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Double implementation of linear cost share equilibrium allocations

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Abstract

In this paper we consider the problem of double implementation of Linear Cost Share Equilibrium (LCSE) allocations by a feasible and continuous mechanism whose Nash allocations and strong Nash allocations coincide with Linear Cost Share Equilibrium allocations. The mechanism presented here allows preferences and initial endowments as well as coalition patterns to be privately observed, a feature missing from much recent work in implementation theory. Since LCSE contains Lindahl equilibrium and Ratio equilibrium as special cases, it doubly implements these two equilibria allocations. Further, if one reinterprets the commodity space, this mechanism also doubly implements Walrasian allocations for private goods economies. Thus, the mechanism given in the paper appears to represent a 'generic' mechanism to doubly implement market-type allocations in private and/or public goods economies.

JEL classification: C72; D61; D78

1. Introduction

This paper considers the problem of double implementation of Linear Cost Share Equilibrium (LCSE) allocations in Nash and strong Nash equilibria by a feasible and continuous mechanism when coalition patterns, preferences, and endowments are unknown to the designer. The important reasons for preferring double implementation over Nash implementation and strong Nash implementation are two-fold: (1) The double implementation covers the case where agents in some coalitions may cooperate and in

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some other coalitions may not, when such information is unknown to the designer. (2) The combining solution concept, which characterizes agents' strategic behavior, may give a state which uses the advantages of both Nash equilibrium and strong Nash equilibrium so that it may be easy to reach and hard to leave.

The notion of Linear Cost Share Equilibrium for public goods economies, which was introduced by Mas-Colell and Silvestre (1989), has the desired property that it yields Pareto efficient allocations even in the increasing returns case and in the presence of externalities (when the equilibrium concept is applied to pure private goods economies). Until recently, for the general equilibrium approach to the efficiency of resource allocation of public goods, the most commonly used general equilibrium notion was the Lindahl equilibrium principle. Many mechanisms have been proposed which implement Lindahl allocations such as those in Hurwicz (1979), Walker (1981), Tian (1989, 1990, 2000), and Li et al. (1995), among others. However, for the general variable returns case, the Lindahl equilibrium principle must occur at a price-taking, profit-maximizing point. This precludes the existence of an equilibrium if increasing returns to scale (IRS) are present. Further, if profits are positive, they must be distributed in accordance with some exogenously given profit distributions. Also, when a firm is owned by the state and the technology of the firm does not display constant returns to scale (CRS), the conventional Lindahl mechanism is problematic since it is not clear how the profits or losses should be distributed. In addition, unlike the Walrasian equilibrium principle, the core equivalence result for the Lindahl equilibrium does not hold in the case of a continuum of agents; the core of a public goods economy can be much larger than the set of Lindahl allocations (cf. Muench, 1972). Some alternative solution concepts for public goods economies have been proposed in the literature, including the Ratio equilibrium notion of Kaneko (1977) and the Generalized Ratio equilibrium notion of Tian (1992) and Tian and Li (1994), as well as the more general solution notions of the various Cost Share equilibria of Mas-Colell and Silvestre (1989). The Cost Share equilibrium notion is not radically different from the Lindahl equilibrium solution. In fact, it coincides with the Lindahl equilibrium notion for the case of convex economies with constant returns to scale.

However, like the Lindahl mechanism, the Cost Share Equilibrium mechanism is not incentive-compatible in the sense that some agents may not reveal their characteristics truthfully (that is, the truth-telling is not a Nash equilibrium). Tian (1993) constructed a mechanism which implements the Cost Share Equilibrium allocations by using Nash equilibrium as a solution concept to describe individuals' self-interested behavior. Nash equilibrium is a strictly non-cooperative notion and is only concerned with single individual deviations such that no one can be improved by unilateral deviation from a prescribed strategy profile. No cooperation among agents is allowed. As a result, although a Nash equilibrium may be easy to reach, it may not be stable in the sense that there may exist a group of agents which can be improved by forming a coalition. Thus it is natural to adopt strong Nash equilibrium which allows all possible cooperation (coalitions) among agents. To have a solution concept combining the properties of Nash and strong Nash equilibria, it is desirable to construct a mechanism which doubly implements a social choice rule by Nash and strong Nash equilibria so that its equilibrium outcomes are not only easy to reach, but also hard to leave. Also, by double implementation, it can cover the situation where agents in some coalitions will cooperate and in some other coalitions will not. Thus the designer does not need to know which coalitions are permissible and, consequently, it allows the possibility for agents to manipulate coalition patterns.

Similar situations prevailed with regard to double implementation of the Walrasian correspondence, Lindahl correspondence, and Ratio equilibrium correspondence in Nash and strong Nash equilibria until Corchon and Wilkie (1996), Peleg (1996a,b), Tian (1996, 2000) presented continuous and feasible mechanisms which doubly implement these correspondences in Nash and strong Nash equilibria.

In this paper we give a feasible and continuous mechanism whose Nash allocations and strong Nash allocations coincide with Linear Cost Share Equilibrium allocations. The mechanism presented here allows not only preferences and initial endowments, but also coalition patterns to be privately observed. Unlike most mechanisms proposed in the literature, our mechanism works not only for three or more agents, but also for two-agent economies, and thus it is a unified mechanism which is irrespective of the number of agents. Also, if one reinterprets the commodity space (cf. Mas-Colell and Silvestre, 1989), our mechanism also doubly implements Walrasian allocations to economies with purely private goods. Thus, our mechanism is sufficiently general to cover Walrasian equilibrium, Lindahl equilibrium, Ratio equilibrium, and cost share equilibrium. In other words, the mechanism in the paper appears to represent a ‘generic’ mechanism to doubly implement competitive-type allocations in private and public goods economies. However, as will be noted, an undesired property of the mechanism is that it is a destruction mechanism in the sense that under-reported endowments are destroyed but not consumed although all agent will report their endowments truthfully at equilibria.

The remainder of the paper is organized as follows. Section 2 sets forth a public goods model and gives the definition of Linear Cost Share Equilibrium. Section 3 presents a mechanism which has the desirable properties mentioned above. Section 4 proves the mechanism doubly implements Linear Cost Share Equilibrium allocations in Nash and strong Nash equilibria. Concluding remarks will be offered in Section 5.

2. The model and linear cost share equilibria

2.1. Economic environments

We will consider public goods economies with $n \geqq 2$ agents, K public goods, and one private good, x being private (as a numeraire) and y public. Denote by $N = \{1,2,\ldots,n\}$ the set of agents. The single private good x can, and probably should, be thought of as a Hicksian composite commodity or money, and public goods y can be thought of as K public projects and are producible from the private good. The technology is given to us as a single cost function $C(y)$. We assume throughout that $C(0)=0$ and that C is increasing (i.e. $C(y') > C(y)$ if $y' > y$),[1] continuous, and convex.

Each agent’s characteristic is denoted by $e_i = (\mathring{w}_i, P_i)$, where $\mathring{w}_i$ is the true initial

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[1] As usual, vector inequalities are defined as follows: Let $a,b \in \mathbb{R}^m$. Then $a \geqq b$ means $a_s \geqq b_s$ for all $s = 1,\ldots,m$; $a \geq b$ means $a \geqq b$ but $a \neq b$; $a > b$ means $a_s > b_s$ for all $s = 1,\ldots,m$.

endowment of the private good and P_i is the strict (irreflexive) preference defined on $\mathbb{R}^{1+K}_{+}$.[2] We assume that $\mathring{w}_i > 0$, and preference P_i is convex[3], continuous, and strictly monotonically increasing in the private good. We further assume that the private good is indispensable (i.e. for all $i \in N$, $(x_i,y)\ P_i\ (0,y')$ for all $x_i \in \mathbb{R}_{++}$, and $y,y' \in \mathbb{R}^K_+$). An economy is the full vector $e = (e_1, \ldots, e_n, C)$ and the set of all such economies is denoted by E.

A state, also called an allocation, of the economy e is a vector $(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^K_+$. A state is *feasible* if

$$\sum_{i=1}^{n} x_i + C(y) \leqq \sum_{i=1}^{n} \mathring{w}_i \tag{1}$$

A *coalition* S is a non-empty subset of N.

A feasible allocation (x,y) can be improved upon by $S \subset N$ if there exists an allocation (x',y') such that: (i) $\Sigma_{i \in S} x'_i + C(y') \leqq \Sigma_{i \in S} \mathring{w}_i$; (ii) $(x'_i,y') P_i (x_i,y)$ for all $i \in S$.

A feasible allocation (x,y) is in the core of e if there does not exist a coalition S that can improve upon (x,y).

An allocation (x,y) is *Pareto-optimal* with respect to the strict preference profile $P = (P_1, \ldots, P_n)$ if it cannot be improved upon by N.

An allocation (x,y) is *individually rational* with respect to the strict preference profile $P = (P_1, \ldots, P_n)$ if it cannot be improved upon by any single individual i.

2.2. Linear cost share equilibria

Let $\Delta^{n-1}_{+} = \{t \in \mathbb{R}^n_+ : \Sigma^n_{i=1} t_i = 1\}$ be the $n-1$ dimensional unit simplex.

An allocation (x^*,y^*) is a *Linear Cost Share Equilibrium (LCSE) allocation* for an economy e if it is feasible and there are $(a^*_1, \ldots, a^*_n) \in \mathbb{R}^{nK}$ with $\Sigma^n_{i=1} a^*_i = 0$ and $(b^*_1, \ldots, b^*_n) \in \Delta^{n-1}_{+}$ such that: (1) $x^*_i + a^*_i \cdot y^* + b^*_i C(y^*) \leqq \mathring{w}_i$ for all $i \in N$; (2) for all $i \in N$, there does not exist (x_i,y) such that $(x_i,y)\ P_i\ (x^*_i,y^*)$ and $x_i + a^*_i \cdot y + b^*_i C(y) \leqq \mathring{w}_i$. Denote by $LCSE(e)$ the set of all such allocations. Note that the Linear Cost Share Equilibrium solution concept does not preclude the presence of increasing returns to scale (IRS). The interpretations of the parameters of the linear cost share system are clear. The b_i parameters are direct cost share parameters while the a_{ij}, which can be positive or negative, are side compensations based on consumption of public goods.

Remark 1. *Mas-Colell and Silvestre (1989) showed that every LCSE allocation is Pareto optimal even in the increasing returns case. Further they showed that, if one reinterprets the commodity space, the equilibrium concept can be applied to economies*

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[2] If we define the binary relation P^*_i in the way that $a\ P^*_i\ b$ if and only if $\neg b\ P_i\ a$ where $\neg$ stands for 'it is not the case that,' then P^*_i is the weak (reflexive) preference. Let concepts used in this paper such as Nash equilibrium and Linear Cost Share Equilibrium allocations be interpreted in terms of the P^*_i. Then the results obtained in this paper for P_i are, in particular, valid for the P^*_i.

[3] P_i is convex if for bundles a,b,c with $0 < \lambda \leqq 1$ and $c = \lambda a + (1-\lambda)b$, the relation $a\ P_i\ b$ implies $c\ P_i\ b$.

with purely private goods or with externalities. Thus, the equilibrium concept can also be viewed as optimality-guaranteeing equilibrium concepts.

Remark 2. *An LCSE allocation does not always result in individually rational allocations. Wilkie (1990) provided such a counter-example. Therefore, an LCSE allocation may not be in the core. However, every interior LCSE allocation belongs to the core of the economy (cf. Weber and Wiesmeth, 1991).*

Given profit share vector $\theta \in \Delta^{n-1}$, an allocation (x^*,y^*) is a *θ-Lindahl equilibrium allocation* for an economy e if it is feasible and there are personalized price vectors, $q_i^* \in \mathbb{R}_+^K$, one for each i, such that: (1) y^* maximizes profits $q^* \cdot y - C(y)$; (2) $x_i^* + q_i^* \cdot y^* \leqq \mathring{w}_i + \theta_i[q^* \cdot y^* - C(y^*)]$ for all $i \in N$; (3) for all $i \in N$, there does not exist (x_i,y) such that $(x_i,y)\ P_i\ (x_i^*,y^*)$ and $x_i + q_i^* \cdot y \leqq \mathring{w}_i + \theta_i[q^* \cdot y^* - C(y^*)]$; (4) $\Sigma_{i=1}^n q_i^* = q^*$. Denote by $L(e;\theta)$ the set of all such allocations.

Mas-Colell and Silvestre (1989) showed that in the convex technology case LCSE allocations are in one-to-one correspondence with Lindahl equilibrium allocations. The correspondence is established by varying the profit share parameters which characterize Lindahl equilibrium allocations, i.e. $LCSE(e) = \cup_{\theta \in \Delta^{n-1}} L(e;\theta)$. Thus, the existence of a LCSE is guaranteed under the same conditions which guarantee the existence of Lindahl equilibria (cf. Foley, 1970; Milleron, 1972; Roberts, 1974). Note that in the constant returns case, a Linear Cost Share Equilibrium allocation reduces to a Lindahl equilibrium allocation.

Remark 3. *Even though the indispensability condition is not necessary for the existence of LCSE, this assumption cannot be dispensed with for feasible implementation. Tian (1988) showed that the Lindahl correspondence violates Maskin's (1997) monotonicity condition without this assumption and thus cannot be Nash-implemented by a feasible mechanism. Since a Linear Cost Share Equilibrium allocation reduces to a Lindahl equilibrium allocation for constant returns economies, they are also necessary conditions for the feasible and continuous implementation of Linear Cost Share Equilibrium allocations.*

3. Mechanism

In the following we will present a feasible and continuous mechanism which doubly implements the LCSE correspondence in Nash and strong Nash equilibrium.

Let M_i denote the ith agent's message domain. Its elements are written as m_i and called messages. Let $M = \prod_{i=1}^n M_i$ denote the message space. The message spaces of agents are defined as follows.

For each $i \in N$, his/her message domain is of the form

$$M_i = (0,\mathring{w}_i] \times \Omega^{nK} \times \Delta_+^{n-1} \times \mathbb{R}^K \times \mathbb{R}_{++} \tag{2}$$

where $\Omega^{nK} = \{(\alpha_1,\ldots,\alpha_n) \in \mathbb{R}^{nk} : \Sigma_{j=1}^n \alpha_j = 0\}$. A generic element of M_i is $m_i =$

$(w_i, \alpha_{i1}, \ldots, \alpha_{in}, \beta_{i1}, \ldots, \beta_{in}, y_i, \gamma_i)$ whose components have the following interpretations. The component w_i denotes a profession of agent i's endowment, the inequality $0 < w_i \leqq \mathring{w}_i$ means that the agent cannot overstate his own endowment; on the other hand, the endowment can be understated, but the claimed endowment w_i must be positive which is necessary to guarantee the feasibility even at disequilibrium points.[4] The intuition here is straightforward: if a mechanism allows agents to overstate their endowments, then it allows for infeasible outcomes — it will sometimes attempt to allocate more than is possible, given the true aggregate endowment. The component $\alpha_i \equiv (\alpha_{i1}, \ldots, \alpha_{in})$ is the side compensation vector profile proposed by individual i. The component $\beta_i \equiv (\beta_{i1}, \ldots, \beta_{in})$ is the direct cost share profile proposed by individual i. The component y_i denotes the proposed level of tax (measured in public goods) that agent i is willing to contribute (a negative y_i means the agent wants to receive compensation from society). The component γ_i is a shrinking index of agent i used to shrink the private good consumption of other agents.

Define the side compensations for consumption of public goods for the ith agent by

$$a_i(m) = \alpha_{i+1,i} \tag{3}$$

where $n + 1$ is to be read as 1. Note that even though $a_i(m)$ is only a function of the α-component, α_{i+1}, announced by agent $i + 1$ for agent i, we can still write it as a function of m without loss of generality.

Define the direct cost share for consumption of public goods for the ith agent by

$$b_i(m) = \beta_{i+1,i} \tag{4}$$

Define a feasible correspondence B: $M \to \to \mathbb{R}^K_+$ by

$$B(m) = \{y \in \mathbb{R}^K_+ : C(y) \leqq \sum_{i=1}^{n} w_i, \text{ and } w_i - a_i(m) \cdot y - b_i(m)C(y) \geqq 0, \ \forall i \in N\} \tag{5}$$

which is clearly non-empty, compact, and convex [by the convexity of $C(\cdot)$] for all $m \in M$. Notice that $0 \in B(m)$ for all $m \in M$. We will show the following lemma in Appendix A.

Lemma 1. *$B(\cdot)$ is continuous on M.*

Let $\tilde{y} = \Sigma_{i=1}^{n} y_i$. Define the outcome function for public goods Y: $M \to \mathbb{R}^K_+$ by

$$Y(m) = \{y : \min_{y \in B(m)} \|y - \tilde{y}\|\} \tag{6}$$

which is the closest point to $\tilde{y}$. Then $Y(m)$ is single-valued and continuous on M.

Define a preliminary private good consumption x_i: $M \to \mathbb{R}_+$ by

$$x_i(m) = \frac{w_i}{1 + \|\alpha_i - \alpha_{i+1}\| + \|\beta_i - \beta_{i+1}\|} - a_i(m) \cdot Y(m) - b_i(m)C(Y(m)) \tag{7}$$

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[4] When goods are physical goods, this requirement can be guaranteed by asking agents to *exhibit* their reported endowments to the designer.

for each $i \in N$.

Define a shrinking index correspondence A: $M \to\to \mathbb{R}_+$ by

$$A(m) = \{\gamma \in \mathbb{R}_+ : \gamma\gamma_i \leq 1,\ \forall i \in N, \text{ and } \gamma\sum_{i=1}^{n}\gamma_i x_i(m) + C(Y(m)) \leqq \sum_{i=1}^{n} w_i\} \tag{8}$$

which is clearly a continuous correspondence with non-empty, compact and convex values.

Let $\bar{\gamma}(m)$ be the largest element of A, i.e. $\bar{\gamma}(m) \in A(m)$, $\bar{\gamma}(m) \geq \gamma$ for all $\gamma \in A(m)$. Thus, $\bar{\gamma}(\cdot)$ is continuous on M.

Finally, define the outcome function for private good consumption $X(m)$: $M \to \mathbb{R}_+$ by

$$X_i(m) = \bar{\gamma}(m)\gamma_i x_i(m) \tag{9}$$

which is agent i's consumption resulting from the strategic configuration m. It may be remarked that, because $\bar{\gamma}(m)\gamma_i \leqq 1$ and $\bar{\gamma}(m)\gamma_i \to 1$ as $\gamma_i \to \infty$, $X_i(m) \leqq x_i(m)$ and $X_i(m) \to x_i(m)$ as $\gamma_i \to \infty$.

Thus the outcome function is continuous and also feasible on M since, by the construction of $B(m)$, $(X(m),Y(m)) \in \mathbb{R}_+^{n+K}$ and

$$\sum_{i=1}^{n} X_i(m) + C(Y(m)) \leqq \sum_{i=1}^{n} w_i \leqq \sum_{i=1}^{n} \mathring{w}_i \tag{10}$$

for all $m \in M$. Note that the last inequality comes from the assumption that agents cannot overstate their endowments.

Denote $h{:}M \to \mathbb{R}_+^{n+K}$ the outcome function, or more explicitly, $h_i(m) = (X_i(m),Y(m))$. Then the mechanism consists of $\langle M,h\rangle$ which is defined on E.

Remark 4. *Note that the mechanism constructed above is a destruction mechanism. That is, the unreported endowments are destroyed but not consumed. One can also construct a withholding mechanism by using the techniques similar to those in Tian (1993).*

A message $m^* = (m_1^*, \ldots, m_n^*) \in M$ is said to be a *Nash equilibrium* of the mechanism $\langle M,h\rangle$ for an economy e if, for each $i \in N$ and $m_i \in M_i$, it is not true that

$$h_i(m_i, m_{-i}^*)\ P_i\ h_i(m^*) \tag{11}$$

where $(m_i, m_{-i}^*) = (m_1^*, \ldots, m_{i-1}^*, m_i, m_{i+1}^*, \ldots, m_n^*)$. $h(m^*)$ is then called a *Nash (equilibrium) allocation* of the mechanism for the economy e. Denote by $N_{M,h}(e)$ the set of all such Nash (equilibrium) allocations.

The mechanism $\langle M,h\rangle$ is said to *Nash-implement* Linear Cost Share Equilibrium allocations $LCSE$ on E, if, for all $e \in E$, $N_{M,h}(e) = LCSE(e)$.

A message $m^* = (m_1^*, \ldots, m_n^*) \in M$ is said to be a *strong Nash equilibrium* of the mechanism $\langle M,h\rangle$ for an economy e if there does not exist any coalition S and $m_S \in \prod_{i \in S} M_i$ such that for all $i \in S$,

$$h_i(m_S, m_{-S}^*)\ P_i\ h_i(m^*) \tag{12}$$

$h(m^*)$ is then called a *strong Nash (equilibrium) allocation* of the mechanism for the economy e. Denote by $SN_{M,h}(e)$ the set of all such strong Nash (equilibrium) allocations.

The mechanism $\langle M,h \rangle$ is said to doubly *implement* the Linear Cost Share Equilibrium allocations on E, if, for all $e \in E$, $SN_{M,h}(e) = N_{M,h}(e) = LCSE(e)$.

Remark 5. *Note that our mechanism works not only for three or more agents, but also for a two-agent world. While most mechanisms which implement market-type social choice correspondences* (*such as Walrasian, Lindahl, Ratio, or LCSE allocations*) *in the existing literature need to distinguish the case of two agents from that of three or more agents, this paper gives a unified mechanism which is independent of the number of agents.*

4. Double implementation

The remainder of this paper is devoted to proving the following theorem.

Theorem 1. *For the class of public goods economies specified by E, the above feasible and continuous mechanism doubly implements the LCSE correspondence in Nash and strong Nash equilibria on E.*

Proof. The proof of Theorem 1 consists of the following three propositions which show the equivalence among Nash allocations, strong Nash allocations, and LCSE allocations. Proposition 1 below proves that every Nash allocation is an LCSE allocation. Proposition 2 below proves that every LCSE allocation is a Nash allocation. Proposition 3 below proves that every Nash equilibrium allocation is a strong Nash equilibrium allocation.

To show these propositions, we first prove the following lemmas.

Lemma 2. *Suppose* $(x_i(m),Y(m))\ P_i\ (x_i,y)$ *for* $i \in N$. *Then agent i can choose a very large* γ_i *such that* $(X_i(m),Y(m))\ P_i(x_i,y)$.

Proof. If agent i declares a large enough γ_i, then $\bar{\gamma}(m)$ becomes very small (since $\bar{\gamma}(m)\gamma_i \leqq 1$) and thus almost nullifies the effect of other agents in $\gamma \Sigma_{i=1}^{n} \gamma_i x_i(m) + C(Y(m)) \leqq \Sigma_{i=1}^{n} w_i$. Thus, $X_i(m) = \bar{\gamma}(m)\gamma_i x_i(m)$ can arbitrarily approach $x_i(m)$ as agent i wishes. From the hypothesis that $(x_i(m),Y(m))\ P_i\ (x_i,y)$ and continuity of preferences, we have $(X_i(m),Y(m))\ P_i\ (x_i,y)$ if agent i chooses a very large γ_i. □

Lemma 3. *If* $(X(m^*),Y(m^*)) \in N_{M,h}(e)$, *then* $X_i(m^*) \in \mathbb{R}_{++}$ *for all* $i \in N$.

Proof. Suppose, by way of contradiction, that $X_i(m^*) = 0$ for some $i \in N$. Let

$$x_i = \frac{1}{1 + \|\alpha_i^* - \alpha_{i+1}^*\| + \|\beta_i^* - \beta_{i+1}^*\|} w_i^*.$$

Then $(x_i,0)\ P_i\ (X_i(m^*),Y(m^*))$ by indispensability of the private good and $w_i^* > 0$. Thus if agent i chooses $y_i = -\Sigma_{j\neq i}^n\ y_j^*$, and keeps other components of the message unchanged, then $0 \in B(m_i,m_{-i}^*)$. Thus $(x_i(m_i,m_{-i}^*),Y(m_i,m_{-i}^*)) = (x_i,0)$ so that $(x_i(m_i,m_{-i}^*),Y(m_i,m_{-i}^*))\ P_i\ (X_i(m^*),Y(m^*))$. Then, by Lemma 2, $(X_i(m_i,m_{-i}^*),Y(m_i,m_{-i}^*))\ P_i\ (X_i(m^*),Y(m^*))$ if agent i chooses a very large γ_i. This contradicts the hypothesis that $(X(m^*),Y(m^*)) \in N_{M,h}(e)$ and thus we must have $X_i(m^*) \in \mathbb{R}_{++}$ for all $i \in N$. □

Lemma 4. *If m^* is a Nash equilibrium, then $\alpha_1^* = \alpha_2^* = \cdots = \alpha_n^*$ and $\beta_1^* = \beta_2^* = \ldots = \beta_n^*$. Therefore $\Sigma_{i\in N} a_i(m^*) = 0$ and $\Sigma_{i\in N} b_i(m^*) = 1$.*

Proof. Suppose, by way of contradiction, that $\alpha_i^* \neq \alpha_{i+1}^*$ and/or $\beta_i^* \neq \beta_{i+1}^*$ for some $i \in N$. Then

$$a_i(m^*)\cdot Y(m^*) + b_i(m)C(Y(m^*)) \leq \frac{1}{1 + \|\alpha_i^* - \alpha_{i+1}^*\| + \|\beta_i^* - \beta_{i+1}^*\|} w_i^* < w_i^*.$$

Let $x_i = w_i^* - a_i(m^*)\cdot Y(m^*) - b_i(m^*)C(Y(m^*))$. Then $x_i > X_i(m^*)$, and thus $(x_i,Y(m^*))\ P_i\ (X_i(m^*),Y(m^*))$ by monotonicity of preferences. Thus, if agent i chooses $\alpha_i = \alpha_{i+1}^*$, $\beta_i = \beta_{i+1}^*$, and keeps other components of the message unchanged, we have $Y(m^*) \in B(m_i,m_{-i}^*)$ and $x_i(m_i,m_{-i}^*) = x_i$. Hence, $(x_i(m_i,m_{-i}^*),Y(m_i,m_{-i}^*)) = (x_i,Y(m^*))$ so that $(x_i(m_i,m_{-i}^*),Y(m_i,m_{-i}^*))\ P_i\ (X_i(m^*),Y(m^*))$. Then, by Lemma 2, $(X_i(m_i,m_{-i}^*),Y(m_i,m_{-i}^*))\ P_i\ (X_i(m^*),Y(m^*))$ if agent i chooses a very large γ_i. This contradicts the hypothesis that $(X(m^*),Y(m^*)) \in N_{M,h}(e)$. Thus we must have $\alpha_1^* = \alpha_2^* = \ldots = \alpha_n^*$ and $\beta_1^* = \beta_2^* = \ldots = \beta_n^*$, and therefore $\Sigma_{i\in N} a_i(m^*) = 0$ and $\Sigma_{i\in N} b_i(m^*) = 1$. □

Lemma 5. *If m^* is a Nash equilibrium, then $w_i^* = \mathring{w}_i$ and consequently $X_i(m^*) + a_i(m^*)\cdot Y(m^*) + b_i(m^*)C(Y(m^*)) = \mathring{w}_i$ for all $i \in N$.*

Proof. Suppose, by way of contradiction, that $w_i^* \neq \mathring{w}_i$ for some $i \in N$. Then $X_i(m^*) + a_i(m^*)\cdot Y(m^*) + b_i(m^*)C(Y(m^*)) \leqq w_i^* < \mathring{w}_i$. Let $x_i = \mathring{w}_i - a_i(m^*)\cdot Y(m^*) - b_i(m^*)C(Y(m^*))$. Then we have $x_i > X_i(m^*)$ and thus $(x_i,Y(m^*))\ P_i\ (X_i(m^*),Y(m^*))$ by monotonicity of preferences. Thus if agent i chooses $w_i = \mathring{w}$, $y_i = Y(m^*) - \Sigma_{j\neq i}^n y_j^*$, and keeps other components of the message unchanged, then $(x_i(m_i,m_{-i}^*),Y(m_i,m_{-i}^*)) = (x_i,Y(m^*))$ so that $(x_i(m_i,m_{-i}^*),Y(m_i,m_{-i}^*))\ P_i\ (X_i(m^*),Y(m^*))$. Therefore, by Lemma 2, $(X_i(m_i,m_{-i}^*),Y(m_i,m_{-i}^*))\ P_i\ (X_i(m^*),Y(m^*))$ if agent i chooses a very large γ_i. This contradicts the hypothesis that $(X(m^*),Y(m^*)) \in N_{M,h}(e)$. So we must have $w_i^* = \mathring{w}_i$. Thus, by the definition of $X_i(m)$, we have $X_i(m^*) = w_i^* - a_i(m^*)\cdot Y(m^*) - b_i(m^*)C(Y(m^*)) = \mathring{w}_i - a_i(m^*)\cdot Y(m^*) - b_i(m^*)C(Y(m^*))$ and therefore $X_i(m^*) + a_i(m^*)\cdot Y(m^*) + b_i(m^*)C(Y(m^*)) = \mathring{w}_i$ for all $i \in N$. □

Lemma 6. *If $(X(m^*),Y(m^*)) \in N_{M,h}(e)$, then $\bar{\gamma}(m^*)\gamma_i^* = 1$ for all $i \in N$ and thus $X(m^*) = x(m^*)$.*

Proof. This is a consequence of Lemma 5. Suppose $\bar{\gamma}(m^*)\gamma_i^* < 1$ for some $i \in N$. Then

$X_i(m^*) = \bar{\gamma}(m^*)\gamma_i^* x_i(m^*) < x_i(m^*)$, and therefore $X_i(m^*) + q_i(m^*) \cdot Y(m^*) < x_i(m^*) + q_i(m^*) \cdot Y(m^*) \leqq \mathring{w}_i$. But this is impossible by Lemma 5. □

Proposition 1. *If the mechanism defined above has a Nash equilibrium* m^*, *then the Nash allocation* $(X(m^*),Y(m^*))$ *is a Linear Cost Share Equilibrium allocation with* $(a_1(m^*),\ldots,a_n(m^*))$ *and* $(b_1(m^*),\ldots,b_n(m^*))$ *as the parameters of the linear cost share system, i.e.* $N_{M,h}(e) \subseteqq LCSE(e)$ *for all* $e \in E$.

Proof. Let m^* be a Nash equilibrium. We need to prove that $(X(m^*),Y(m^*))$ is an LCSE allocation with $(a_1(m^*),\ldots,a_n(m^*))$ and $(b_1(m^*),\ldots,b_n(m^*))$ as the parameters of the linear cost share system. Note that the mechanism is feasible, $\Sigma_{i=1}^n a_i(m^*) = 0$, and $\Sigma_{i=1}^n b_i(m^*) = 1$ as well as $X_i(m^*) + a_i(m^*) \cdot Y(m^*) + b_i(m^*)C(Y(m^*)) = \mathring{w}_i$ for all $i \in N$ by Lemmas 4 and 5. So we only need to show that each individual is maximizing his/her preferences. Suppose, by way of contradiction, that there is some $(x_i,y) \in \mathbb{R}_+^{1+K}$ such that $(x_i,y)\ P_i\ (X_i(m^*),Y(m^*))$ and $x_i + a_i(m^*) \cdot y + b_i(m^*)C(y) \leqq \mathring{w}_i$. Let:

$$x_{\lambda i} = \lambda x_i + (1-\lambda)X_i(m^*)$$

$$y_\lambda = \lambda y + (1-\lambda)Y(m^*)$$

Then by the convexity of preferences we have $(x_{\lambda i},y_\lambda)\ P_i\ (X_i(m^*),Y(m^*))$ for any $0 < \lambda < 1$. Also $(x_{\lambda i},y_\lambda) \in \mathbb{R}_+^{1+K}$ and $x_{\lambda i} + a_i(m^*) \cdot y_\lambda + b_i(m^*)C(y_\lambda) \leqq \mathring{w}_i$ by convexity of the cost function and non-negativity of $b_i(m^*)$. Now suppose that player i chooses $y_i = y_\lambda - \Sigma_{j \neq i}^n y_j^*$, and keeps w_i^*, a_i^*, b_i^*, and γ_i^* unchanged. Since $w_j^* - a_j(m^*) \cdot Y(m^*) - b_j(m^*)C(Y(m^*)) > 0$ for all $j \in N$ by noting the fact that $X(m^*) > 0$, by the continuity of the cost function and outcome functions, we have $w_j^* - a_j(m_i,m_{-i}^*) \cdot y_\lambda - b_j(m_i,m_{-i}^*)C(y_\lambda) > 0$ for all $j \in N$ as λ is sufficiently small. Hence $y_\lambda \in B(m_i,m_{-i}^*)$ and therefore $Y(m_i,m_{-i}^*) = y_\lambda$ as well as, by Lemma 6 and the convexity of $C(\cdot)$, $X_i(m_{-i}^*,m_i) = x_i(m_{-i}^*,m_i) = \mathring{w}_i - a_i(m^*) \cdot Y(m_{-i}^*,m_i) - b_i(m^*)C(Y(m_{-i}^*,m_i)) = \mathring{w}_i - a_i(m^*) \cdot y_\lambda - b_i(m^*)C(y_\lambda) \geqq \lambda[\mathring{w}_i - a_i(m^*) \cdot y - b_i(m^*)C(y)] + (1-\lambda)[\mathring{w}_i - a_i(m^*) \cdot Y(m^*) - b_i(m^*)C(Y(m^*))] = \lambda x_i + (1-\lambda)X_i(m^*) = x_{i\lambda}$. From $(x_{i\lambda},y_\lambda)P_i$ $(X_i(m^*),Y(m^*))$, we have:

$$(X_i(m_{-i}^*,m_i),Y(m_{-i}^*,m_i))\ P_i\ (X_i(m^*),Y(m^*))$$

This contradicts the hypothesis that $(X(m^*),Y(m^*)) \in N_{M,h}(e)$. □

Proposition 2. *If* (x^*,y^*) *is a LCSE allocation with* $(a_1^*,\ldots,a_n^*)$ *and* $(b_1^*,\ldots,b_n^*)$ *as the parameters of the linear cost share system, then there is a Nash equilibrium* m^* *such that* $X_i(m^*) = x_i^*$, $a_i(m^*) = a_i^*$, *and* $b_i(m^*) = b_i^*$, *for all* $i \in N$, $Y(m^*) = y^*$, *i.e.* $LCSE(e) \subseteqq N_{M,h}(e)$ *for all* $e \in E$.

Proof. We first note that $x^* \in \mathbb{R}_{++}^n$ by the assumption that the private good is indispensable. We need to show that there is a message m^* such that (x^*,y^*) is a Nash equilibrium allocation. Let $\alpha_i^* = (a_1^*,\ldots,a_n^*)$, $\beta_i^* = (b_1^*,\ldots,b_n^*)$, $w_i^* = \mathring{w}_i$, $y_i^* = y^*/n$, and $\gamma_i^* = 1$ for all $i \in N$. Then, $a_i(m^*) = a_i^*$, $b_i(m^*) = b_i^*$, $Y(m^*) = y^*$, and $X_i(m^*) = x_i^*$, for all $i \in N$. Notice that agent i cannot change $a_i(m^*)$ and $b_i(m^*)$ by changing m_i. Then, $(a_i(m_i,m^*_{-i}),b_i(m_i,m^*_{-i})) = (a_i(m^*),b_i(m^*))$ for all $m_i \in M_i$. Also, $(X(m_i,m^*_{-i}), Y(m_i,m^*_{-i})) \in \mathbb{R}_+^{1+K}$ and $X_i(m_i,m^*_{-i}) + a_i(m^*) \cdot Y(m_i,m^*_{-i}) + b_i(m^*)C(Y(m_i,m^*_{-i})) \leqq \mathring{w}_i$ for all $i \in N$ and $m_i \in M_i$. Therefore, we know that it is not true that:

$$(X_i(m_i,m^*_{-i}),Y(m_i,m^*_{-i}))\ P_i\ (X_i(m^*),Y(m^*))$$

for otherwise it contradicts the fact that $(X_i(m^*),Y(m^*))$ is a LCSE allocation. □

Proposition 3. *Every Nash equilibrium m^* of the mechanism defined above is a strong Nash equilibrium, that is $N_{M,h}(e) \subseteqq SN_{M,h}(e)$ for all $e \in E$.*

Proof. Let m^* be a Nash equilibrium. By Proposition 1, we know that $(X(m^*),Y(m^*))$ is a Linear Cost Share Equilibrium allocation with $(a_1(m^*),\dots,a_n(m^*))$ and $(b_1(m^*),\dots,b_n(m^*))$ as the parameters of the linear cost share system. Then $(X(m^*),Y(m^*))$ is Pareto optimal and thus the coalition N cannot be improved upon by any $m \in M$. Now for any coalition S with $\emptyset \neq S \neq N$, choose $i \in S$ such that $i+1 \not\in S$. Then no strategy played by S can change the budget set of i since $a_i(m)$ and $b_i(m)$ are determined by $a_{i+1,i}$ and $b_{i+1,i}$, respectively. Furthermore, because $(X(m^*),Y(m^*)) \in LCSE(e)$, it is P_i-maximal in the budget set of i, and thus S cannot improve upon $(X(m^*),Y(m^*))$. □

Since every strong Nash equilibrium is clearly a Nash equilibrium, by combining Propositions 1–3, we know that $N_{M,h}(e) = LCSE(e)$ for all $e \in E$ and thus the proof of Theorem 1 is completed. □

5. Concluding remarks

In this paper, we have presented a general market-type mechanism which doubly implements the Linear Cost Share Equilibrium allocations when coalition patterns, preferences and endowments are private information and unknown to the designer. The mechanism is well-behaved in the sense that it is feasible and continuous. Furthermore, unlike most mechanisms proposed in the literature, it gives a unified mechanism which is independent of the number of agents. Also, if one reinterprets the commodity space, our mechanism results in Pareto efficient allocations for private goods economies even with externalities at Nash equilibria without considering the profit-maximization principle. Thus, our mechanism is sufficiently general to implement Walrasian equilibrium for pure exchange economies and Lindahl equilibrium or Linear Cost Share Equilibrium allocations for public goods economies. In other words, the mechanism in the paper appears to represent a unified mechanism to implement market-type allocations in private and public goods economies.

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Appendix A. Proof of Lemma 1

It is clear that $B(\cdot)$ has closed graph by the continuity of $a_i(\cdot)$, $b_i(\cdot)$, and $C(\cdot)$. Since the range space of the correspondence $B(\cdot)$ is bounded by the set $\{y \in \mathbb{R}^K_+ : C(y) \leqq \Sigma_{i=1}^n w_i\}$, it is compact. Thus, $B(\cdot)$ is upper hemi-continuous on M. So we only need to show that $B(m)$ is also lower hemi-continuous at every $m \in M$. Let $m \in M$, $y \in B(m)$, and let $\{m_k\}$ be a sequence such that $m_k \to m$, where $m_k = (m_1^k, \ldots, m_n^k)$ and $m_i^k = (w_i^k, \alpha_i^k, \beta_i^k, y_i^k, \gamma_i^k)$. We want to prove that there is a sequence $\{y_k\}$ such that $y_k \to y$, and, for all k, $y_k \in B(m_k)$, i.e. $y_k \in \mathbb{R}^K_+$, $w_i^k - a_i(m_k) \cdot y_k - b_i(m_k)C(y_k) \geqq 0$ for all $i \in N$, and $C(y_k) \leqq \Sigma_{i \in N} w_i^k$. We first prove that there is a sequence $\{\hat{y}_k\}$ such that $\hat{y}_k \to y$, and, for all k, $\hat{y}_k \in \mathbb{R}^K_+$, $w_i^k - a_i(m_k) \cdot \hat{y}_k - b_i(m_k)C(\hat{y}_k) \geqq 0$ for all $i \in N$.

Let $N' = \{i \in N : w_i - a_i(m) \cdot y - b_i(m)C(y) = 0\}$. Two cases will be considered.

Case 1: $N' = \emptyset$, i.e. $w_i - a_i(m) \cdot y - b_i(m)C(y) > 0$ for all $i \in N$. Then, by the continuity of $a_i(\cdot)$ and $b_i(\cdot)$, for all k larger than a certain integer k', we have $w_i^k - a_i(m_k) \cdot y - b_i(m_k)C(y) > 0$. Let $\hat{y}_k = y$ for all $k > k'$ and $\hat{y}_k = 0$ for $k \leqq k'$. Then, $\hat{y}_k \to y$, and, for all k, $\hat{y}_k \in \mathbb{R}^K_+$ and $w_i^k - a_i(m_k) \cdot \hat{y}_k - b_i(m_k)C(\hat{y}_k) \geqq 0$ for all $i \in N$.

Case 2: $N' \neq \emptyset$. Then $w_i - a_i(m) \cdot y - b_i(m)C(y) = 0$ for all $i \in N'$. Note that, since $w_i > 0$, we must have $a_i(m) \cdot y + b_i(m)C(y) > 0$, and thus, by the continuity of $a_i(\cdot)$ and $b_i(\cdot)$, $f_k(y) \equiv a_i(m_k) \cdot y + b_i(m_k)C(y) > 0$ for all k larger than a certain integer k'. For each $k \geqq k'$ and $i \in N'$, let $\lambda_{ik} = \frac{w_i^k}{f_k(y)}$, let

$$\hat{y}_{ik} = \begin{cases} \lambda_{ik} y & \text{if } \dfrac{w_i^k}{f_k(y)} \leqq 1 \\ y & \text{otherwise} \end{cases}$$

and let $\hat{y}_k = \min_{i \in N'}\{\hat{y}_{ik}\} = y \min_{i \in N'}\{1, \lambda_{ik}\}$. Then $\hat{y}_k \leqq \hat{y}_{ik} \leqq y$. Also, since

$$\lambda_{ik} = \frac{w_i^k}{a_i(m_k) \cdot y + b_i(m_k)C(y)} \to \frac{w_i}{a_i(m) \cdot y + b_i(m)C(y)} = 1$$

for all $i \in N'$, we have $\hat{y}_k \to y$ and $f_k(\hat{y}_{ik}) > 0$ for all k larger than a certain integer k''. Now we claim that $\hat{y}_k$ also satisfies $w_i^k - a_i(m_k) \cdot \hat{y}_k - b_i(m_k)C(\hat{y}_k) \geqq 0$ for all $i \in N$ and $k \geqq \max\{k', k''\}$. Indeed, for each $i \in N'$, if $\lambda_{ik} = w_i^k / f_k(y) \leqq 1$, then $\hat{y}_k \leqq \hat{y}_{ik} = \lambda_{ik} y$, and thus we have $f_k(\hat{y}_k) \leqq f_k(\hat{y}_{ik}) = f_k(\lambda_{ik} y) \leqq \lambda_{ik} f_k(y) = w_i^k$. This is because $f_k(y)$ is convex with $f_k(0) = 0$ by the convexity of $C(\cdot)$ and $C(0) = 0$, and thus $f_k(\lambda_{ik} y) \leqq \lambda_{ik} f_k(y)$. So the second inequality holds. To see the first inequality also holds, writing $\hat{y}_k = \lambda \hat{y}_{ik}$ for some

$\lambda \leqq 1$ (because $\hat{y}_k \leqq \hat{y}_{ik}$ and $\hat{y}_k$ and $\hat{y}_{ik}$ are both proportional to y, such a λ exists), we have $f_k(\hat{y}_k) = f_k(\lambda \hat{y}_{ik}) \leqq \lambda f_k(\hat{y}_{ik}) \leqq f_k(\hat{y}_{ik})$ by noting that $f_k(\hat{y}_{ik}) > 0$ for $k \geqq k''$. Consequently, we have $w_i^k - a_i(m_k) \cdot y - b_i(m_k)C(\hat{y}_k) \geqq 0$.

Now, for each $i \in N'$, if $w_i^k / f_k(y) > 1$, i.e. $a_i(m_k) \cdot y + b_i(m_k)C(y) < w_i^k$, then $\hat{y}_k \leqq \hat{y}_{ik} = y$, and thus $f_k(\hat{y}_k) \leqq f_k(\hat{y}_{ik}) = f_k(y) < w_i^k$, as above, by the convexity of $f_k(\cdot)$ and $f_k(0) = 0$. Consequently, we have $w_i^k - a_i(m_k) \cdot \hat{y}_k - b_i(m_k)C(\hat{y}_k) > 0$.

For all $i \in N \backslash N'$, since $w_i^k - a_i(m_k) \cdot y - b_i(m_k)C(y) > 0$, we have $w_i^k - a_i(m_k) \cdot \hat{y}_k - b_i(m_k)C(\hat{y}_k) > 0$ for all k larger than a certain integer k''' by the continuity of $C(\cdot)$. Thus, for all $k \geqq \max\{k', k'', k'''\}$ and $i \in N$, we have $w_i^k - a_i(m_k) \cdot \hat{y}_k - b_i(m_k)C(\hat{y}_k) \geqq 0$.

We now show that there is a sequence $\{\bar{y}_k\}$ such that $\bar{y}_k \rightarrow y$, and, for all k, $\bar{y}_k \in \mathbb{R}_+^K$ and $C(\bar{y}_k) \leqq \Sigma_{i \in N} w_i^k$. Again, two cases will be considered.

Case (i): $C(y) < \Sigma_{i \in N} w_i$. Then, for all k larger than a certain integer k', we have $C(y) < \Sigma_{i \in N} w_i^k$. Let $\bar{y}_k = y$ for all $k > k'$ and $\bar{y}_k = 0$ for $k \leqq k'$. Then, $\bar{y}_k \rightarrow y$, and, for all k, $\bar{y}_k \in \mathbb{R}_+^K$ and $C(\bar{y}_k) \leqq \Sigma_{i \in N} w_i^k$.

Case (ii): $C(y) = \Sigma_{i \in N} w_i$. Define $\bar{y}_k$ as follows:

$$\bar{y}_k = \begin{cases} \dfrac{\sum_{i \in N} w_i^k}{C(y)} y & \text{if } \dfrac{\sum_{i \in N} w_i^k}{C(y)} \leqq 1 \\ y & \text{otherwise} \end{cases}$$

Then $\bar{y}_k \leqq y$. Also, since

$$\frac{\sum_{i \in N} w_i^k}{C(y)} \rightarrow \frac{\sum_{i \in N} w_i}{C(y)} = 1,$$

we have $\bar{y}_k \rightarrow y$. We now claim that $\bar{y}_k$ satisfies $C(\bar{y}_k) \leqq \Sigma_{i \in N} w_i^k$. Indeed, if $C(y) \geqq \Sigma_{i \in N} w_i^k$, i.e.

$$\frac{\sum_{i \in N} w_i^k}{C(y)} \leqq 1,$$

then

$$\bar{y}_k = \frac{\sum_{i \in N} w_i^k}{C(y)} y$$

and thus

$$C(\bar{y}_k) \leqq \frac{\sum_{i \in N} w_i^k}{C(y)} C(y) = \sum_{i \in N} w_i^k$$

by the convexity of $C(\cdot)$ and $C(0) = 0$. If $C(y) < \Sigma_{i \in N} w_i^k$, i.e.

$$\frac{\sum_{i \in N} w_i^k}{C(y)} > 1,$$

then $\bar{y}_k = y$ and thus $C(\bar{y}_k) = C(y) < \Sigma_{i \in N} w_i^k$.

Thus, in both cases, there is a sequence $\{\bar{y}_k\}$ such that $\bar{y}_k \rightarrow y$, and, for all k, $C(\bar{y}_k) \leqq \Sigma_{i \in N} w_i^k$.

Finally, let $y'_k = \min(\bar{y}_k, \hat{y}_k)$. Then $y'_k \rightarrow y$ since $\bar{y}_k \rightarrow y$ and $\hat{y}_k \rightarrow y$. Also, for every k larger than a certain integer $\bar{k}$, we have $y'_k \geqq 0$, $C(y'_k) \leqq C(\bar{y}_k) \leqq \Sigma_{i \in N} w_i^k$ by the monotonicity of $C(\cdot)$, and, as above, by the convexity of $f_k(\cdot)$, $f_k(0) = 0$, and $0 < f_k(y'_k) \leqq f_k(\hat{y}_k)$, we have $a_i(m_k) \cdot y'_k + b_i(m_k)C(y'_k) \leqq a_i(m_k) \cdot \hat{y}_k + b_i(m_k)C(\hat{y}_k) \leqq w_i^k$ which implies that $w_i^k - a_i(m_k) \cdot y'_k - b_i(m_k)C(y'_k) \geqq 0$ for all $i \in N$. Let $y_k = y'_k$ for all $k > \bar{k}$ and $y_k = 0$ for $k \leqq \bar{k}$. Then, $y_k \rightarrow y$, and $y_k \in B(m_k)$ for all k. Therefore, the sequence $\{y_k\}$ has all the desired properties. So $B(m)$ is lower hemi-continuous at every $m \in M$. $\square$

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