

# The Unique Informational Efficiency of the Lindahl Allocation Process in Economies with Public Goods\*

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## Abstract

This paper investigates the informational requirements of resource allocation processes in public goods economies with any number of firms and commodities. We show that the Lindahl mechanism is informationally efficient in the sense that it uses the smallest message space among smooth resource allocation processes that are informationally decentralized and realize Pareto optimal allocations over the class of public goods economies where Lindahl equilibria exist. Furthermore, we show that the Lindahl mechanism is the unique informationally efficient decentralized mechanism that realizes Pareto efficient and individually rational allocations in public goods economies with Cobb-Douglas utility functions and quadratic production functions.

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## 1 Introduction

Since the pioneering work of Hurwicz (1960) and Mount and Reiter (1974), there has been a lot of work on studying the informational requirements of decentralized resource allocation mechanisms. The focus in this literature has particularly been on the dimension of the message space being used for communication among agents. These informational requirements depend upon two basic components: the class and types of economic environments over which a mechanism is supposed to operate and the particular outcomes that a mechanism is required to realize. This paper will study the informational requirements of resource allocation mechanisms that select Pareto optimal allocations for public goods economies with general convex production technologies and any number of producers and goods.

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The interest in studying the informational requirements and the design of resource allocation mechanisms was greatly stimulated by the “socialist controversy” — the debate over the feasibility of central planning between Mises-Hayek and Lange-Lerner. In line with the prevailing tradition, interest in this area was focused on Pareto-optimality and informationally decentralized decision making. Allocative efficiency (Pareto optimality) and informational efficiency are two highly desired properties for an economic mechanism to have. Pareto optimality requires resources be allocated efficiently while informational efficiency requires an economic system have the minimal informational cost of operation.

The notion of an allocation mechanism was first formalized by Hurwicz (1960). Such a mechanism can be viewed as an abstract planning procedure; it consists of a message space in which communication takes place, rules by which the agents form messages, and an outcome function that translates messages into outcomes (allocations of resources). Mechanisms are imagined to operate iteratively. Attention, however, may be focused on mechanisms that have stationary or equilibrium messages for each possible economic environment. A mechanism realizes a prespecified welfare criterion (called social choice rule, or social choice correspondence) if the outcomes given by the outcome function agree with the welfare criterion at the stationary messages. The realization theory studies the question of how much communication must be provided to realize a given performance, or more precisely, studies the minimal informational cost of operating a given performance in terms of the size of the message space and determines which economic system or social choice rule is informationally the most efficient in the sense that a minimal informational cost is used to operate the system. Such studies can be found in Hurwicz (1972, 1977), Mount and Reiter (1974), Calsamiglia (1977), Walker (1977), Sato (1981), Hurwicz, Reiter, and Saari (1985), Calsamiglia and Kirman (1993), Tian (1990, 1994, 2004, 2006) among others.

One of the well-known results in this literature establishes the minimality of the competitive mechanism in using information for pure exchange economies. Hurwicz (1972), Mount and Reiter (1974), Walker (1977), Hurwicz (1986b) among others proved that, for pure exchange private goods economies, the Walrasian allocation process is the informationally efficient process in the sense that any smooth informationally decentralized allocation mechanism that achieves Pareto optimal allocations must use information as least as large as the competitive mechanism, i.e., the competitive allocation process has a message space of minimal dimension among smooth resource allocation processes that are privacy preserving (informationally decentralized) and non-wasteful (i.e., yielding Pareto efficient allocations).<sup>1</sup> For brevity, these results have been referred to as the Efficiency Theorem. Jordan (1982) and Calsamiglia and Kirman (1993) further provided the Uniqueness Theorem for private goods pure exchange economies. Jordan (1982)

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<sup>1</sup>The term “smoothness” used here is not referred as the usual differentiability of a function. Instead, the smoothness of a mechanism referees that the stationary message correspondence is either locally threaded or if the inverse of the stationary message correspondence has a Lipschitzian-continuous selection in the subset. This terminology was used by Hurwicz (1999). We will give the definition of the local threadedness below.

proved that the competitive allocation process is uniquely informationally efficient. Calsamiglia and Kirman (1993) proved the equal income Walrasian mechanism is uniquely informationally efficient among all allocations mechanisms that realize fair allocations. Recently, Tian (2004) investigates the informational requirements of resource allocation processes in pure exchange economies with consumption externalities. It is shown that the distributive Lindahl mechanism is a uniquely informationally efficient allocation process that is informationally decentralized and realizes Pareto efficient allocations over the class of economies that include non-malevolent economies. Tian (2006) further proved the unique informational efficiency of the competitive market mechanism for private ownership production economies. These efficiency and uniqueness results are of fundamental importance from the point of view of political economy. They show the uniqueness of the competitive market mechanism in terms of allocative efficiency and informational efficiency for private goods economies.

The concept of Lindahl equilibrium in economies with public goods is, in many ways, a natural generalization of the Walrasian equilibrium notion in private goods economies, with attention to the well-known duality that reverses the role of prices and quantities between private and public goods, and between Walrasian and Lindahl allocations. In the Walrasian case, prices must be equalized while quantities are individualized; in the Lindahl case the quantities of the public good must be the same for everyone, while prices charged for public goods are individualized. In addition, the concepts of Walrasian and Lindahl equilibria are both relevant to private-ownership economies. Furthermore, they are characterized by purely price-taking behavior on the part of agents. It is essentially this property that one can also exploit to define the Lindahl process as an informationally decentralized process.

For the class of public goods economies, Sato (1981) obtained a similar result showing that the Lindahl allocation process has a message space of minimal dimension among a certain class of resource allocation processes that are privacy preserving and non-wasteful. However, Sato (1981) only dealt with the class of public goods economies with just a single producer, and in such a case, a special class of economies is constructed using a class of linear production sets for the producer. Quite clearly, more complex production sets must be devised when the number of firms increases. So one of the purpose of the paper is to fill this gap, although our main purpose in the paper is to establish the unique informational efficiency of the Lindahl mechanism.

In this paper we establish the informational optimality and uniqueness of the Lindahl mechanism for public goods economies with any number of producers. The task of this paper is three-fold. First, we establish the lower bound of information, as measured by the Fréchet topological size of the message space, that is required to guarantee an informationally decentralized mechanism to realize Pareto efficient allocations over the class of public goods economies. Theorem 1 shows that any smooth informationally decentralized mechanism that realizes Pareto efficient allocations on a class of public goods economies that includes a test family of Cobb-Douglas utility functions and quadratic production functions as a subclass has a message space of dimension no smaller than  $(L + K - 1)I + (L + K)J$ , where  $I$  is the number of consumers,  $J$

is the number of firms,  $L$  is the number of private goods, and  $K$  is the number of public goods.

Second, we establish the informational optimality of the Lindahl mechanism. Theorem 2 shows that the lower bound is exactly the size of the message space of the Lindahl mechanism, and thus any smooth informationally decentralized mechanism that realizes Pareto efficient allocations over the class of public goods economies in which Lindahl equilibria exist has a message space of dimension no smaller than the one for the Lindahl allocation mechanism, and thus the Lindahl mechanism is informationally the most efficient process among smooth privacy preserving and non-wasteful resource allocation mechanisms.

Third, we show that the Lindahl mechanism is actually the unique informationally efficient process that realizes Pareto efficient and individually rational allocations over the class of public goods economies with Cobb-Douglas utility functions and quadratic production functions. Theorem 3 shows that any informationally decentralized, individually rational, and non-wasteful mechanism with the  $(L+K-1)I+(L+L)J$ -dimensional message space and a continuous single-valued stationary message function is essentially the Lindahl mechanism on the test family with Cobb-Douglas utility functions and quadratic production functions. Thus, any other economic institution that achieves Pareto efficient and individually rational allocations for public goods economies must use a message space whose informational size is bigger than that of the Lindahl mechanism.

In an unpublished paper, Nayak (1982) attempted to establish the informational efficiency of the Lindahl mechanism for public goods economies. However, he considered an unusual class of production technology sets that results in positive outputs with zero inputs. Nevertheless, to the author's knowledge, there is no "Uniqueness Theorem" on the Lindahl mechanism for public goods economies in the literature. It may also be worthwhile to mention that the proof of Lemma 6 relies on the local homology of manifolds. However, no knowledge of algebraic topology is required to understand the statements of the other lemmas and theorems.

The remainder of this paper is as follows. In Section 2, we provide a formal description of the model. We specify public goods economic environments with any number of goods and firms, and give notation and definitions on resource allocation, performance correspondence, outcome function, allocation mechanism, etc. Section 3 establishes a lower bound of the size of the message space that is required to guarantee that a smooth informationally decentralized mechanism that realizes Pareto efficient allocations on the class of public goods economies. Section 4 gives an Efficiency Theorem on the allocative efficiency and informational efficiency of the Lindahl mechanism for the class of public goods economies where Lindahl equilibria exist. Section 5 gives a Uniqueness Theorem that shows that only the Lindahl mechanism is informationally efficient over the class of public goods economies with Cobb-Douglas utility functions and quadratic production functions. Concluding remarks are presented in section 6.

## 2 Model

In this section we will give notation, definitions, and a framework that will be used in the paper.

### 2.1 Public Goods Economic Environments

Consider public goods economies with  $L$  private goods,  $K$  public goods,  $I$  consumers (characterized by their consumption sets, preferences, and endowments), and  $J$  firms (characterized by their production sets). It will often be convenient to distinguish a vector representing a private commodity bundle with an index  $\rho$ , a vector of public goods with an index  $\sigma$ . Throughout this paper, subscripts are used to index consumers or firms, and superscripts are used to index goods unless otherwise stated. By an agent, we will mean either a consumer or a producer, thus there are  $N := I + J \geq 2$  agents.<sup>2</sup>

For the  $i$ th consumer, his characteristic is denoted by  $e_i = (X_i, w_i, R_i)$ , where  $X_i \subset \mathbb{R}^{L+K}$  is his consumption set,  $w_i \in \mathbb{R}^L$  is his initial endowments of the private goods, and  $R_i$  is a preference ordering on  $X_i$  which is assumed to be strictly monotonically increasing, convex<sup>3</sup>, and continuous. Let  $P_i$  be the strict preference (asymmetric part) of  $R_i$ . We assume that there are no initial endowments of public goods, and the public goods will be produced from private goods. For producer  $j$ , his characteristic is denoted by  $e_j = (\mathcal{Y}_j)$  where  $\mathcal{Y}_j \subset \mathbb{R}^{L+K}$  is his production possibility set. We assume that, for  $j = I + 1, \dots, N$ ,  $\mathcal{Y}_j$  is nonempty, closed, convex, and  $0 \in \mathcal{Y}_j$ . Denote by  $E_i$  the set of the  $i$ -th agent's characteristics.

An economy is the full vector  $e = (e_1, \dots, e_I, e_{I+1}, \dots, e_N)$  and the set of all such production economies is denoted by  $E$  that is endowed with the product topology.

### 2.2 Allocations

Let  $x_i = (x_i^\rho, x_i^\sigma)$  denote a consumption bundle of commodities by consumer  $i$ , where  $x_i^\rho$  is the net exchange of private goods and  $x_i^\sigma$  is a consumption vector of the public goods by consumer  $i$ . Denote by  $x = (x_1, \dots, x_I)$  a (net) consumption distribution. A consumption distribution  $x$  is said to be *individually feasible* if  $(x_i^\rho + w_i, x_i^\sigma) \in X_i$  for all  $i = 1, \dots, I$ . Similarly, let  $y_j = (y_j^\rho, y_j^\sigma)$  denote producer  $j$ 's (net) output vector that has positive components for outputs and negative ones for inputs. Here  $y_j^\rho$  is a production vector of the private goods and  $y_j^\sigma$  is a production vector of the public goods by producer  $j$ . Note that, by the assumption of no public goods inputs,  $y_j^\sigma \geq 0$ . Denote by  $y = (y_{I+1}, \dots, y_N)$  a production plan. A production plan  $y$  is said to be *individually feasible* if  $y_j \in \mathcal{Y}_j$  for all  $j = I + 1, \dots, I + J$ .

An allocation of the economy  $e$  is a vector  $z := (x, y) \in \mathbb{R}^{N(L+K)}$  with  $(x_i^\rho + w_i, x_i^\sigma) \in X_i$  for  $i = 1, \dots, I$  and  $y_j \in \mathcal{Y}_j$  for  $j = I + 1, \dots, I + J$ . An allocation  $z = (x, y)$  is said to be *consistent*

<sup>2</sup>As usual, vector inequalities are defined as follows: Let  $a, b \in \mathbb{R}^m$ . Then  $a \geq b$  means  $a_s \geq b_s$  for all  $s = 1, \dots, m$ ;  $a \geq b$  means  $a \geq b$  but  $a \neq b$ ;  $a > b$  means  $a_s > b_s$  for all  $s = 1, \dots, m$ .

<sup>3</sup> $R_i$  is convex if for bundles  $a, b, c$  with  $0 < \lambda \leq 1$  and  $c = \lambda a + (1 - \lambda)b$ , the relation  $a P_i b$  implies  $c P_i b$ . Note that the term "convex" is defined as in Debreu (1959), not as in some recent textbooks.

if

$$\sum_{i=1}^I x_i^\rho = \sum_{j=I+1}^N y_j^\rho, \quad (1)$$

and

$$x_i^\sigma = \sum_{j=I+1}^N y_j^\sigma, \quad i = 1, \dots, I, \quad (2)$$

i.e., the total net demand for the private goods by consumers is equal to the total net supply of private goods by producers, and the consumption of the public goods by each consumer is equal to the total production of public goods by producers, and thus all consumers consume the same amounts of public goods.

An allocation  $z = (x, y)$  is said to be *feasible* if it is consistent and individually feasible for all individuals.

An allocation  $z = (x, y)$  is said to be *Pareto efficient* if it is feasible and there does not exist another feasible allocation  $z' = (x', y')$  such that  $(x_i^{\rho'} + w_i, x_i^{\sigma'}) R_i(x_i^\rho + w_i, x_i^\sigma)$  for all  $i = 1, \dots, I$  and  $(x_i^{\rho'} + w_i, x_i^{\sigma'}) P_i(x_i^\rho + w_i, x_i^\sigma)$  for some  $i = 1, \dots, I$ . Denote by  $P(e)$  the set of all such allocations.

An important characterization of a Pareto optimal allocation is associated with the following concept.

A price system  $(p, q) = (p, q_1, \dots, q_I) \in \mathbb{R}_+^{L+IK}$  is called a vector of *efficiency prices* for a Pareto optimal allocation  $(x, y)$  if

- (a)  $p \cdot x_i^\rho + q_i \cdot x_i^\sigma \leq p \cdot x_i^{\rho'} + q_i \cdot x_i^{\sigma'}$  for all  $i = 1, \dots, I$  and all  $x_i'$  such that  $(x_i^{\rho'} + w_i, x_i^{\sigma'}) \in X_i$  and  $(x_i^{\rho'} + w_i, x_i^{\sigma'}) R_i(x_i^\rho + w_i, x_i^\sigma)$ ;
- (b)  $p \cdot y_j^\rho + \hat{q} \cdot y_j^\sigma \geq p \cdot y_j^{\rho'} + \hat{q} \cdot y_j^{\sigma'}$  for all  $y_j' \in \mathcal{Y}_j$ ,  $j = I+1, \dots, N$ . Here  $\hat{q} = \sum_{i=1}^I q_i$ .

Similar to Debreu (1959, p. 93), we may call  $(x, y)$  an equilibrium relative to the price system  $p$ . It is well known that under certain regularity conditions such as convexity, continuity, etc, as we assumed in the paper, every Pareto optimal allocation  $(x, y)$  has an efficiency price associated with it (see Foley (1970) and Milleron (1972)). Note that by the strict monotonicity of preferences, we must have  $(p, q) \in \mathbb{R}_{++}^{L+IK}$

It is perhaps not obvious what the appropriate generalization of the individual rationality concept should be for public goods economies in the presence of decreasing returns to scale. It is natural to seek a distribution (called the reference distribution) that would play a role analogous to that played by the initial endowment in the case of constant returns. The reference distribution then should depend on the environment as well as how much was produced, by whom, and other factors. Thus, similar to Hurwicz (1979), we introduce the following definition of individual rationality of an allocation for public goods economies, which includes the usual individual rationality for public goods economies with constant returns as a special case.

An allocation  $z = (x, y)$  is said to be *individually rational* with respect to the fixed share guarantee structure  $\gamma_i(e; \theta)$  if  $(x_i^\rho + w_i, x_i^\sigma) R_i(\gamma_i(e; \theta) + w_i, 0)$  for all  $i = 1, \dots, I$ . Here,  $\gamma_i(e; \theta)$

is given by

$$\gamma_i(e; \theta) = \frac{\sum_{j=I+1}^N \theta_{ij} [p \cdot y_j^\rho + \hat{q} \cdot y_j^\sigma]}{p \cdot w_i} w_i, \quad i = 1, \dots, I, \quad (3)$$

where  $(p; q)$  is an efficiency price system for  $e$  and the  $\theta_{ij}$  are non-negative fractions such that  $\sum_{i=1}^n \theta_{ij} = 1$  for  $j = I + 1, \dots, N$ . Denote by  $I_\theta(e)$  the set of all such allocations.

Now we define the Lindahl equilibria of a private ownership economy with public goods in which the  $i$ -th consumer owns the share  $\theta_{ij}$  of the  $j$ -th producer, and is, consequently entitled to the corresponding fraction of its profits. Thus, the ownership structure can be denoted by the matrix  $\theta = (\theta_{ij})$ . Denote by  $\Theta$  the set of all such ownership structures.

An allocation  $z = (x, y) = (x_1, x_2, \dots, x_I, y_{I+1}, y_{I+2}, \dots, y_N) \in \mathbb{R}_{++}^{I(L+K)} \times \mathcal{Y}$  is a  $\theta$ -Lindahl allocation for an economy  $e$  if it is feasible and there is a price system  $(p, q) = (p, q_1, \dots, q_I)$  with the price vector  $p \in \mathbb{R}_{++}^L$  and personalized price vectors  $q_i \in \mathbb{R}_{++}^K$ , one for each  $i$ , such that:

- (1)  $p \cdot x_i^\rho + q_i \cdot x_i^\sigma = \sum_{j=I+1}^N \theta_{ij} [p \cdot y_j^\rho + \hat{q} \cdot y_j^\sigma]$  for all  $i = 1, \dots, I$ ;
- (2)  $(x_i^{\rho'} + w_i, x_i^{\sigma'}) P_i (x_i^\rho + w_i, x_i^\sigma)$  implies  $p \cdot x_i^{\rho'} + q_i \cdot x_i^{\sigma'} > \sum_{j=I+1}^N \theta_{ij} [p \cdot y_j^\rho + \hat{q} \cdot y_j^\sigma]$  for all  $i = 1, \dots, I$ ;
- (3)  $p \cdot y_j^\rho + \hat{q} \cdot y_j^\sigma \geq p \cdot y_j^{\rho'} + \hat{q} \cdot y_j^{\sigma'}$  for all  $y_j' \in \mathcal{Y}_j$  and  $j = I + 1, \dots, N$ .

Here  $\hat{q} = \sum_{i=1}^I q_i$ . Denote by  $L_\theta(e)$  the set of all such allocations, and by  $\mathcal{L}_\theta(e)$  the set of all such price-allocation triple  $(p, q, z)$ .

It may be remarked that, every  $\theta$ -Lindahl allocation is clearly individually rational with respect to  $\gamma_i(e; \theta)$ , and also, by the strict monotonicity of preferences, it is Pareto efficient. Thus we have  $L_\theta(e) \subset I_\theta(e) \cap P(e)$  for all  $e \in E$ .

### 2.3 Allocation Mechanisms

Let  $Z = \{(x, y) \in \mathbb{R}^{(L+K)(I+J)} : \sum_{i=1}^I x_i^\rho = \sum_{j=I+1}^N y_j^\rho \ \& \ x_i^\sigma = \sum_{j=I+1}^N y_j^\sigma \ (i = 1, \dots, I)\}$  and let  $F$  be a social choice rule, i.e., a correspondence from  $E$  to  $Z$ . Following Mount and Reiter (1974), a *message process* is a pair  $\langle M, \mu \rangle$ , where  $M$  is a set of abstract messages and called message space, and  $\mu : E \rightarrow M$  is a stationary or equilibrium message correspondence that assigns to every economy  $e$  the set of stationary (equilibrium) messages. An *allocation mechanism (process)* is a triple  $\langle M, \mu, h \rangle$  defined on  $E$ , where  $h : M \rightarrow Z$  is the outcome function that assigns every equilibrium message  $m \in \mu(e)$  to the corresponding trade  $z \in Z$ .

An allocation mechanism  $\langle M, \mu, h \rangle$ , defined on  $E$ , *realizes* the social choice rule  $F$ , if for all  $e \in E$ ,  $\mu(e) \neq \emptyset$  and  $h(m) \in F(e)$  for all  $m \in \mu(e)$ .

In this paper, informational properties will be investigated for a class of mechanisms that realize Pareto efficient outcomes. Let  $\mathcal{P}(e)$  be a set of Pareto efficient allocations for  $e \in E$ . An allocation mechanism  $\langle M, \mu, h \rangle$  is said to be *non-wasteful* on  $E$  with respect to  $\mathcal{P}$  if for all  $e \in E$ ,  $\mu(e) \neq \emptyset$  and  $h(m) \in \mathcal{P}(e)$  for all  $m \in \mu(e)$ . If an allocation mechanism  $\langle M, \mu, h \rangle$  is non-wasteful on  $E$  with respect to  $P$ , the set of all Pareto efficient outcomes, then it is said simply to be non-wasteful on  $E$ .

An allocation mechanism  $\langle M, \mu, h \rangle$  is said to be *privacy-preserving or informationally decentralized* on  $E$  if there exists a correspondence  $\mu_i : E_i \rightarrow M$  for each  $i$  such that  $\mu(e) = \bigcap_{i=1}^n \mu_i(e_i)$  for all  $e \in E$ .

Thus, when a mechanism is privacy-preserving, each individual's messages are dependent on the environments only through the characteristics of the individual and the individual does not need to know the characteristics of the other individuals.

**Remark 1** This important feature of the communication process implies that the so called “crossing condition” has to be satisfied. Mount and Reiter (Lemma 5, 1974) showed that an allocation mechanism  $\langle M, \mu, h \rangle$  is privacy-preserving on  $E$  if and only if for every  $i$  and every  $e$  and  $e'$  in  $E$ ,  $\mu(e) \cap \mu(e') = \mu(e'_i, e_{-i}) \cap \mu(e_i, e'_{-i})$ , where  $(e'_i, e_{-i}) = (e_1, \dots, e_{i-1}, e'_i, e_{i+1}, \dots, e_N)$ , i.e., the  $i$ th element of  $e$  is replaced by  $e'_i$ . Thus, if two economies have the same equilibrium message, then any “crossed economy” in which one agent from one of the two initial economies is “switched” with the agent from the other must have the same equilibrium message. Hence, for a given mechanism, if two economies have the same equilibrium message  $m$ , the mechanism leads to the same outcome for both, and further, this outcome must also be the outcome of the mechanism for any of the crossed economies because of the crossing condition.

Let  $\langle M, \mu, h \rangle$  be an allocation mechanism on  $E$ . The stationary message correspondence  $\mu$  is said to be locally threaded at  $e \in E$  if it has a locally continuous single-valued selection at  $e$ . That is, there is a neighborhood  $N(e) \subset E$  and a continuous function  $f : N(e) \rightarrow M$  such that  $f(e') \in \mu(e')$  for all  $e' \in N(e)$ . The stationary message correspondence  $\mu$  is said to be locally threaded on  $E$  if it is locally threaded at every  $e \in E$ .

## 2.4 The Lindahl Process

We now give a privacy-preserving process that realizes the Lindahl correspondence  $L_\theta$ , and in which messages consist of prices and trades of all agents. To do so, we restrict ourselves to the subset, denoted by  $E^L$ , of production economies on which  $L(e) \neq \emptyset$  for all  $e \in E^L$ . For convenience, in this section, we normalize the price system by making the first private goods the numeraire so that  $p^1 = 1$ .

Define the demand correspondence of consumer  $i$  ( $i = 1, \dots, I$ )  $D_i : \mathbb{R}_{++}^L \times \mathbb{R}_{++}^{IK} \times \Theta \times \mathbb{R}_+^J \times E_i$  by

$$D_i(p, q, \theta, \pi_{I+1}, \dots, \pi_N, e_i) = \tag{4}$$

$$\{x_i : (x_i^\rho + w_i, x_i^\sigma) \in X_i, p \cdot x_i^\rho + q_i \cdot x_i^\sigma = \sum_{j=I+1}^N \theta_{ij} \pi_j$$

$$(x_i^{\rho'} + w_i, x_i^{\sigma'}) \in P_i(x_i^\rho + w_i, x_i^\sigma) \text{ implies } p \cdot x_i^{\rho'} + q_i \cdot x_i^{\sigma'} > \sum_{j=I+1}^N \theta_{ij} \pi_j, \quad \} \tag{5}$$

where  $\pi_j$  is the profit of firm  $j$  ( $j = I + 1, \dots, N$ ).

Define the supply correspondence of producer  $j$  ( $j = I + 1, \dots, N$ )  $S_j : \mathbb{R}_{++}^L \times \mathbb{R}_{++}^{IK} \times E_j$  by

$$S_j(p, q, e_j) = \{y_j : y_j \in \mathcal{Y}_j, p \cdot y_j^\rho + \hat{q} \cdot y_j^\sigma \geq p \cdot y_j^{\rho'} + \hat{q} \cdot y_j^{\sigma'} \forall y_j' \in \mathcal{Y}_j\}. \quad (6)$$

Note that  $(p, q, x, y)$  is a  $\theta$ -Lindahl equilibrium for economy  $e$  with the private ownership structure  $\theta$  if  $p \in \mathbb{R}_{++}^L$ ,  $q \in \mathbb{R}_{++}^{IK}$ ,  $x_i \in D_i(p, q, \theta, p \cdot y_{I+1}^\rho + \hat{q} \cdot y_{I+1}^\sigma, \dots, p \cdot y_N^\rho + \hat{q} \cdot y_N^\sigma)$  for  $i = 1, \dots, I$ ,  $y_j \in S_j(p, q, e_j)$  for  $j = I + 1, \dots, N$ , and the allocation  $(x, y)$  is consistent.

The Lindahl process  $\langle M_L, \mu_L, h_L \rangle$  is defined as follows.

Define  $M_L = \mathbb{R}_{++}^L \times \mathbb{R}_{++}^{IK} \times Z$ .

Define  $\mu_L : E \rightarrow M_L$  by

$$\mu_L(e) = \bigcap_{i=1}^N \mu_{Li}(e_i), \quad (7)$$

where  $\mu_{Li} : E_i \rightarrow M_L$  is defined as follows:

- (1) For  $i = 1, \dots, I$ ,  $\mu_{Li}(e_i) = \{(p, q, x, y) : p \in \mathbb{R}_{++}^L, q \in \mathbb{R}_{++}^{IK}, x_i \in D_i(p, q, \theta, p \cdot y_{I+1}^\rho + \hat{q} \cdot y_{I+1}^\sigma, \dots, p \cdot y_N^\rho + \hat{q} \cdot y_N^\sigma, e_i), \sum_{i=1}^I x_i^\rho = \sum_{j=I+1}^N y_j^\rho$ , and  $x_i^\sigma = \sum_{j=I+1}^N y_j^\sigma$ ,  $i = 1, \dots, I\}$ .
- (2) For  $i = I + 1, \dots, N$ ,  $\mu_{Li}(e_i) = \{(p, q, x, y) : p \in \mathbb{R}_{++}^L, q \in \mathbb{R}_{++}^{IK}, y_i \in S_i(p, q, e_i), \sum_{i=1}^I x_i^\rho = \sum_{j=I+1}^N y_j^\rho$ , and  $x_i^\sigma = \sum_{j=I+1}^N y_j^\sigma$ ,  $i = 1, \dots, I\}$ .

Thus, we have  $\mu_L(e) = \mathcal{L}_\theta(e)$  for all  $e \in E$ .

Finally, the Lindahl outcome function  $h_L : M_L \rightarrow Z$  is defined by

$$h_L(p, q, x, y) = (x, y), \quad (8)$$

which is an element in  $L_\theta(e)$ .

The Lindahl process can be viewed as a formalization of resource allocation, which is non-wasteful and individually rational with respect to the fixed share guarantee structure  $\gamma_i(e; \theta)$ . The Lindahl message process is privacy-preserving by the construction of the Lindahl process.

**Remark 2** For a given private ownership structure matrix  $\theta$ , since an element,  $m = (p, q, x_1, \dots, x_I, y_{I+1}, \dots, y_N) \in \mathbb{R}_{++}^L \times \mathbb{R}_{++}^{IK} \times \mathbb{R}^{N(L+K)}$ , of the Lindahl message space  $M_L$  satisfies the conditions  $p^1 = 1$ ,  $\sum_{i=1}^I x_i^\rho = \sum_{j=I+1}^N y_j^\rho$ ,  $x_i^\sigma = \sum_{j=I+1}^N y_j^\sigma$ ,  $p \cdot x_i^\rho + q_i \cdot x_i^\sigma = \sum_{j=I+1}^N \theta_{ij} [p \cdot y_j^\rho + \hat{q} \cdot y_j^\sigma]$  for  $i = 1, \dots, I$ , and one of these equations is not independent, any Lindahl message is contained within a Euclidean space of dimension  $(L + IK + IL + IK + JL + JK) - (1 + L + IK + I) + 1 = (L + K - 1)I + (L + K)J$  and thus, an upper bound on the Euclidean dimension of  $M_L$  is  $(L + K - 1)I + (L + K)J$ .

## 2.5 Informational Size of Message Spaces

The notion of informational size can be considered as a concept that characterizes the relative sizes of topological spaces that are used to convey information in the resource allocation process.

It would be natural to consider that a space, say  $S$ , has more information than the other space  $T$  whenever  $S$  is topologically “larger” than  $T$ . This suggests the following definition, which was introduced by Walker (1977).

Let  $S$  and  $T$  be two topological spaces. The space  $S$  is said to have as much information as the space  $T$  by the Fréchet ordering, denoted by  $S \geq_F T$ , if  $T$  can be embedded homeomorphically in  $S$ , i.e., if there is a subspace of  $S$  which is homeomorphic to  $T$ .

Let  $S$  and  $T$  be two topological spaces and let  $\psi : T \rightarrow S$  be a correspondence. The correspondence  $\psi$  is said to be injective if  $\psi(t) \cap \psi(t') \neq \emptyset$  implies  $t = t'$  for any  $t, t' \in T$ . That is, the inverse,  $(\psi)^{-1}$ , of  $\psi$  is a single-valued function.

A topological space  $M$  is an  $n$ -dimensional manifold if it is locally homeomorphic to  $\mathbb{R}^n$ .

## 2.6 Cobb-Douglas-Quadratic Economies

To establish the informational efficiency of the Lindahl mechanism, we will adopt a standard approach that is widely used in the realization literature: For a set of admissible economies and a smooth informationally decentralized mechanism realizing a social choice correspondence, if one can find a (parametrized) subset (test family) of the set such that the subset is of dimension  $n$ , and the stationary message correspondence is injective, that is, if the inverse of the stationary message correspondence is single-valued, then the dimension of the message space required for an informationally decentralized mechanism to realize the social choice correspondence cannot be lower than  $n$  on the subset. Thus, it cannot be lower than  $n$  for any superset of the subset, and in particular, for the entire class of economies. It is this result that was used by Hurwicz (1977), Mount and Reiter (1974), Walker (1977), Sato (1981), Calsamiglia and Kirman (1993) among others to show the minimal dimension and thus informational efficiency of the competitive mechanism, Lindahl mechanism, and the equal-income Walrasian mechanism over the various classes of economic environments. It is also this result that was used by Calsamiglia (1977) and Hurwicz (1999) to show the non-existence of a smooth finite-dimensional message space mechanism that realize Pareto efficient allocations in certain economies with increasing returns and economies with production externalities that result in non-convex production sets. It is the same result that will be used in the present paper to establish the lower bound of the size of the message space required for an informationally decentralized and non-wasteful smooth mechanism on the test family that we will specify below, and consequently over the entire class of public goods economies with general convex preferences and production sets.

The test family, denoted by  $E^{cq} = \prod_{i=1}^N E_i^{cq}$ , are a special class of public goods economies, where preference orderings are characterized by Cobb-Douglas utility functions, and efficient production technology are characterized by quadratic functions.

For  $i = 1, \dots, I$ , consumer  $i$ 's admissible economic characteristics in  $E_i^{cq}$  are given by the set of all  $e_i = (X_i, w_i, R_i)$  such that  $X_i = \mathbb{R}_+^{L+K}$ ,  $w_i > 0$ , and  $R_i$  is represented by a Cobb-Douglas utility function  $u(\cdot, a_i, c_i)$  with  $a_i \in \mathbb{R}_+^{L-1}$  and  $c_i \in \mathbb{R}_+^K$  such that  $u(x_i^\rho + w_i, x_i^\sigma, a_i, c_i) =$

$$(x_i^{\rho^1} + w_i^1) \prod_{l=2}^L (x_i^{\rho^l} + w_i^l)^{a_i^l} \prod_{k=1}^K (x_i^{\sigma^k})^{c_i^k}.$$

For  $i = I + 1, \dots, N$ , producer  $i$ 's admissible economic characteristics are given by the set of all  $e_i = \mathcal{Y}(b_i, d_i)$  such that

$$\begin{aligned} \mathcal{Y}(b_i, d_i) &= \{y_i \in \mathbb{R}^L : b_i^1 y_i^1 + \sum_{l=2}^L [y_i^{\rho^l} + \frac{b_i^l}{2} (y_i^{\rho^l})^2] + \sum_{k=1}^K [y_i^{\sigma^k} + \frac{d_i^k}{2} (y_i^{\sigma^k})^2] \leq 0 \\ &\quad 0 \leq y_i^{\rho^l} \leq \frac{1}{b_i^l} \text{ for } l = 2, \dots, L \\ &\quad 0 \leq y_i^{\sigma^k} \leq \frac{1}{d_i^k} \text{ for } k = 1, \dots, K\}, \end{aligned} \tag{9}$$

where  $b_i = (b_i^1, \dots, b_i^L)$  with  $b_i^l > 1$  for  $l = 1, \dots, L$ , and  $d_i = (d_i^1, \dots, d_i^K)$  with  $d_i^k > 1$  for  $k = 1, \dots, K$ . It is clear that any economy in  $E^{cq}$  is fully specified by the parameters  $a = (a_2, \dots, a_I)$ ,  $b = (b_{I+1}, \dots, b_N)$ ,  $c = (c_1, \dots, c_I)$ , and  $d = (d_{I+1}, \dots, d_N)$ . Furthermore, production sets are nonempty, closed, and convex by noting that  $0 \in \mathcal{Y}(b_i, d_i)$  and their efficient points are represented by quadratic production functions in which  $y_i^{\rho^1}$  is an input, and all other components of  $y_i$  are outputs.

Given an initial endowment  $\bar{w} \in \mathbb{R}_{++}^{LI}$ , with  $\bar{w}_i^1 \geq 2(L + K - 1)J$ , define a subset  $\bar{E}^{cq}$  of  $E^{cq}$  by  $\bar{E}^{cq} = \{e \in E^{cd} : w_i = \bar{w}_i \forall i = 1, \dots, I\}$ . That is, endowments are constant over  $\bar{E}^{cq}$ .

A topology is introduced to the class  $\bar{E}^{cq}$  as follows. Let  $\|\cdot\|$  be the usual Euclidean norm on  $\mathbb{R}^{L+K}$ . For each consumer  $i$ , ( $i = 1, \dots, I$ ), define a metric  $\delta$  on  $\bar{E}_i^{cq}$  by  $\delta[u_i(\cdot, a_i, c_i), u(\cdot, \bar{a}_i, \bar{c}_i)] = \|a_i - \bar{a}_i\| + \|c_i - \bar{c}_i\|$ . Note that, since endowments are fixed over  $\bar{E}_i^{cq}$ , this defines a topology on  $\bar{E}_i^{cq}$ . Similarly, for each producer  $i$ , ( $i = I + 1, \dots, N$ ), define a metric  $\delta$  on  $\bar{E}_i^{cq}$  by  $\delta[\mathcal{Y}(b_i, d_i), \mathcal{Y}(\bar{b}_i, \bar{d}_i)] = \|b_i - \bar{b}_i\| + \|d_i - \bar{d}_i\|$ . We may endow  $\bar{E}^{cq}$  with the product topology of the  $\bar{E}_i^{cq}$  ( $i = 1, \dots, N$ ) and we call this the parameter topology, which will be denoted by  $\mathcal{T}_p$ . Then it is clear that the topological space  $(\bar{E}^{cq}, \mathcal{T}_p)$  is homeomorphic to the  $(L + K - 1)I + (L + K)J$  dimensional Euclidean space  $\mathbb{R}^{(L+K-1)I+(L+K)J}$ .

### 3 The Lower Bound of Informational Requirements of Mechanisms

In this section we establish a lower bound (the minimal amount) of information, as measured by the Fréchet information size of the message space, that is required to guarantee that an informationally decentralized mechanism realizes Pareto efficient allocations on,  $E$ , the class of public goods economies.

As usual, to establish the efficiency results, we need to impose the interiority assumption that Pareto efficient allocations are interior. A sufficient condition that guarantees interior outcomes is that a mechanism is individually rational. In fact, a mechanism that gives everything to a single individual yields Pareto efficient outcomes and no information about prices is needed. Thus, given a class  $E$  of economies that includes  $E^{cq}$ , we define an optimality correspondence

$\mathcal{P} : E \rightarrow Z$  such that the restriction  $\mathcal{P}|_{E^{cq}}$  associates with  $e \in E^{cq}$  the set of  $\mathcal{P}(e)$  of all the Pareto efficient allocations that assign strictly positive consumption to every consumer.

The following lemma, which is based on the test family of Cobb–Douglas–Quadratic economies  $\bar{E}^{cq}$  specified in the above section, is central in finding the lower bound of informational requirements of resource allocation processes by the Fréchet ordering.

**Lemma 1** *Suppose  $\langle M, \mu, h \rangle$  is an allocation mechanism on the special class of economies  $\bar{E}^{cq}$  such that:*

- (i) *it is informationally decentralized;*
- (ii) *it is non-wasteful with respect to  $\mathcal{P}$ .*

*Then, the stationary message correspondence  $\mu$  is injective on  $\bar{E}^{cq}$ . That is, its inverse is a single-valued mapping on  $\mu(\bar{E}^{cq})$ .*

Proof. Suppose that there is a message  $m \in \mu(e) \cap \mu(\bar{e})$  for  $e, \bar{e} \in \bar{E}^{cq}$ . It will be proved that  $e = \bar{e}$ . Since  $\mu$  is a privacy-preserving correspondence,

$$\mu(e) \cap \mu(\bar{e}) = \mu(\bar{e}_i, e_{-i}) \cap \mu(e_i, \bar{e}_{-i}) \quad (10)$$

for all  $i = 1, \dots, N$  by Remark 1, and hence, in particular,

$$m \in \mu(e) \cap \mu(\bar{e}_i, e_{-i}) \quad (11)$$

for all  $i = 1, \dots, N$ . Let  $z = (x, y) = h(m)$ . Since the process  $\langle M, \mu, h \rangle$  is non-wasteful with respect to  $\mathcal{P}$ ,  $z = h(m)$  and (11) imply that  $z \in \mathcal{P}(e) \cap \mathcal{P}(\bar{e}_i, e_{-i})$ . Since Cobb–Douglas utility functions  $u_i(x)$  are strictly quasi-concave and production functions defined by efficient points of production sets,  $-y_j^{\rho^1} = \frac{1}{b_j^1} \sum_{l=2}^L (y_j^{\rho^l} + \frac{b_j^l}{2} (y_j^{\rho^l})^2) + \frac{1}{b_j^1} \sum_{k=1}^K (y_j^{\sigma^k} + \frac{d_j^k}{2} (y_j^{\sigma^k})^2)$ , are strictly convex, by the usual Lagrangian method of constrained maximization,  $z \in \mathcal{P}(e)$  implies

$$\frac{a_i^l (x_i^{\rho^1} + \bar{w}_i^1)}{(x_i^{\rho^l} + \bar{w}_i^l)} = \frac{1 + b_j^l y_j^{\rho^l}}{b_j^1} \quad l = 2, \dots, L, i = 1, \dots, I, j = I + 1, \dots, N, \quad (12)$$

$$\sum_{i=1}^I \frac{c_i^k (x_i^{\rho^1} + \bar{w}_i^1)}{x_i^{\sigma^k}} = \frac{1 + d_j^k y_j^{\sigma^k}}{b_j^1} \quad k = 1, \dots, K, j = I + 1, \dots, N, \quad (13)$$

and

$$b_j^1 y_j^1 = - \sum_{l=2}^L [y_j^{\rho^l} + \frac{b_j^l}{2} (y_j^{\rho^l})^2] - \sum_{k=1}^K [y_j^{\sigma^k} + \frac{d_j^k}{2} (y_j^{\sigma^k})^2] \quad j = I + 1, \dots, N. \quad (14)$$

(12) and (13) are well-known conditions for Pareto efficiency for economies with public goods. At Pareto optimality, (12) means the marginal rate of substitution between two private goods for each consumer  $i$  should be equal to the marginal rate of technical substitution between the two goods for all producers  $j$ , and (13) means the sum of marginal rate of substitutions between

a public good and a private good for all consumers should be equal to the marginal rate of technical substitution between these two goods for all producers  $j$ .

Similarly,  $z \in \mathcal{P}(\bar{e}_i, e_{-i})$  implies

$$\frac{\bar{a}_i^l(x_i^{\rho 1} + \bar{w}_i^1)}{(x_i^{\rho l} + \bar{w}_i^l)} = \frac{1 + b_j^l y_j^{\rho l}}{b_j^1} \quad l = 2, \dots, L, i = 1, \dots, I, j = I + 1, \dots, N, \quad (15)$$

$$\frac{\bar{c}_i^k(x_i^{\rho 1} + \bar{w}_i^1)}{x_i^{\sigma k}} + \sum_{s \neq i}^I \frac{c_s^k(x_s^{\rho 1} + \bar{w}_s^1)}{x_s^{\sigma k}} = \frac{1 + d_j^k y_j^{\sigma k}}{b_j^1} \quad k = 1, \dots, K, j = I + 1, \dots, N. \quad (16)$$

From equations (12) and (15), we have

$$a_i^l = \bar{a}_i^l \quad l = 2, \dots, L, i = 1, \dots, I, \quad (17)$$

and from equations (13) and (16), we have

$$\frac{c_i^k(x_i^{\rho 1} + \bar{w}_i^1)}{x_i^{\sigma k}} = \frac{\bar{c}_i^k(x_i^{\rho 1} + \bar{w}_i^1)}{x_i^{\sigma k}} \quad (18)$$

and thus we have

$$c_i^k = \bar{c}_i^k \quad k = 1, \dots, K, i = 1, \dots, I, \quad (19)$$

i.e.,  $a = \bar{a}$  and  $c = \bar{c}$ .

As for producers,  $z \in \mathcal{P}(\bar{e}_j, e_{-j})$  implies

$$\frac{a_i^l(x_i^{\rho 1} + \bar{w}_i^1)}{(x_i^{\rho l} + \bar{w}_i^l)} = \frac{1 + \bar{b}_j^l y_j^{\rho l}}{\bar{b}_j^1} \quad l = 2, \dots, L, i = 1, \dots, I, j = I + 1, \dots, N, \quad (20)$$

$$\sum_{i=1}^I \frac{c_i^k(x_i^{\rho 1} + \bar{w}_i^1)}{x_i^{\sigma k}} = \frac{1 + \bar{d}_j^k y_j^{\sigma k}}{\bar{b}_j^1} \quad k = 1, \dots, K, j = I + 1, \dots, N, \quad (21)$$

and

$$\bar{b}_j^1 y_j^1 = - \sum_{l=2}^L [y_j^{\rho l} + \frac{\bar{b}_j^l}{2} (y_j^{\rho l})^2] - \sum_{k=1}^K [y_j^{\sigma k} + \frac{\bar{d}_j^k}{2} (y_j^{\sigma k})^2] \quad j = I + 1, \dots, N. \quad (22)$$

From equations (12) and (20), we derive

$$\frac{b_j^1}{\bar{b}_j^1} = \frac{1 + b_j^l y_j^{\rho l}}{1 + \bar{b}_j^l y_j^{\rho l}} \quad l = 2, \dots, L, j = I + 1, \dots, N. \quad (23)$$

Also, from equations (13) and (21), we derive

$$\frac{b_j^1}{\bar{b}_j^1} = \frac{1 + d_j^k y_j^{\sigma k}}{1 + \bar{d}_j^k y_j^{\sigma k}} \quad k = 1, \dots, K, j = I + 1, \dots, N. \quad (24)$$

From equations (14) and (22), we derive

$$\frac{b_j^1}{\bar{b}_j^1} = \frac{\sum_{l=2}^L (1 + \frac{b_j^l}{2} y_j^{\rho l}) y_j^{\rho l} + \sum_{k=1}^K (1 + \frac{d_j^k}{2} y_j^{\sigma k}) y_j^{\sigma k}}{\sum_{l=2}^L (1 + \frac{\bar{b}_j^l}{2} y_j^{\rho l}) y_j^{\rho l} + \sum_{k=1}^K (1 + \frac{\bar{d}_j^k}{2} y_j^{\sigma k}) y_j^{\sigma k}} \quad j = I + 1, \dots, N. \quad (25)$$

Thus, from equations (23) and (25), we have

$$\frac{1 + b_j^l y_j^{\rho l}}{\sum_{l=2}^L (1 + \frac{b_j^l}{2} y_j^{\rho l}) y_j^{\rho l} + \sum_{k=1}^K (1 + \frac{d_j^k}{2} y_j^{\sigma k}) y_j^{\sigma k}} = \frac{1 + \bar{b}_j^l y_j^{\rho l}}{\sum_{l=2}^L (1 + \frac{\bar{b}_j^l}{2} y_j^{\rho l}) y_j^{\rho l} + \sum_{k=1}^K (1 + \frac{\bar{d}_j^k}{2} y_j^{\sigma k}) y_j^{\sigma k}} \quad (26)$$

for  $l = 2, \dots, L$ ,  $j = I + 1, \dots, N$ , and from equations (24) and (25), we have

$$\frac{1 + d_j^k y_j^{\sigma k}}{\sum_{l=2}^L (1 + \frac{b_j^l}{2} y_j^{\rho l}) y_j^{\rho l} + \sum_{k=1}^K (1 + \frac{d_j^k}{2} y_j^{\sigma k}) y_j^{\sigma k}} = \frac{1 + \bar{d}_j^k y_j^{\sigma k}}{\sum_{l=2}^L (1 + \frac{\bar{b}_j^l}{2} y_j^{\rho l}) y_j^{\rho l} + \sum_{k=1}^K (1 + \frac{\bar{d}_j^k}{2} y_j^{\sigma k}) y_j^{\sigma k}} \quad (27)$$

for  $l = 2, \dots, L$ ,  $j = I + 1, \dots, N$ .

Multiplying  $y_j^{\rho l}$  on the both sides of equation (26) and  $y_j^{\sigma k}$  on the both sides of equation (27), and then making summations over these two equations, we have

$$\frac{\sum_{l=2}^L (1 + b_j^l y_j^{\rho l}) y_j^{\rho l} + \sum_{k=1}^K (1 + d_j^k y_j^{\sigma k}) y_j^{\sigma k}}{\sum_{l=2}^L (1 + \frac{b_j^l}{2} y_j^{\rho l}) y_j^{\rho l} + \sum_{k=1}^K (1 + \frac{d_j^k}{2} y_j^{\sigma k}) y_j^{\sigma k}} = \frac{\sum_{l=2}^L (1 + \bar{b}_j^l y_j^{\rho l}) y_j^{\rho l} + \sum_{k=1}^K (1 + \bar{d}_j^k y_j^{\sigma k}) y_j^{\sigma k}}{\sum_{l=2}^L (1 + \frac{\bar{b}_j^l}{2} y_j^{\rho l}) y_j^{\rho l} + \sum_{k=1}^K (1 + \frac{\bar{d}_j^k}{2} y_j^{\sigma k}) y_j^{\sigma k}} \quad (28)$$

for  $j = I + 1, \dots, N$ . Simplifying equation (28), we have

$$\sum_{l=2}^L b_j^l (y_j^{\rho l})^2 + \sum_{k=1}^K d_j^k (y_j^{\sigma k})^2 = \sum_{l=2}^L \bar{b}_j^l (y_j^{\rho l})^2 + \sum_{k=1}^K \bar{d}_j^k (y_j^{\sigma k})^2 \quad j = I + 1, \dots, N. \quad (29)$$

Multiplying  $1/2$  and adding  $\sum_{l=2}^L y_j^{\rho l} + \sum_{k=1}^K y_j^{\sigma k}$  on the both sides of equation (29), and then applying equations (14) and (22), we have

$$b_j^1 y_j^{\rho 1} = \bar{b}_j^1 y_j^{\rho 1} \quad j = I + 1, \dots, N, \quad (30)$$

which implies

$$b_j^1 = \bar{b}_j^1 \quad j = I + 1, \dots, N. \quad (31)$$

Finally, from equations (23), (24) and (31), we have

$$b_j^l = \bar{b}_j^l \quad l = 2, \dots, L, j = I + 1, \dots, N. \quad (32)$$

and

$$d_j^k = \bar{d}_j^k \quad k = 1, \dots, K, j = I + 1, \dots, N. \quad (33)$$

Thus, we have proved

$$b_j = \bar{b}_j \quad j = I + 1, \dots, N, \quad (34)$$

and

$$d_j = \bar{d}_j \quad j = I + 1, \dots, N, \quad (35)$$

which means  $b = \bar{b}$  and  $d = \bar{d}$ . Thus, equations (17), (19), (34), and (35) mean that  $e = \bar{e}$ . Consequently, the inverse of the stationary message correspondence,  $(\mu)^{-1}$  is a single-valued mapping from  $\mu(\bar{E}^{cq})$  to  $\bar{E}^{cq}$ . Q.E.D.

The following theorem establishes a lower bound of the Fréchet ordering informational size of messages spaces of any smooth allocation mechanism that is informationally decentralized and non-wasteful over the class of economies  $E$ .

**Theorem 1 (Informational Boundedness Theorem)** *Suppose that  $\langle M, \mu, h \rangle$  is an allocation mechanism on the class of public goods economies  $E$  such that:*

- (i) *it is informationally decentralized;*
- (ii) *it is non-wasteful with respect to  $\mathcal{P}$ ;*
- (iii)  *$M$  is a Hausdorff topological space;*
- (iv)  *$\mu$  is locally threaded at some point  $e \in \bar{E}^{cq}$ .*

*Then, the size of the message space  $M$  is at least as large as  $\mathbb{R}^{(L+K-1)I+(L+K)J}$ , that is,  $M \geq_F \mathbb{R}^{(L+K-1)I+(L+K)J}$ .*

*Proof.* As was noted above,  $\bar{E}^{cq}$  is homeomorphic to  $\mathbb{R}^{(L+K-1)I+(L+K)J}$ . Hence, it suffices to show  $M \geq_F \bar{E}^{cq}$ .

By the injectiveness of Lemma 1, we know that the restriction  $\mu|_{\bar{E}^{cq}}$  of the stationary message correspondence  $\mu$  to  $\bar{E}^{cq}$  is an injective correspondence. Since  $\mu$  is locally threaded at  $e \in \bar{E}^{cq}$ , there exists a neighborhood  $N(e)$  of  $e$  and a continuous function  $f : N(e) \rightarrow M$  such that  $f(e') \in \mu(e')$  for all  $e' \in N(e)$ . Then  $f$  is a continuous injection from  $N(e)$  into  $M$ . Since  $\mu$  is an injective correspondence from  $\bar{E}^{cq}$  into  $M$ , thus  $f$  is a continuous one-to-one function on  $N(e)$ .

Since  $\bar{E}^{cq}$  is homeomorphic to  $\mathbb{R}^{(L+K-1)I+(L+K)J}$ , there exists a compact set  $\bar{N}(e) \subset N(e)$  with nonempty interior point. Also, since  $f$  is a continuous one-to-one function on  $N(e)$ ,  $f$  is a continuous one-to-one function from the compact space  $\bar{N}(e)$  onto a Hausdorff topological space  $f(\bar{N}(e))$ . Hence, it follows that the restriction  $f|_{\bar{N}(e)}$  is a homeomorphic imbedding on  $\bar{N}(e)$  by Theorem 5.8 in Kelley (1955, p. 141). Choose an open ball  $\dot{N}(e) \subset \bar{N}(e)$ . Then  $\dot{N}(e)$  and  $f(\dot{N}(e))$  are homeomorphic by a homeomorphism  $f|_{\dot{N}(e)} : \dot{N}(e) \rightarrow f(\dot{N}(e))$ . This, together with the fact that  $\bar{E}^{cq}$  is homeomorphic to its open ball  $\dot{N}(e)$ , implies that  $\bar{E}^{cq}$  is homeomorphic to  $f(\dot{N}(e)) \subset M$ , implying that  $\mu_L(E_L) =_F \bar{E}^{cd}$  can be homeomorphically imbedded in  $\mu(E_L)$ . Hence, it follows that  $M \geq_F \bar{E}^{cd} =_F \mathbb{R}^{(L+K-1)I+(L+K)J}$ . Q.E.D.

## 4 Informational Efficiency of Lindahl Mechanism

In the previous section, we found that the lower bound of the Fréchet informational size of message spaces for smooth allocation mechanisms that are privacy-preserving and non-wasteful over the class  $E$  of public goods economies that includes  $\bar{E}^{cq}$  is the  $(L + K - 1)I + (L + K)J$ -dimensional Euclidean space  $\mathbb{R}^{(L+K-1)I+(L+K)J}$ . In this section we assert that the lower bound is exactly the size of the message space of the Lindahl mechanism, and thus the Lindahl mechanism is informationally efficient among all smooth resource allocation mechanisms that are informationally decentralized and non-wasteful over the set  $E^L$  of production economies on which  $L(e) \neq \emptyset$  for all  $e \in E^L$ .

From Remark 2, we know that the upper bound dimension of the message space of the Lindahl mechanism is also  $(L + K - 1)I + (L + K)J$ . As a result, if we can show that this

upper bound can be reached on the restriction of the message space of the Lindahl mechanism to the test family  $\bar{E}^{cq}$  of Cobb-Douglas–Quadratic economies, i.e., if we can show that  $\mu_L|_{\bar{E}^{cq}}$  is homeomorphic to the  $(L+K-1)I+(L+K)J$ -dimensional Euclidean space  $\mathbb{R}^{(L+K-1)I+(L+K)J}$ , then we know that the Fréchet informational size of the message space of the Lindahl mechanism is  $\mathbb{R}^{(L+K-1)I+(L+K)J}$  and thus the Lindahl mechanism is informationally efficient among all resource allocation mechanisms that are informationally decentralized and non-wasteful over the class of economies in which Lindahl equilibria exist. Hence, to show the informational efficiency of the Lindahl mechanism, it suffices for us to show that this upper bound can be actually reached on the test family of economies for the Lindahl mechanism.

We will first state the following lemmas that shows that the Lindahl mechanism is single-valued and continuous so that it is locally threaded on the test family set  $\bar{E}^{cq}$  of Cobb-Douglas–Quadratic economies.

**Lemma 2** *For any given private ownership structure matrix  $\theta$ , every economy in  $\bar{E}^{cq}$  has a unique  $\theta$ -Lindahl equilibrium, i.e.,  $L_\theta(e)$  is a single-valued mapping from  $\bar{E}^{cq}$  to  $Z$ .*

*Proof.* To show the existence and uniqueness of  $\theta$ -Lindahl equilibrium, we want to first derive the supply and demand functions of agents.

Produce  $i$  ( $i = I + 1, \dots, N$ ) chooses his production plan so as to maximize profit within  $\mathcal{Y}(b_i, d_i)$ . Thus, he solves the following profit maximizing problem:

$$\max[p \cdot y_i^\rho + \hat{q} \cdot y_i^\sigma]$$

subject to

$$b_i^1 y_i^{\rho 1} + \sum_{l=2}^L [y_i^{\rho l} + \frac{b_i^l}{2} (y_i^{\rho l})^2] + \sum_{k=1}^K [y_i^{\sigma k} + \frac{d_i^k}{2} (y_i^{\sigma k})^2] = 0, \quad (36)$$

$$0 \leq y_i^{\rho l} \leq \frac{1}{b_i^l} \text{ for all } l = 2, \dots, L,$$

and

$$0 \leq y_i^{\sigma k} \leq \frac{1}{d_i^k} \text{ for all } k = 1, \dots, K.$$

An interior solution  $y$  must satisfy the following first-order conditions:

$$p^1 = \lambda_i b_i^1 \quad (37)$$

$$p^l = \lambda_i (1 + b_i^l y_i^l), \quad l = 2, \dots, L, \quad (38)$$

where  $\lambda_i$  is a Lagrange multiplier. From (37) and (38), and the restrictions  $0 \leq y_i^{\rho l} \leq \frac{1}{b_i^l}$  for  $l = 2, \dots, L$  and  $0 \leq y_i^{\sigma k} \leq \frac{1}{d_i^k}$  for  $k = 1, \dots, K$ , we can obtain the supply functions

$$y_i^{\rho l}(p, q) = \begin{cases} \frac{1}{b_i^l} & \text{if } \frac{p^l}{p^1} \geq \frac{2}{b_i^l} \\ \frac{b_i^l p^l}{b_i^l p^1} - \frac{1}{b_i^l} & \text{if } \frac{1}{b_i^l} < \frac{p^l}{p^1} < \frac{2}{b_i^l} \\ 0 & \text{if } \frac{p^l}{p^1} \leq \frac{1}{b_i^l} \end{cases}$$

for  $l = 2, \dots, L$ ,

$$y_i^{\sigma k}(p, q) = \begin{cases} \frac{1}{d_i^k} & \text{if } \frac{q_i^k}{p^1} \geq \frac{2}{b_i^1} \\ \frac{b_i^1 q_i^k}{d_i^k p^1} - \frac{1}{d_i^k} & \text{if } \frac{1}{b_i^1} < \frac{q_i^k}{p^1} < \frac{2}{b_i^1} \\ 0 & \text{if } \frac{q_i^k}{p^1} \leq \frac{1}{b_i^1} \end{cases}$$

for  $k = 1, \dots, K$  and  $i = 1, \dots, I$ , and thus, by (36),

$$y_i^{\rho 1}(p, q) = -\frac{1}{b_i^1} \sum_{l=2}^L [y_i^{\rho l}(p, q) + \frac{b_i^l}{2} (y_i^{\rho l}(p, q))^2] - \frac{1}{b_i^1} \sum_{k=1}^K [y_i^{\sigma k}(p, q) + \frac{d_i^k}{2} (y_i^{\sigma k}(p, q))^2]. \quad (39)$$

It may be remarked that  $-3(L+K-1)/2 \leq y_i^1(p, q) \leq 0$  for all  $(p, q) \in \mathbb{R}_{++}^{L+K}$ . Indeed, for  $l = 2, \dots, L$ , if  $\frac{1}{b_i^1} < \frac{p^l}{p^1} < \frac{2}{b_i^1}$ , then  $y_i^{\rho l}(p, q) = \frac{b_i^1 p^l}{b_i^l p^1} - \frac{1}{b_i^1} < \frac{1}{b_i^1}$  which means  $\frac{b_i^1 p^l}{p^1} < 2$ , and thus

$$\begin{aligned} y_i^{\rho l}(p, q) + \frac{b_i^l}{2} (y_i^{\rho l}(p, q))^2 &= \left[ 1 + \frac{b_i^l}{2} y_i^{\rho l}(p, q) \right] y_i^{\rho l}(p, q) \\ &= \left[ 1 + \frac{b_i^l}{2} \left( \frac{b_i^1 p^l}{b_i^l p^1} - \frac{1}{b_i^1} \right) \right] \left[ \frac{b_i^1 p^l}{b_i^l p^1} - \frac{1}{b_i^1} \right] \\ &= \frac{1}{2b_i^l} \left( 1 + \frac{b_i^1 p^l}{p^1} \right) \left( \frac{b_i^1 p^l}{p^1} - 1 \right) \\ &= \frac{1}{2b_i^l} \left[ \left( \frac{b_i^1 p^l}{p^1} \right)^2 - 1 \right] < 3/2 \end{aligned} \quad (40)$$

by noting that  $b_i^l > 1$  and  $\frac{b_i^1 p^l}{p^1} < 2$ . If  $\frac{p^l}{p^1} \geq \frac{2}{b_i^1}$ , then  $y_i^{\rho l}(p, q) = \frac{1}{b_i^1}$  and thus  $y_i^{\rho l}(p, q) + \frac{b_i^l}{2} (y_i^{\rho l}(p, q))^2 = \frac{1}{b_i^1} + \frac{1}{2b_i^l} < 3/2$ . If  $\frac{p^l}{p^1} \leq \frac{1}{b_i^1}$ , then  $y_i^{\rho l}(p, q) = 0$  by (39). Thus, for each  $l = 2, \dots, L$ ,  $0 \leq y_i^{\rho l}(p, q) + \frac{b_i^l}{2} (y_i^{\rho l}(p, q))^2 < 3/2$  for all  $(p, q) \in \mathbb{R}_{++}^{L+K}$ . Similarly, we can show  $0 \leq y_i^{\sigma k}(p, q) < 3/2$  for  $k = 1, \dots, K$  and for all  $(p, q) \in \mathbb{R}_{++}^{L+K}$ . Therefore by (39), we have  $-3(L+K-1)/2 < y_i^{\rho 1}(p, q) \leq 0$  for all  $(p, q) \in \mathbb{R}_{++}^{L+K}$ .

Consumer  $i$  (for  $i = 1, \dots, I$ ) chooses his consumption so as to maximize his utility subject to his budget constraint. Since all utility functions are Cobb-Douglas, the net demand functions for private goods are given by

$$x_i^{\rho 1}(p, q) = \frac{1}{p^1 (1 + \sum_{l=2}^L a_i^l + \sum_{k=1}^L c_i^k)} \left( p \cdot \bar{w}_i + \sum_{j=I+1}^N \theta_{ij} [p \cdot y_j^{\rho}(p, q) + \hat{q} \cdot y_j^{\sigma}(p, q)] \right) - \bar{w}_i^1, \quad (41)$$

and

$$x_i^{\rho l}(p, q) = \frac{a_i^l}{p_i^l (1 + \sum_{l=2}^L a_i^l + \sum_{k=1}^L c_i^k)} \left( p \cdot \bar{w}_i + \sum_{j=I+1}^N \theta_{ij} [p \cdot y_j^{\rho}(p, q) + \hat{q} \cdot y_j^{\sigma}(p, q)] \right) - \bar{w}_i^l \quad (42)$$

for  $l = 2, \dots, L$ . The demand functions for public goods are given by

$$x_i^{\sigma k}(p, q) = \frac{c_i^k}{q_i^k (1 + \sum_{l=2}^L a_i^l + \sum_{k=1}^L c_i^k)} \left( p \cdot \bar{w}_i + \sum_{j=I+1}^N \theta_{ij} [p \cdot y_j^{\rho}(p, q) + \hat{q} \cdot y_j^{\sigma}(p, q)] \right). \quad (43)$$

Define the aggregate net excess demand function for private goods by

$$\hat{z}^\rho(p, q) = \sum_{i=1}^I x_i^\rho(p, q) - \sum_{i=I+1}^N y_i^\rho(p, q). \quad (44)$$

Define the  $i$ -the consumer's excess demand function for public goods by

$$\hat{z}_i^\sigma(p, q) = x_i^\sigma(p, q) - \sum_{i=I+1}^N y_i^\sigma(p, q) \quad (45)$$

for  $i = 1, \dots, I$ .

To show the existence and uniqueness of Lindahl equilibrium, we define a transformed economy  $e'$  with only private commodities such that there is one-to-one correspondence between the transformed private goods economy  $e'$  and the original public goods economy  $e$ . This approach is standard and has been adopted by Foley (1970) and Milleron (1972) to show the existence of Lindahl equilibrium. Extend the commodity space by considering each consumer's bundle of public goods as a separate group of commodities. In this  $KI + L$  space, extend the sets  $X_i$  by writing zeros for all public good components not corresponding to the  $i$ -th consumer. The transformed production sets are defined by  $\mathcal{Y}'(b_j, d_j) = \{(y_j^\rho, y_{j1}^\sigma), \dots, (y_j^\rho, y_{jI}^\sigma) : y_{j1}^\sigma = \dots = y_{jI}^\sigma = y_j^\sigma \text{ \& } (y_j^\rho, y_j^\sigma) \in \mathcal{Y}(b_j, d_j)\}$  for  $j = I + 1, \dots, I + J$ .

Notice that, since every consumer's budget constraint holds with equality, and the demand and supply functions are clearly continuous, the aggregate excess demand function  $\hat{z}(p, q) = (\hat{z}^\rho, \hat{z}_1^\sigma, \dots, \hat{z}_I^\sigma)$  is continuous and satisfies Walras's Law, i.e.,  $p \cdot \hat{z}^\rho(p, q) + q \cdot \hat{z}^\sigma(p, q) = 0$  for all  $p \in \mathbb{R}_{++}^L$ . Thus, by the existence theorem on Walrasian equilibrium (cf. Varian (1992)), there exists some  $(p, q) \in \mathbb{R}_{++}^L$  such that  $\hat{z}(p, q) \leq 0$ , which means  $(p, q, \hat{z}(p, q))$  is a  $\theta$ -Walrasian equilibrium for the transformed economy  $e'$  and hence  $(p, q, \hat{z}(p, q))$  is a  $\theta$ -Lindahl equilibrium for the original public goods economy  $e \in \bar{E}^{cq}$ .<sup>4</sup>

Now we show that every economy  $e \in \bar{E}^{cq}$  has a unique  $\theta$ -Lindahl equilibrium, or, it is equivalent to show that the corresponding transformed economy  $e'$  has a unique  $\theta$ -Walrasian equilibrium. For this, it suffices to show that all goods for the economy  $e'$  are gross substitutes at any price  $(p, q) \in \mathbb{R}_{++}^{L+IK}$ , i.e., an increase in price,  $s$ , brings about an increase in the excess demand for good  $t$ . When  $\hat{z}$  is differentiable, the gross substitutes condition becomes  $\frac{\partial \hat{z}^t(p, q)}{\partial p^s} > 0$  for  $t \neq s$ .

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<sup>4</sup>Another way to show the existence of a  $\theta$ -Lindahl equilibrium is to apply the existence theorem of Milleron (1972) directly by noting that  $\mathcal{Y}_i$  is closed and convex,  $0 \in \mathcal{Y}_i$ ,  $(-\mathbb{R}_+^L) \in \mathcal{Y}_i$  and  $\mathcal{Y}_i \cap (-\mathcal{Y}_i) \subset \{0\}$ .

For each  $j = I + 1, \dots, N$ , from (39), if  $\frac{1}{b_i^l} < \frac{p^l}{p^1} < \frac{2}{b_i^l}$ , we have

$$\frac{\partial y_j^{\rho l}(p, q)}{\partial p^l} = \frac{b_j^1}{b_j^l p^1} > 0 \quad l = 2, \dots, L, \quad (46)$$

$$\frac{\partial y_j^{\rho l}(p, q)}{\partial p^s} = 0 \quad s \neq l, l \neq 1, \quad (47)$$

$$\frac{\partial y_j^{\rho l}(p, q)}{\partial p^1} = -\frac{b_j^1 p^l}{b_j^l (p^1)^2} < 0 \quad l = 2, \dots, L, \quad (48)$$

$$\frac{\partial y_j^{\rho l}(p, q)}{\partial q_i^k} = 0 \quad i = 1, \dots, I, k = 1, \dots, K, l = 2, \dots, L. \quad (49)$$

From (39), if  $\frac{1}{b_i^l} < \frac{q_i^l}{p^1} < \frac{2}{b_i^l}$ , we have

$$\frac{\partial y_j^{\sigma k}(p, q)}{\partial q_i^k} = \frac{b_j^1}{d_j^k p^1} > 0 \quad i = 1, \dots, I, k = 1, \dots, K, \quad (50)$$

$$\frac{\partial y_j^{\sigma k}(p, q)}{\partial q_i^t} = 0 \quad t \neq k, k = 1, \dots, K, i = 1, \dots, I, \quad (51)$$

$$\frac{\partial y_j^{\sigma k}(p, q)}{\partial p^1} = -\frac{b_j^1 q_i^k}{d_j^k (p^1)^2} < 0 \quad k = 1, \dots, K, i = 1, \dots, I, \quad (52)$$

$$\frac{\partial y_j^{\sigma k}(p, q)}{\partial p^s} = 0 \quad s = 1, \dots, L, k = 1, \dots, K. \quad (53)$$

As a result, from (39), (48), and (52), we have

$$\frac{\partial y_j^{\rho 1}(p, q)}{\partial p^1} = -\frac{1}{b_j^1} \sum_{l=2}^L [1 + b_j^l y_j^{\rho l}] \frac{\partial y_j^{\rho l}}{\partial p^1} - \frac{1}{b_j^1} \sum_{k=1}^K [1 + d_j^k y_j^{\sigma k}] \frac{\partial y_j^{\sigma k}}{\partial p^1} > 0, \quad (54)$$

$$\frac{\partial y_j^{\rho 1}(p, q)}{\partial p^l} = -\frac{1}{b_j^1} [1 + b_j^l y_j^{\rho l}] \frac{\partial y_j^{\rho l}}{\partial p^l} < 0, \quad l = 2, \dots, L, \quad (55)$$

$$\frac{\partial y_j^{\rho 1}(p, q)}{\partial q_i^k} = -\frac{1}{b_j^1} [1 + d_j^k y_j^{\sigma k}] \frac{\partial y_j^{\sigma k}}{\partial q_i^k} < 0 \quad k = 1, \dots, K, i = 1, \dots, I. \quad (56)$$

When  $\frac{p^l}{p^1} \leq \frac{1}{b_j^1}$  (resp.  $\frac{q_i^k}{p^1} \leq \frac{1}{b_j^1}$ ) or  $\frac{p^l}{p^1} \geq \frac{2}{b_j^1}$  (resp.  $\frac{q_i^k}{p^1} \geq \frac{2}{b_j^1}$ ),  $y_j^{\rho l}(p, q)$  (resp.  $y_j^{\sigma k}(p, q)$ ) are constant functions for  $l = 2, \dots, L$  (resp. for  $k = 1, \dots, K$  and  $i = 1, \dots, I$ ). Thus,  $y_j^{\rho l}(p, q)$  is a nonincreasing function in  $p^s$  and  $q_i^t$  for any  $l \neq s, t = 1, \dots, K$ , and any  $(p, q) \in \mathbb{R}_{++}^{L+IK}$ , and  $y_j^{\sigma k}(p, q)$  is a nonincreasing function in  $q_i^t$  and  $p^l$  for any  $k \neq t, l = 1, \dots, L$ , and any  $(p, q) \in \mathbb{R}_{++}^{L+IK}$ .

Note that, by Hotelling's Lemma (cf. Varian (1992, p. 43),  $\frac{\partial [p \cdot y_j^{\rho}(p, q) + \hat{q} \cdot y_j^{\sigma}(p, q)]}{\partial p^s} = y_j^{\rho s}(p, q)$ , and  $\frac{\partial [p \cdot y_j^{\rho}(p, q) + \hat{q} \cdot y_j^{\sigma}(p, q)]}{\partial q_i^t} = y_j^{\sigma t}(p, q)$ . Also, note that  $y_j^{\rho l} \geq 0$  for  $l = 2, \dots, L$ ,  $y_j^{\sigma k} \geq 0$  for  $k = 1, \dots, K$ , and  $-3(L + K - 1)/2 < y_j^{\rho 1} \leq 0$  for all  $i = 1, \dots, I$ . Then, for each  $i = 1, \dots, I$ , from (41), we have

$$\frac{\partial x_i^{\rho l}(p, q)}{\partial p^s} = \frac{a_i^l}{p_i^l (1 + \sum_{l=2}^L a_i^l + \sum_{k=1}^L c_i^k)} \left[ \bar{w}_i^s + \sum_{j=I+1}^N \theta_{ij} y_j^{\rho s}(p, q) \right] > 0 \quad (57)$$

for  $l \neq s, s \neq 1$  by noting that  $y_j^{\rho s}(p, q) \geq 0$  for all  $s = 2, \dots, L$ ,

$$\frac{\partial x_i^{\rho l}(p, q)}{\partial q_i^t} = \frac{a_i^l}{p_i^l(1 + \sum_{l=2}^L a_i^l + \sum_{k=1}^L c_i^k)} \left[ \bar{w}_i^s + \sum_{j=I+1}^N \theta_{ij} y_j^{\sigma t}(p, q) \right] > 0 \quad (58)$$

for  $t = 1, \dots, K, i = 1, \dots, I$  by noting that  $y_j^{\sigma k}(p, q) \geq 0$  for all  $k = 1, \dots, K$ , and

$$\begin{aligned} \frac{\partial x_i^{\rho l}(p, q)}{\partial p^1} &= \frac{a_i^1}{p^l(1 + \sum_{l=2}^L a_i^l + \sum_{k=1}^L c_i^k)} \left[ \bar{w}_i^1 + \sum_{j=I+1}^N \theta_{ij} y_j^{\rho 1}(p, q) \right] \\ &> \frac{a_i^l}{p^l(1 + \sum_{l=2}^L a_i^l + \sum_{k=1}^L c_i^k)} \left[ \bar{w}_i^1 + \sum_{j=I+1}^N \theta_{ij} (-3(L + K - 1)/2) \right] \\ &> \frac{a_i^l}{p^l(1 + \sum_{l=2}^L a_i^l + \sum_{k=1}^L c_i^k)} [2(L + K - 1)J - 3(L + K - 1)J/2] > 0 \end{aligned} \quad (59)$$

for  $l \neq 1$ . Similarly, we can show that  $\frac{\partial x_i^{\sigma k}(p, q)}{\partial q_i^t} > 0$  for all  $v = 1, \dots, I, t \neq k$ , and  $k = 1, \dots, K$ , and  $\frac{\partial x_i^{\sigma k}(p, q)}{\partial p^s} > 0$  for all  $s = 1, \dots, L$  and  $k = 1, \dots, K$ . Thus, the net demand function  $x_i^s(p, q)$  is an increasing function and the supply function  $y_i^s(p, q)$  is nonincreasing function in price  $t \neq s$  and for every  $(p, q) \in \mathbb{R}_{++}^{L+IK}$ . Therefore, an increase in price,  $t$ , brings about an increase in the excess demand for good  $s$ , and thus all goods are gross substitutes. Hence, the  $\theta$ -Walrasian equilibrium must be unique for the transformed economy  $e'$  (cf. Varian (1992)), and consequently  $\theta$ -Lindahl equilibrium is unique for the original economy  $e$ . Q.E.D.

**Lemma 3** *Let  $\mu_{cq}$  be the Lindahl equilibrium message correspondence on  $\bar{E}^{cq}$ . The  $\mu_{cq}$  is a continuous function.*

*Proof.* By Lemma 2, we know  $\mu_{cq} = (p, x, y)$  is a (single-valued) function. Also, from (39), (39), (39), (41), (42), and (43), we know that the demand function  $x(p, q; \xi)$  and supply function  $y(p, q; \xi)$  are continuous in  $(p, q)$  and  $\xi := (a, b, c, d)$ . So we only need to show the price system  $(p, q)$  is a continuous function on  $\bar{E}^{cq}$ . Since the demand function  $x(p, q; \xi)$  and supply function  $y(p, q; \xi)$  are homogenous of degree zero in  $(p, q)$ , we can normalize the price system as an element in the compact simplex set  $\bar{\Delta}^{L+IK-1} = \{(p, q) \in \mathbb{R}_+^{L+IK} : \sum_{l=1}^L p^l + \sum_{i=1}^I \sum_{k=1}^K q_i^k = 1\}$ .

Let  $\{e(k)\}$  be a sequence in  $\bar{E}^{cq}$  and  $e(k) \rightarrow e \in \bar{E}^{cq}$ . Since any economy in  $\bar{E}^{cq}$  is fully specified by the parameter vector  $\xi$ ,  $e(k) \rightarrow e$  implies  $\xi(k) \rightarrow \xi$ .

Let  $\mu_{cq} = (p, q, x(p, q; \xi), y(p, q; \xi))$  and  $\mu_{cq}(k) = (p(k), q(k), x(p(k), q(k); \xi(k)), y(p(k), q(k); \xi(k)))$ .

Then we have,  $\hat{z}(p, q; \xi) = 0$  and  $\hat{z}(p(k); \xi(k)) = 0$ , e.i.,

$$\sum_{i=1}^I x_i^p(p, q; \xi) = \sum_{j=I+1}^N y_j^p(p, q; \xi), \quad (60)$$

$$x_i^\sigma(p, q; \xi) = \sum_{j=I+1}^N y_j^\sigma(p, q; \xi) \quad i = 1, \dots, I, \quad (61)$$

$$\sum_{i=1}^I x_i^p(p(k); \xi(k)) = \sum_{j=I+1}^N y_j^p(p(k); \xi(k)), \quad (62)$$

$$x_i^\sigma(p(k); \xi(k)) = \sum_{j=I+1}^N y_j^\sigma(p(k); \xi(k)) \quad i = 1, \dots, I. \quad (63)$$

Since the sequence  $\{p(k)\}$  is contained in the compact set  $\bar{\Delta}^{L+IK-1}$ , there exists a convergent subsequence  $\{p(k_t), q(k_t)\}$  which converges to, say,  $(\bar{p}, \bar{q}) \in \bar{\Delta}^{L+IK-1}$  and  $\hat{z}(p(k_t), q(k_t); \xi(k_t)) = 0$ . Since  $x_i(p(k), q(k); \xi(k))$  and  $y_j(p(k), q(k); \xi(k))$  are continuous in  $\xi$ ,  $\hat{z}(p, q; \xi)$  is continuous in  $\xi$  and thus we have  $\hat{z}(p(k_t), q(k_t); \xi(k_t)) \rightarrow \hat{z}(\bar{p}, \bar{q}, \xi)$  as  $k_t \rightarrow \infty$  and  $\xi(k_t) \rightarrow \xi$ . However, since every  $e \in \bar{E}^{cq}$  has the unique  $\theta$ -Lindahl equilibrium price system  $(p, q)$  which is completely determined by  $\hat{z}(p, q; \xi) = 0$ , so we must have  $\bar{p} = p$  and  $\bar{q} = q$ . Q.E.D.

**Lemma 4** *Let  $\mu_L$  be the Lindahl equilibrium message correspondence on  $E^L$ , where  $E^L \subset E$  so that  $L(e) \neq \emptyset$  for all  $e \in E^L$ . Then  $\mu_L(E^L)$  is homeomorphic to  $\bar{E}^{cq}$ .*

Proof. Let  $\mu_{cq}$  be the restriction of  $\mu_L$  to  $\bar{E}^{cq}$ . By Lemmas 2 and 3, we know that  $\mu_{cq}$  is a continuous function. We want to show that the inverse of  $\mu_{cq}$ ,  $(\mu_{cq})^{-1}$  is also a function.

Let  $m \in \mu_{cq}(\bar{E}^{cq})$  and let  $e, e' \in (\mu_{cq})^{-1}(m)$ . Then  $m \in \mu_{cq}(e) \cap \mu_{cq}(e') = \mu_c(e) \cap \mu_c(e') = \mu_c(e'_i, e_{-i}) \cap \mu_c(e_i, e'_{-i})$  for all  $i = 1, \dots, N$  by Remark 1. Let  $z = h_{cd} \in L(\bar{E}^{cq})$  be the Lindahl outcome function. Since  $u_i$  is monotonically increasing, we know  $z$  is Pareto efficient by the First Theorem of Welfare Economics. Then, the allocation process  $\langle M_L, \mu_{cq}, h_{cq} \rangle$  is privacy-preserving and non-wasteful over  $\bar{E}^{cq}$  with respect to  $\mathcal{P}$ . Then, by Lemma 1,  $e = e'$  and thus  $(\mu_{cq})^{-1}$  is a function. Therefore,  $\mu_{cq}$  is a continuous one-to-one function on  $\bar{E}^{cq}$ .

Since every  $e$  is fully characterized by  $(a, b, c, d) \in \mathbb{R}_{++}^{(L+K-1)I+(L+K)J}$ ,  $\bar{E}^{cq}$  is homeomorphic to the finite-dimensional Euclidean space  $\mathbb{R}^{(L+K-1)I+(L+K)J}$ . Thus, it must be homeomorphic to any open ball centered on any of its points, and also locally compact. It follows that for any  $e \in \bar{E}^{cq}$ , we can find a neighborhood  $N(e)$  of  $e$  and a compact set  $\bar{N}(e) \subset N(e)$  with a nonempty interior point. Since  $\mu_{cd}$  is a continuous one-to-one function on  $N(e)$ ,  $\mu_{cd}$  is a continuous one-to-one function from the compact space  $\bar{N}(e)$  onto an Euclidean (and hence Hausdorff topological) space  $\mu_{cd}(\bar{N}(e))$ . Hence, it follows that the restriction  $\mu_{cd}$  restricted to  $N(e)$  is a homeomorphic imbedding on  $\bar{N}(e)$  by Theorem 5.8 in Kelley (1955, p. 141). Choose an open ball  $\overset{\circ}{N}(e) \subset \bar{N}(e)$ . Then  $\overset{\circ}{N}(e)$  and  $\mu_{cd}(\overset{\circ}{N}(e))$  are homeomorphic by a homeomorphism  $\mu_{cd}|_{\overset{\circ}{N}(e)} : \overset{\circ}{N}(e) \rightarrow \mu_{cd}(\overset{\circ}{N}(e))$ . This, together with the fact that  $\bar{E}^{cq}$  is homeomorphic to its

open ball  $\dot{N}(e)$ , implies that  $\bar{E}^{cq}$  is homeomorphic to  $\mu_{cd}(\dot{N}(e)) \subset M_L$ , implying that  $\mu_{cd}(E^L)$  can be homeomorphically imbedded in  $\mu_L(E^L)$ .

Finally, by Remark 2, the Lindahl message space  $M_L$  is contained within an Euclidean space of dimension  $(L + K - 1)I + (L + K)J$ . This necessarily implies that  $M_L$  and thus  $\mu_L(\bar{E}^{cq})$  is homeomorphic to  $\mathbb{R}^{(L+K-1)I+(L+K)J}$  because this restriction  $\mu_{cd}(\bar{E}^{cq})$  is homeomorphic to  $\mathbb{R}^{(L+K-1)I+(L+K)J}$ , and consequently,  $\mu_L(E^L)$  is homeomorphic to  $\bar{E}^{cq}$ . Q.E.D.

From the above lemmas and Theorem 1, we have the following theorem that establish the informational efficiency of the Lindahl mechanism within the class of all smooth resource allocation mechanisms that are informationally decentralized and non-wasteful over the class of production economies  $E^L$  on which  $L(e) \neq \emptyset$  for all  $e \in E^L$ .

**Theorem 2 (Informational Efficiency Theorem)** *Suppose that  $\langle M, \mu, h \rangle$  is an allocation mechanism on the class  $E^L$  of public goods economies  $E^L$  such that:*

- (i) *it is informationally decentralized;*
- (ii) *it is non-wasteful with respect to  $\mathcal{P}$ ;*
- (iii)  *$M$  is a Hausdorff topological space;*
- (iv)  *$\mu$  is locally threaded at some point  $e \in \bar{E}^{cq}$ .*

*Then, the size of the message space  $M$  is at least as large as that of the Lindahl mechanism, that is,  $M \geq_F M_L =_F \mathbb{R}^{(L+K-1)I+(L+K)J}$ .*

*Proof.* Let  $\langle M_L, \mu_L, h_L \rangle$  be the Lindahl mechanism and let  $\bar{E}^{cq}$  be the special class of public goods economies defined in the previous section. Since  $L(e) \neq \emptyset$  for all  $e \in E^L$ , the Lindahl mechanism is well defined. Furthermore, since  $u_i$  is locally non-satiated on  $E$  by assumption, we know  $(x, y)$  is Pareto efficient by the First Theorem of Welfare Economics. Then, the Lindahl process  $\langle M_L, \mu_L, h_L \rangle$  is privacy-preserving and non-wasteful over  $E^L$ .

Also, as noted above,  $\bar{E}^{cq}$  is homeomorphic to  $\mathbb{R}^{(L+K-1)I+(L+K)J}$ . Also by Lemma 4,  $\mu_L(\bar{E}^{cq})$  and  $M_L$  are homeomorphic to  $\bar{E}^{cq}$ . Thus, by Theorem 1, we have  $M \geq_F M_L =_F \mathbb{R}^{(L+K-1)I+(L+K)J}$ . Q.E.D.

## 5 The Uniqueness Theorem

In this section we establish that the Lindahl allocation process is the only informationally efficient decentralized mechanism for public goods economies among smooth mechanisms that achieve Pareto optimal and individually rational allocations. We first show the following lemmas.

**Lemma 5** *Suppose that  $\langle M, \mu, h \rangle$  is an allocation mechanism on the class of public goods economies  $E^{cq}$  such that:*

- (i) it is informationally decentralized;
- (ii) it is non-wasteful with respect to  $\mathcal{P}$ ;
- (iii) it is individually rational with respect to the fixed share guarantee structure  $\gamma_i(e; \theta)$ .

Then, there is a function  $\phi : \mu(E^{cq}) \rightarrow \mathbb{R}_{++}^{L+IK} \times Z$  defined by  $\phi(m) = (p, q, x, y)$  where  $(x, y) = h(m)$  and  $(p, q)$  is an efficiency price system at the allocation  $(x, y)$ . In particular,  $(p, q_i)$  is proportional to  $Du_i(w_i + x_i^\rho, x_i^\sigma)$  for consumer  $i = 1, \dots, I$ , and  $(p, \hat{q})$  is proportional to  $DT(y_i)$  for producer  $i = I + 1, \dots, N$  for each  $e \in E^{cq}$  with  $m \in \mu(e)$ , where  $T(y_i) = b_i^1 y_i^1 + \sum_{l=2}^L (y_i^{\rho l} + \frac{b_i^l}{2} (y_i^{\rho l})^2) + \sum_{k=1}^K (y_i^{\sigma k} + \frac{d_i^k}{2} (y_i^{\sigma k})^2) = 0$ .

Proof. Let  $e \in E^{cq}$ , let  $m \in \mu(e)$ , and let  $(x, y) = h(m)$ . Since  $(x, y)$  is Pareto optimal, there exists an efficiency price system  $(p, q) \in \mathbb{R}_{++}^{L+IK}$  so that  $\phi(m) = (p, q, x, y)$  is an equilibrium relative to the price system  $(p, q)$ . Also, since  $(x, y)$  is individually rational with respect to the fixed share guarantee structure  $\gamma_i(e; \theta)$  and utility functions  $u_i(x)$  are Cobb-Douglas, we have  $(w_i + x_i^\rho, x_i^\sigma) \in \mathbb{R}_{++}^{L+K}$  by noting that  $[p \cdot y_j^\rho + \hat{q} \cdot y_j^\sigma] \geq 0$  for all  $j = I + 1, \dots, N$  and  $(\gamma_i(e; \theta) + w_i) \in \mathbb{R}_{++}^L$  for all  $i = 1, \dots, I$ . Also, since  $y_i^1 = -\frac{1}{b_i^1} \sum_{l=2}^L (y_i^{\rho l} + \frac{b_i^l}{2} (y_i^{\rho l})^2) - \frac{1}{b_i^1} \sum_{k=1}^K (y_i^{\sigma k} + \frac{d_i^k}{2} (y_i^{\sigma k})^2)$  is strictly concave,  $y_i$  is an interior point of  $\mathcal{Y}(c_i, d_i)$  for all  $i = I + 1, \dots, N$ . Therefore,  $(p, q_i)$  must be proportional to  $Du_i(w_i + x_i^\rho, x_i^\sigma)$  for  $i = 1, \dots, I$ , and  $(p, \hat{q})$  must be proportional to  $DT(y_i)$  for  $i = I + 1, \dots, N$ , where  $Du_i$  and  $DT$  denote the partial derivative vectors local tangent spaces. Let  $e' \in E^{cq}$  be any other environment with  $m \in \mu(e')$ . Since  $\langle M, \mu, h \rangle$  is privacy-preserving, by Remark 1, we have  $m \in \mu(e'_i, e_{-i})$  for each  $i = 1, \dots, N$ . Therefore,  $(x, y) \in h[\mu(e'_i, e_{-i})]$  for each  $i = 1, \dots, N$ , and thus  $(p, q_i)$  is proportional to  $Du'_i(w_i + x_i^\rho, x_i^\sigma)$  for  $i = 1, \dots, I$ , and  $(p, \hat{q})$  is proportional to  $DT'(y_i)$  for  $i = I + 1, \dots, N$ . Thus,  $\phi$  is well defined. Q.E.D.

Next we want to show that the equilibrium message cannot reveal any more than the allocation  $(x, y)$  and the efficiency price vector  $(p, q)$ . In other words, if the message space  $M$  has minimal dimension and  $\mu$  is a continuous function, then  $\phi$  is a one-to-one mapping.

**Lemma 6** Suppose that  $\langle M, \mu, h \rangle$  is an allocation mechanism on the class of public goods economies  $E^{cq}$  such that:

- (i) it is informationally decentralized;
- (ii) it is non-wasteful with respect to  $\mathcal{P}$ ;
- (iii) it is individually rational with respect to a given fixed share guarantee structure  $\gamma_i(e; \theta)$ ;
- (iv)  $M$  is a  $(L + K - 1)I + (L + K)J$  dimensional manifold;
- (v)  $\mu$  is a continuous function on  $E^{cq}$ .

Let  $\bar{e} \in E^{cq}$  and let  $(\bar{p}, \bar{q}, \bar{x}, \bar{y}) = \phi[\mu(\bar{e})]$ , where  $\phi$  is defined as Lemma 5. If  $e^*$  is any environment such that  $w_i^* + \bar{x}_i^\rho > 0$ , and  $Du_i^*(w_i^* + \bar{x}_i^\rho, \bar{x}_i^\sigma)$  is proportional to  $(\bar{p}, \bar{q}_i)$  for consumer  $i = 1, \dots, I$ , and  $DT^*(\bar{y}_i)$  is proportional to  $(\bar{p}, \sum_{i=1}^I \bar{q}_i)$  for producer  $i = I + 1, \dots, N$ , then  $\mu(e^*) = \mu(\bar{e})$ . In particular,  $\phi$  is a one-to-one function.

Proof. For  $\bar{e} \in E^{cq}$ , define the set  $\bar{E}^{cq} = \{(w, a, b, c, d) \in E^{cq} : w_i = \bar{w}_i \forall i = 1, \dots, I\}$ . Note that, by Lemma 1, for each  $e, e' \in \bar{E}^{cq}$ , if for some  $(x, y) \in Z$ ,  $Du_i(\bar{w}_i + x_i^\rho, x_i^\sigma)$  is proportional to  $Du'_i(\bar{w}_i + x_i^\rho, x_i^\sigma)$  for consumer  $i = 1, \dots, I$ , and  $DT_i(y)$  is proportional to  $DT'_i(y_i)$  for producer  $i = I + 1, \dots, N$ , then  $a = a', b = b', c = c',$  and  $d = d'$ . Thus, by Lemma 5,  $\phi \cdot \mu$  is one-to-one on  $\bar{E}^{cq}$  and so  $\mu$  is one-to-one on  $\bar{E}^{cq}$ .

Since  $\bar{E}^{cq}$  is homeomorphic to  $\mathbb{R}^{(L+K-1)I+(L+K)J}$ , we can consider  $\bar{E}^{cq}$  as an open subset of  $\mathbb{R}^{(L+K-1)I+(L+K)J}$ . Let  $N(\bar{e})$  be an open neighborhood of  $\bar{e}$  in  $\bar{E}^{cq}$  and let  $\bar{m} = \mu(\bar{e})$ . Since  $M$  is a  $(L + K - 1)I + (L + K)J$  dimensional manifold,  $M$  and  $\bar{E}^{cd}$  are manifolds of the same dimension. Also, since  $\mu$  is a one-to-one function on  $N(\bar{e})$ ,  $V(\bar{m}) \equiv \mu(N(\bar{e}))$  is an open neighborhood of  $\bar{m}$  and  $\mu$  is a homeomorphism on  $N(\bar{e})$  to  $V(\bar{m})$  by Exercise 18.10 in Greenberg (1967, p. 82). Let  $n = (L + K - 1)I + (L + K)J$ , and let  $H_n(M, M - \bar{m})$  denote the  $n$ -th singular homology module of  $M$  relative to  $M - \bar{m}$ , with integral coefficients. Then  $H_n(M, M - \bar{m})$  is isomorphic to  $Z$  (denoted by  $H_n(M, M - \bar{m}) \cong Z$ ), the set of integers, and the homomorphism  $i_* : H_n(V(\bar{m}), V(\bar{m}) - \bar{m}) \rightarrow H_n(M, M - \bar{m})$  induced by the inclusion  $i : (V(\bar{m}), V(\bar{m}) - \bar{m}) \rightarrow (M, M - \bar{m})$  is isomorphism (see Greenberg (1967, p. 111)).

Also, by the hypothesis,  $D \ln u_i^*(w_i^* + \bar{x}_i^\rho, \bar{x}_i^\sigma)$  is proportional to  $\ln \bar{u}_i(\bar{w}_i + \bar{x}_i^\rho, \bar{x}_i^\sigma)$  for consumer  $i = 1, \dots, I$ , and  $DT^*(\bar{y}_i)$  is proportional to  $D\bar{T}(\bar{y}_i)$  for producer  $i = I + 1, \dots, N$  so there is some  $\lambda_i^* > 0$ , one for each  $i$ , such that for consumer  $i = 1, \dots, I$

$$\begin{aligned} \frac{a_i^{*l}}{w_i^{*l} + \bar{x}_i^{\rho l}} &= \lambda_i^* \frac{\bar{a}_i^l}{\bar{w}_i^l + \bar{x}_i^{\rho l}}, \quad l = 2, \dots, L, \\ c_i^{*k} &= \lambda_i^* \bar{c}_i^k, \end{aligned}$$

and for producer  $i = I + 1, \dots, N$ ,

$$\begin{aligned} b_i^{*l} &= \begin{cases} \lambda_i^* \bar{b}_i^l & \text{if } l = 1 \\ \frac{\lambda_i^* (1 + \bar{b}_i^l y_i^{\rho l}) - 1}{y_i^{\rho l}} & \text{if } l = 2, \dots, L \end{cases}, \\ d_i^{*k} &= \frac{\lambda_i^* (1 + \bar{d}_i^k y_i^{\sigma k}) - 1}{y_i^{\sigma k}}. \end{aligned}$$

Define the function  $G : N(\bar{e}) \times [0, 1] \rightarrow E^{cq}$  by  $G((\bar{w}, a, b, c, d), t) = (w', a', b', c', d')$ , where

$$\begin{aligned}
w'_i &= tw_i^* + (1-t)\bar{w}_i, \\
a'_i &= [t\bar{a}_i^l + (1-t)a_i^l] \frac{w_i^l + \bar{x}_i^{\rho l}}{\bar{w}_i^l + \bar{x}_i^{\rho l}} [t\lambda_i^* + (1-t)] \quad i = 1, \dots, I, l = 2, \dots, L, \\
c'_i &= [t\bar{c}_i^k + (1-t)c_i^k] [t\lambda_i^* + (1-t)] \quad i = 1, \dots, I, k = 1, \dots, K, \\
b'_i &= \begin{cases} [t\bar{b}_i^1 + (1-t)b_i^1] [t\lambda_i^* + (1-t)] & \text{if } l = 1 \\ \frac{[t\lambda_i^* + (1-t)][1 + (t\bar{b}_i^l + (1-t)b_i^l)y_i^{\rho l}] - 1}{y_i^{\rho l}} & l = 2, \dots, L \end{cases} \quad i = I+1, \dots, N \\
d'_i &= \frac{[t\lambda_i^* + (1-t)][1 + (t\bar{d}_i^k + (1-t)d_i^k)y_i^{\sigma k}] - 1}{y_i^{\sigma k}} \quad k = 1, \dots, K. \tag{64}
\end{aligned}$$

Then,  $G(\cdot, 0)$  is the inclusion map by noting that  $G((\bar{w}, a, b, c, d), 0) = (\bar{w}, a, b, c, d)$  for all  $(\bar{w}, a, b, c, d) \in N(\bar{e})$ , and  $G(\cdot, 1)$  is the constant map on  $N(\bar{e})$  to  $e^*$  by noting that  $G((\bar{w}, a, b, c, d), 1) = (w^*, a^*, b^*, c^*, d^*)$  for all  $(\bar{w}, a, b, c, d) \in N(\bar{e})$ . Let  $t \in [0, 1]$ ,  $(\bar{w}, a, b, c, d) \in N(\bar{e})$  with  $(\bar{w}, a, b, c, d) \neq \bar{e}$ , and  $(w', a', b', c', d') = G(\bar{w}, a, b, c, d, t)$ . Then  $\frac{a_i^l}{w_i^l + \bar{x}_i^{\rho l}}$  is proportional to  $\frac{t\bar{a}_i^l + (1-t)a_i^l}{\bar{w}_i^l + \bar{x}_i^{\rho l}}$  for  $i = 1, \dots, I$  and  $l = 2, \dots, L$ ,  $c_i^k$  is proportional to  $t\bar{c}_i^k + (1-t)c_i^k$  for  $i = 1, \dots, I$  and  $k = 1, \dots, K$ ,  $b_i^1$  is proportional to  $[t\bar{b}_i^1 + (1-t)b_i^1]$ ,  $[1 + b_i^l \bar{y}_i^{\rho l}]$  is proportional to  $[1 + (t\bar{b}_i^l + (1-t)b_i^l)\bar{y}_i^{\rho l}]$  for  $l = 2, \dots, L$  and  $i = I+1, \dots, N$ , and  $[1 + d_i^k \bar{y}_i^{\sigma k}]$  is proportional to  $[1 + (t\bar{d}_i^k + (1-t)d_i^k)\bar{y}_i^{\sigma k}]$  for  $k = 1, \dots, K$  and  $i = I+1, \dots, N$ . Then environment  $(\bar{w}, t\bar{a} + (1-t)a, t\bar{b} + (1-t)b, t\bar{c} + (1-t)c, t\bar{d} + (1-t)d) \in \bar{E}^{cq}$ , so by the argument in the first paragraph, for each  $t < 1$ ,  $\mu[G(e, t)] = \bar{m}$  only if  $e = \bar{e}$ .

Now, suppose by way of contradiction that  $\mu(e^*) \neq \bar{m}$ . Then  $\mu[G(e, t)] \neq \bar{m}$  whenever  $e \neq \bar{e}$ . Define  $G' : (V(\bar{m}), V(\bar{m}) - \bar{m}) \times [0, 1] \rightarrow (M, M - \bar{m})$  by  $G'(m, t) = \mu[G[\mu^{-1}(m), t]]$ . Then  $G'$  is a homotopy between the inclusion  $i : (V(\bar{m}), V(\bar{m}) - \bar{m}) \rightarrow (M, M - \bar{m})$  and the constant map  $j : (V(\bar{m}), V(\bar{m}) - \bar{m}) \rightarrow (m^*, M - \bar{m})$ , where  $m^* = \mu(e^*)$ . Hence,  $i_* = j_*$  and consequently  $i_*$  is the zero homomorphism, which contradicts the fact that  $i_*$  is isomorphic to  $Z$ . Q.E.D.

**Theorem 3 (The Uniqueness Theorem)** Suppose that  $\langle M, \mu, h \rangle$  is an allocation mechanism on the class of public goods economies  $E^{cq}$  such that:

- (i) it is informationally decentralized;
- (ii) it is non-wasteful with respect to  $\mathcal{P}$ ;
- (iii) it is individually rational with respect to the fixed share guarantee structure  $\gamma_i(e; \theta)$ ;
- (iv)  $M$  is a  $(L + K - 1)I + (L + K)J$  dimensional manifold;
- (v)  $\mu$  is a continuous function on  $E^{cq}$ .

Then, there is a homeomorphism  $\phi$  on  $\mu(E^{cq})$  to  $M_L$  such that

- (a)  $\mu_L = \phi \cdot \mu$ ;
- (b)  $h_L \cdot \phi = h$ .

The conclusion of the theorem is summarized in the following commutative homeomorphism diagram:

$$\begin{array}{ccccc}
E^{cq} & & \xrightarrow{\mu_L} & & M_L \\
\downarrow \mu & \nearrow \phi & & \nwarrow \phi^{-1} & \downarrow h_L \\
\mu(E^{cq}) & & \xrightarrow{h} & & Z
\end{array}$$

Proof. Let  $\phi : M \rightarrow \mathbb{R}_{++}^{L+IK} \times Z$  be the function defined as Lemma 5. We first show that  $\mu_L = \phi \cdot \mu$ . Suppose by way of contradiction that  $\mu_L(e) \neq \phi[\mu(e)]$  for some  $e \in E^{cq}$ . Let  $(p, q, x, y) = \phi[\mu(e)]$ . Then  $(x, y)$  is not a  $\theta$ -Lindahl equilibrium allocation, i.e.,  $(x, y) \notin L_\theta(e)$ . Thus, for some  $i$ , we have  $p \cdot x_i^\rho + q_i \cdot x_i^\sigma < \sum_{j=I+1}^{I+I} \theta_{ij} [p \cdot y_j^\rho + \hat{q} \cdot y_j^\sigma]$ . Let  $w_i^* = (1/t)(w_i + x_i^\rho) - x_i^\rho$ , let  $e_i^* = (w_i^*, a_i, b_i, c_i, d_i)$ , let  $e^* = (e_i^*, e_{-i})$ , and let  $g(t) = u_i[(1-t)(w_i + x_i^\rho) + t(w_i + \gamma_i(e^*; \theta)), x_i^\sigma] - u_i(w_i + x_i^\rho, x_i^\sigma)$  where  $0 < t < 1$ . Then,  $g(t) \rightarrow 0$  as  $t \rightarrow 0$ , and  $dg/dt = D_\rho u_i[(1-t)(w_i + x_i^\rho) + t(w_i + \gamma_i(e^*; \theta)), x_i^\sigma](\gamma_i(e^*; \theta) - x_i^\rho) \rightarrow D_\rho u_i[w_i + x_i^\rho, x_i^\sigma]((w_i + x_i^\rho) \frac{\sum_{j=I+1}^{I+I} \theta_{ij} [p \cdot y_j^\rho + \hat{q} \cdot y_j^\sigma]}{p \cdot (w_i + x_i^\rho)} - x_i^\rho) = \lambda p \cdot [(w_i + x_i^\rho) \frac{\sum_{j=I+1}^{I+I} \theta_{ij} [p \cdot y_j^\rho + \hat{q} \cdot y_j^\sigma]}{p \cdot (w_i + x_i^\rho)} - x_i^\rho] = \lambda (\sum_{j=I+1}^{I+I} \theta_{ij} [p \cdot y_j^\rho + \hat{q} \cdot y_j^\sigma] - p \cdot x_i^\rho) > 0$  for some  $\lambda > 0$  as  $t \rightarrow 0$  by noting that  $D_\rho u_i[w_i + x_i^\rho, x_i^\sigma]$  is proportional to  $p$  and  $p \cdot x_i^\rho \leq p \cdot x_i^\rho + q_i \cdot x_i^\sigma < \sum_{j=I+1}^{I+I} \theta_{ij} [p \cdot y_j^\rho + \hat{q} \cdot y_j^\sigma]$ . Here  $D_\rho u_i(\cdot)$  is the vector of partial derivatives with respect to  $x_i^\rho$ . Thus we have  $g(t) = u_i[(1-t)(w_i + x_i^\rho) + t(w_i + \gamma_i(e^*; \theta)), x_i^\sigma] - u_i(w_i + x_i^\rho, x_i^\sigma) > 0$ , i.e.,  $u_i[(1-t)(w_i + x_i^\rho) + t(w_i + \gamma_i(e^*; \theta)), x_i^\sigma] > u_i(w_i + x_i^\rho, x_i^\sigma)$  when  $t$  is a sufficiently small positive number by noting that  $g(\cdot)$  is continuous and  $g(t) \rightarrow 0$  as  $t \rightarrow 0$ . Then, multiplying  $1/t$  on the both sides of  $u_i[(1-t)(w_i + x_i^\rho) + t(w_i + \gamma_i(e^*; \theta)), x_i^\sigma] > u_i(w_i + x_i^\rho, x_i^\sigma)$  and by the homogeneity of preferences represented by  $u_i(\cdot)$ , we have  $u_i((1/t)(w_i + x_i^\rho), x_i^\sigma) < u_i((1/t)(w_i + x_i^\rho) - x_i^\rho + \gamma_i(e^*; \theta), x_i^\sigma)$  when  $t$  is a sufficiently small positive number. Thus, since  $\langle M, \mu, h \rangle$  is individually rational with respect to the fixed share guarantee structure  $\gamma_i(e^*; \theta)$ , we must have  $\mu(e^*) \neq \mu(e)$  otherwise we have  $u_i(w_i^* + x_i^\rho, x_i^\sigma) = u_i((1/t)(w_i + x_i^\rho), x_i^\sigma) < u_i((1/t)(w_i + x_i^\rho) - x_i^\rho + \gamma_i(e^*; \theta), x_i^\sigma) = u_i(w_i^* + \gamma_i(e^*; \theta), x_i^\sigma)$  which contradicts the hypothesis that  $(x, y) = h[\mu(e^*)]$  is individually rational with respect to the fixed share guarantee structure  $\gamma_i(e^*; \theta)$ . However,  $(x, y)$  is Pareto optimal for  $e^*$ , and  $Du_i(w_i^* + x_i^\rho, x_i^\sigma)$  is proportional to  $(p, q_i)$ , Lemma 6 implies that  $\mu(e^*) = \mu(e)$ , a contradiction. So we must have  $\mu_L = \phi \cdot \mu$ . Furthermore, since  $h_L$  is the projection  $(p, q, x, y) \rightarrow (x, y)$ , it follows that  $h_L \cdot \phi = h$ .

Now we show that  $\phi$  is a homeomorphism on  $\mu(E^{cq})$  to  $M_L$ . By Lemma 6,  $\phi$  is a one-to-one mapping. Also, since  $h_L \cdot \phi = h$ , the range of  $\phi$  is  $M_L$ . So it only remains to show that  $\phi$  and  $\phi^{-1}$  are continuous. To show that  $\phi^{-1}$  is continuous, let  $\{m(k)\}$  be a sequence in  $M_L$ , which converges to some  $m \in M_L$  with  $m(k) = \mu(e(k))$  for all  $k$  and  $m \in \mu(e)$ . Since  $\phi^{-1} \cdot \mu_L = \mu$ ,  $\phi^{-1}(m(k)) = \mu(e(k))$  for all  $k$  and  $\phi^{-1}(m) = \mu(e)$ . Since  $\mu$  is continuous,  $\mu(e(k))$  converges to  $\mu(e)$ , so  $\phi^{-1}$  is continuous. Since  $M_L$  and  $M$  are manifolds of the same dimension,  $\phi^{-1}$  is a homeomorphism on  $M_L$  to  $\phi^{-1}(M_L) = \mu(E^{cq})$  by Exercise 18.10 in Greenberg (1967. p. 82). Q.E.D.

**Remark 3** The Uniqueness Theorem obtained above is based on the class of Cobb-Douglas–Quadratic production economies. As in Jordan (1982), a similar example may be constructed to show that, for the Uniqueness Theorem, it is necessary to require the relation Cobb-Douglas–Quadratic environments and equilibrium messages be single-valued and continuous. The single-valuedness requirement cannot be extended much beyond the Cobb-Douglas–Quadratic class without conflicting with the multiplicity of Lindahl equilibrium. Nevertheless, similar to Jordan (1982), we can extend the above Uniqueness Theorem to more general classes of economic environments if we impose the additional regularity assumptions that the message space is connected and that the set of messages associated with Cobb-Douglas–Quadratic environments is a closed subset of the message space.

## 6 Concluding Remarks

In this paper, it has been shown that the minimal dimension of message space for privacy preserving and non-wasteful resource allocation processes on the neoclassical public goods economies with any number of producers and commodities is the dimension of the Lindahl message space that equals  $(L+K-1)I+(L+K)J$ , and thus the Lindahl mechanism is informationally the most efficient decentralized mechanism. Furthermore, it is shown that the Lindahl mechanism is the unique informationally efficient and decentralized mechanism that realizes Pareto efficient and individually allocations over the class of the public goods economies with Cobb-Douglas utility functions and Quadratic production functions. This unique informational efficiency result on the Lindahl mechanism for public goods economies is a kind of impossibility theorem: it implies that there exists no other privacy preserving, individually rational, and non-wasteful resource mechanism that uses a message space whose informational size is smaller than, or the same as, that of the Lindahl message space.

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