq 0 \text{ and } b_i^2 - 4a_i c_i < 0.$$

Assume that $u_i(e) = g_i(e_i) - v_i(z)$ is concave over $\prod_{i \in I}[0, e_i^0]$ with $e_i^0 \geq \frac{b_i}{2a_i}$. We now show that Assumption 2 is satisfied. Consider $\lambda \in \Delta$ and $\overline{e} \in X$ defined as follows:

$$\lambda_{i,S} = \frac{1}{\#S} \ \forall S \in \Im \text{ and } \overline{e}_i = \frac{b_i}{2a_i}, \ \forall i \in I.$$

If $\overline{z} = \sum_{i=1}^{n} \frac{b_i}{2a_i} = \frac{b}{2a}$, then $\overline{e} = (\frac{b_1}{2a_1}, ..., \frac{b_n}{2a_n})$ is a strong Nash equilibrium. Indeed, we have $g_i(e_i) = a_i e_i^2 - b_i e_i + c_i$ and $v_i(z) = az^2 - bz + c$, then $g_i^{'}(\overline{e}_i) = 0$ and $v_i^{'}(\overline{z}) = 0$. Thus (4.3) holds.

[7] The solutions of the following system are within the set $\prod_{i \in S}[0, e_i^0]$, $j \in S$ and $\lambda_S \in \Delta_S$:

$$\lambda_{j,S} g_j^{'}(e_j) = \sum_{h \in S} \lambda_{h,S} v_h^{'}(\sum_{i \in I} e_i).$$

4.2 Simple Oligopoly Static Model

This subsection is dedicated to examining a simple oligopoly model. We first recall the Cournot model in which the firms are quantity choosers producing a homogeneous good.

Let p be the market price of a perfectly homogeneous good produced by the n firms of an industry ($I = \{1, ..., n\}$), q_i be the sales of the i-th firm, $q = (q_1, ..., q_n)$, and let $Q = \sum_{i=1}^{n} q_i$ be the total sales in the market. The inverse demand function is $p = F(Q)$. The cost for the i-th firm is given by $C_i(q_i)$. The profit of the i-th firm is then given by $\psi_i(q) = q_i F(Q) - C_i(q_i)$.

Define a noncooperative game $G = \langle X_i, \psi_i \rangle_{i \in I}$ as follows. Let $X_i = [0, \overline{q}_i]$, $X = \prod_{i \in I} [0, \overline{q}_i]$, $X_S = \prod_{i \in S} [0, \overline{q}_i]$, for each $S \in \Im$, and $\psi = (\psi_1, ..., \psi_n)$.

Again, we want to provide some sufficient conditions that guarantee the existence of strong Nash equilibrium. To do so, we make the following assumptions.

Assumption 3. $F(Q)$ and $C_i(q_i)$ are continuous and nonnegative on $Q \in [0, +\infty)$ and $q_i \in [0, +\infty)$, respectively.

Assumption 4. There exists $\overline{q}_i > 0$, $i = 1, ..., n$ such that $\psi_i(q)$ is concave over $\prod_{i \in I} [0, \overline{q}_i]$.

Assumption 5. There exist $\lambda \in \Delta$ and $q \in X$ such that

$$\lambda_{j,S} C_j^{'}(q_j) = \lambda_{j,S} F(\sum_{i \in I} q_i) + F^{'}(\sum_{i \in I} q_i) \sum_{h \in S} \lambda_{h,S} q_h, \ \forall j \in S, \ \forall S \in \Im. \tag{4.4}$$

Then, by Theorem 3.1, we have the following proposition.

PROPOSITION 4.2 *Suppose Assumptions 3,4 and 5 are satisfied.* [8] *Then, the game* $G = \langle X_i, \psi_i \rangle_{i \in I}$ *possesses a strong Nash equilibrium.*

EXAMPLE 4.2 Consider a game with $I = \{1, 2, ..., n\}$, $q = (q_1, ..., q_n)$, $Q = \sum_{i=1}^{n} q_i$, and

$$F(Q) = aQ^2 - bQ + c, \ C_i(q_i) = \theta_i q_i^2 \text{ for } i = 1, ..., n$$
$$\text{with } b^2 - 4ac < 0 \text{ and } a, b, \theta_i > 0 \text{ for } i = 1, ..., n.$$

Suppose that $\psi_i(q) = q_i F(Q) - C_i(q_i)$ is concave over $\prod_{i \in I} [0, q_i^0]$ with $q_i^0 \geq \frac{4ac - b^2}{8a\theta_i}$.

[8] The solutions of the following system are within the set $\prod_{i \in S} [0, e_i^0]$, $j \in S$ and $\lambda_S \in \Delta_S$:

$$\lambda_{j,S} C_j^{'}(q_j) = \lambda_{j,S} F(\sum_{i \in I} q_i) + F^{'}(\sum_{i \in I} q_i) \sum_{h \in S} \lambda_{h,S} q_h.$$

If $(4ac-b^2)\sum_{i=1}^{n}\frac{1}{\theta_i}=4b$, then there exists $\overline{q}=(\frac{4ac-b^2}{8a\theta_1},...,\frac{4ac-b^2}{8a\theta_n})$ such that Assumption 5 is satisfied. To see this, let $\lambda_{i,S}=\frac{1}{\#S}\ \ \forall S\in\Im$ and $\overline{q}_i=\frac{4ac-b^2}{8a\theta_i},\ \forall i\in I$. Then $\sum_{i=1}^{n}\overline{q}_i=\frac{b}{2a}$, *i.e.* $F^{'}(\overline{q})=0$.

Since $F^{'}(\overline{q})=0$, then system (4.4) becomes:

$$2\theta_i\overline{q}_i=\frac{4ac-b^2}{4a},\ i\in I.$$

Thus, $\overline{q}_i=\frac{4ac-b^2}{8a\theta_i}$, $i\in I$ such that $F^{'}(\overline{q})=0$. This condition is equivalent to $(4ac-b^2)\sum_{i=1}^{n}\frac{1}{\theta_i}=4b$. Therefore, $\overline{q}=(\frac{4ac-b^2}{8a\theta_1},...,\frac{4ac-b^2}{8a\theta_n})$ is a strong Nash equilibrium.

5 Conclusion

There is no general theorem on the existence of strong Nash equilibrium available in the literature. In the present paper we filled this gap by proposing some the existence results on strong Nash equilibria in general games. We provide a condition, called the coalition consistence property which, together with the concavity and continuity of payoffs, permits the existence of strong Nash equilibria in games with compact and convex strategy spaces. The coalition consistency property is a general condition that cannot be dispensed with for the existence of strong Nash equilibrium. It is satisfied in many economic games and relatively easy to check.

We also characterized the existence of strong Nash equilibria by providing a necessary and sufficient condition. Moreover, we suggest a procedure that can be used to efficiently compute strong Nash equilibrium. Our results would be useful for solving theoretical and practical problems from various domains. The results are illustrated with applications to economies with multilateral environmental externalities and to the simple oligopoly static model.

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