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# Incentive Mechanism Design for Production Economies with Both Private and Public Ownerships*

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In this paper we deal with the problem of incentive mechanism design which yields efficient allocations for general mixed ownership economies when preferences, individual endowments, and coalition patterns among individuals are unknown to the planner. We do so by doubly implementing the proportional solution for economies with any number of private sector and public sector commodities and any number of individuals as well as the coexistence of privately and publicly owned firms with general convex production possibility sets. Furthermore, the mechanisms work not only for three or more agents, but also for two-agent economies. *Journal of Economic Literature* Classification Numbers: C72, D61, D71, D82. © 2000 Academic Press

## 1. INTRODUCTION

The notions of self-interested behavior, decentralized decision making, and incentive-compatibility are basic and key concepts in studying various economic systems or organizations. However, most investigations have been concerned with analyzing the performance of an economic organization or institution only under the restriction of private ownership. Until recently, for the general equilibrium approach to the efficiency of resource allocation of private goods, the most commonly used general equilibrium notion

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was the Walrasian equilibrium principle which is a general equilibrium concept for private-ownership institutions. Since the Walrasian mechanism, in general, is not incentive-compatible even for classical economic environments when the number of agents is finite, many incentive mechanisms have been proposed to implement Walrasian allocations at Nash equilibrium and/or strong Nash equilibrium points such as those in Hurwicz (1979), Schmeidler (1980), Hurwicz *et al.* (1995), Postlewaite and Wettstein (1989), Tian (1989, 1992, 1996), Hong (1995), and Peleg (1996a, 1996b) among others.

The Walrasian equilibrium principle, however, has a limited scope. It must take profit shares as exogenously given and, further, no publicly owned firms are allowed in the economy under consideration. In their paper, Roemer and Silvestre (1993) introduced the notion of the proportional solution for general convex economies with both private and public ownership with endogenous profit distributions.[1] The notion of the proportional solution has desired properties in that it yields Pareto efficient allocations. It is also a generalization of Walrasian equilibrium to the case of an economy with public as well as private endowments, in the sense that, if there are no public endowments, the solution is the conventional Walrasian solution, but when there are public endowments, it is equivalent to the Walrasian solution with *endogenous* profit shares distributed among consumers in proportion to their purchase of public sector commodities. Thus the solution concept gives an endogenous theory of profit distribution for economies with coexistence of privately and publicly owned firms. However, like the Walrasian mechanism, the proportional solution mechanism is not incentive-compatible either. As Roemer and Silvestre indicate in their paper, they do not consider the issues of asymmetric information and incentive compatibility of the solution concept.

In this paper we attempt to fill this gap. We will deal with the problem of the incentive mechanism design which results in Pareto efficient allocations for mixed ownership economies. We do so by investigating the incentive aspect of the proportional solution for general convex economies. We allow not only preferences and individual endowments but also coalition patterns among individuals to be unknown to the planner. We will propose a mechanism which doubly implements the proportional solution. That

[1]For pure public goods economies, some solution concepts have also been proposed in the literature as solution concepts for public/state ownership, including the ratio equilibrium notion of Kaneko (1977) and the generalized ratio equilibrium notion of Tian and Li (1994), as well as the more general solution notions of the various cost share equilibria of Mas-Colell and Silvestre (1989). A version of the proportional solution for public goods economies actually coincides, under some conditions, with the notion of balanced linear cost share equilibrium presented in Mas-Colell and Silvestre (1989).

is, we will give mechanisms in which not only Nash allocations but also strong Nash allocations coincide with proportional allocations. By double implementation, the solution can cover the situation where agents in some coalitions will cooperate and in some other coalitions will not. Thus, the designer does not need to know which coalitions are permissible and, consequently, the possibility is allowed for agents to manipulate coalition patterns.

Note that the mechanisms proposed in the paper also have some nice properties in that they use feasible and continuous outcome functions, and further, have message spaces of finite dimension. In addition, our double implementation result is very general. We allow any number of private sector commodities and public sector commodities, any number of firms and individuals, and coexistence of privately and publicly owned firms with general convex production possibility sets. Furthermore, our mechanisms work not only for three or more agents, but also for two-agent economies, and thus they are unified mechanisms which are irrespective of the number of agents. Although Suh (1995) and Yoshihara (1999) considered double implementation of the proportional solution, they only considered a very special case of economic environments with only one private sector commodity and one public sector commodity, no privately owned firms, and one publicly owned firm. In addition, the mechanisms they constructed are not continuous. Furthermore, the method they used to construct the mechanisms is quite different from the one we adopt. Also, they need to assume that the production function is differentiable, while we allow for general production possibility sets.

It may also be remarked that, because of the endogenous profit distributions which depend on both prices and quantities at equilibrium, and the double implementation requirement, as well as the generality of economic environments under consideration, the implementation problem of the proportional solution is much harder than the one for the conventional Walrasian solution; and the mechanisms to be constructed are quite different from the existing ones. As such, some of the techniques used in implementing other market-like equilibrium solutions such as Walrasian equilibrium for private goods economies or Lindahl equilibrium for public goods economies may not be applicable, and thus some new techniques are needed.

The remainder of this paper is as follows. Section 2 sets forth a general mixed ownership model and gives the notion of a proportional solution, and provides notation and definitions used for mechanism design. Section 3 gives a mechanism which has the desirable properties mentioned above when preferences, individual endowments, and coalition patterns are unknown to the designer, but production possibility sets are known. We then prove that this mechanism doubly implements the proportional solution. Concluding remarks are presented in Section 4.

## 2. THE MODEL

### 2.1. *Economic Environments*

We consider a mixed ownership economy in which there are $n \geqq 2$ consumers, $F$ privately owned firms, one publicly owned firm, and $H$ commodities which are partitioned into two groups denoted by $\rho$ and $\mu$.[2] There are $L$ private sector commodities in group $\rho$ and $K$ public sector commodities in group $\mu$ (thus $L + K = H$). Throughout this paper subscripts are used to index consumers or firms, and superscripts are used to index goods unless otherwise stated. Denote by $N = \{1, 2, \ldots, n\}$ the set of consumers, $x^\rho$ the consumption of private sector goods which are either initially owned by individuals or produced by privately owned firms, and $x^\mu$ the consumption of public sector goods which are either initially owned or produced by a publicly owned firm. Production technologies of privately owned firms and of the publicly owned firm are given by $\mathcal{Y}_1, \ldots, \mathcal{Y}_f, \ldots, \mathcal{Y}_F$ and $\mathcal{Y}_C$, respectively. Here "$C$" is for "collectively owned." We assume that, for $g = 1, \ldots, F, C$, $\mathcal{Y}_g$ is nonempty, closed, convex, and $0 \in \mathcal{Y}_g$. Note that, since privately owned firms only produce goods in group $\rho$ and the publicly owned firm only produces goods in group $\mu$, we have implicitly assumed that $y_f^\mu \leqq 0$ for any $y_f = (y_f^\rho, y_f^\mu) \in \mathcal{Y}_f$ and $y_C^\rho \leqq 0$ for any $y_C = (y_C^\rho, y_C^\mu) \in \mathcal{Y}_C$.

Each agent's characteristic is denoted by $e_i = (C_i, w_i, (\theta_{i1}, \ldots, \theta_{iF}), R_i)$, where $C_i = \mathbb{R}_+^H$ is the consumption set; $w_i = (w_i^\rho, w_i^\mu) \in \mathbb{R}_+^H$ is the true initial endowment vector of commodities; $\theta_{if} \in \mathbb{R}_+$ are the profit shares of private firms $f$, $f = 1, \ldots, F$, satisfying $\sum_{i=1}^n \theta_{if} = 1$; and $R_i$ is the preference ordering defined on $\mathbb{R}_+^H$. Let $P_i$ denote the asymmetric part of $R_i$ (i.e., $a\ P_i\ b$ if and only if $a\ R_i\ b$, but not $b\ R_i\ a$). We assume that $w_i^\rho > 0$, $w_i^\mu = 0$, and preference ordering $R_i$ is strictly monotonically increasing, continuous, and convex.[3] The public sector initially owns the nonnegative vector $w_C = (w_C^\rho, w_C^\mu) \in \mathbb{R}_+^H$ with $w_C^\rho = 0$ and $w_C^\mu > 0$.[4] Thus, we have $\sum_{i=1}^n w_i + w_C > 0$. For detailed discussions on the setting of the economic environment, see Roemer and Silvestre (1993).

[2]As usual, vector inequalities are defined as follows: Let $a, b \in \mathbb{R}^m$. Then $a \geqq b$ means $a_s \geqq b_s$ for all $s = 1, \ldots, m$; $a \geq b$ means $a \geqq b$ but $a \neq b$; $a > b$ means $a_s > b_s$ for all $s = 1, \ldots, m$.

[3]$R_i$ is convex if, for bundles $a$ and $b$, $a\ P_i\ b$ implies $\lambda a + (1 - \lambda) b\ P_i\ b$ for all $0 < \lambda \leqq 1$.

[4]Here we just follow Roemer and Silvestre (1993) by assuming that $w_i^\mu = 0$ and $w_C^\rho = 0$ although our implementation results hold for the general case of $w_i^\mu \geqq 0$ and $w_C^\rho \geqq 0$.

An economy is the full vector $e = (e_1, \ldots, e_n, \mathcal{Y}_1, \ldots, \mathcal{Y}_F, \mathcal{Y}_C)$ and the set of all such economies is denoted by $E$.

An allocation of the economy $e$ is a vector $(x_1, \ldots, x_n, y_1, \ldots, y_F, y_C) \in \mathbb{R}^{H(n+F+1)}$ such that: (1) $x := (x_1, \ldots, x_n) \in \mathbb{R}^{nH}_+$, and (2) $y_g \in \mathcal{Y}_g$ for $g = 1, F, C$. Denote $y_P = (y_1, \ldots, y_F)$ ("$P$" for privately owned firms), and $y = (y_P, y_C)$.

An allocation $(x, y)$ is *feasible* if

$$\sum_{i=1}^{n} x_i \leqq \sum_{i=1}^{n} w_i + w_C + \sum_{f=1}^{F} y_f + y_C. \tag{1}$$

Denote the aggregate endowment, consumption, and production by $\hat{w} = \sum_{i=1}^{n} w_i$, $\hat{x} = \sum_{i=1}^{n} x_i$, $\hat{y}_P = \sum_{f=1}^{F} y_f$, respectively.[5] Then the feasibility condition can be notationly written as

$$\hat{x} \leqq \hat{w} + w_C + \hat{y}_P + y_C$$

or more specifically as

$$\hat{x}^{\rho} \leqq \hat{w}^{\rho} + \hat{y}^{\rho}_P + y^{\rho}_C \tag{2}$$

$$\hat{x}^{\mu} \leqq w^{\mu}_C + \hat{y}^{\mu}_P + y^{\mu}_C \tag{3}$$

by noting that $\hat{w}^{\mu} = 0$ and $\hat{w}^{\rho}_C = 0$.

An allocation $(x, y)$ is *Pareto optimal* with respect to the preference profile $R = (R_1, \ldots, R_n)$ if it is feasible and there is no other feasible allocation $(x', y')$ such that $x'_i \, P_i x_i$ for all $i \in N$.

## 2.2. *Proportional Solution*

To define the proportional solution, let us first define the efficiency prices for Pareto optimal allocations.

The nonzero vector $p^* \in \mathbb{R}^H_+$ is a vector of *efficiency prices* for a Pareto optimal allocation $(x^*, y^*_P, y^*_C)$ if

(a) $p^* \cdot x^*_i \leqq p^* \cdot x_i$ for all $i \in N$ and $x_i \in \mathbb{R}^H_+$ such that $x_i \, R_i \, x^*_i$;

(b) $p^* \cdot y^*_g \geqq p^* \cdot y_g$ for all $y_g \in \mathcal{Y}_g$, $g = 1, \ldots, F, C$;

(c) $p^* \cdot \hat{z}^* = 0$.

Here $\hat{z}^* = \hat{x}^* - \hat{w} - w_C - \hat{y}^*_P - y^*_C$.

[5]For notational convenience, "$\hat{a}$" will be used throughout the paper to denote the sum of vectors $a_i$, i.e., $\hat{a} := \sum a_l$.

Given a price vector $p = (p^\rho, p^\mu)$ and an allocation $(x, y_P, y_C)$, a proportional share $\theta_{iC}$ is defined by[6]

$$\theta_{iC} = \begin{cases} \frac{p^\mu \cdot x_i^\mu - \sum_{f=1}^F \theta_{if} p^\mu \cdot y_f^\mu}{p^\mu \cdot \hat{x}^\mu - p^\mu \cdot \hat{y}_f^\mu} & \text{if } p^\mu \cdot \hat{x}^\mu > p^\mu \cdot \hat{y}_f^\mu \\ \frac{1}{n} & \text{if } p^\mu \cdot \hat{x}^\mu = p^\mu \cdot \hat{y}_f^\mu \end{cases}, \tag{4}$$

Note that $\theta_{iC} \geqq 0$ since $-y_f^\mu \geqq 0$, and $\sum_{i=1}^n \theta_{iC} = 1$.

An allocation $(x^*, y^*)$ is a *proportional solution* for an economy $e \in E$ if:

(i) $(x^*, y^*)$ is Pareto optimal;

(ii) there exists a vector of efficiency prices $p^*$ for $(x^*, y^*)$ such that

$$p^{*\rho} \cdot x_i^{*\rho} = p^{*\rho} \cdot w_i^\rho + \sum_{f=1}^F \theta_{if} p^{*\rho} \cdot y_f^{*\rho} + \theta_{iC} p^{*\rho} \cdot \left[\hat{x}^{*\rho} - \hat{w}^\rho - \hat{y}_P^\rho\right]. \tag{5}$$

Denote by $PR(e)$ the set of all such proportional allocations. Note that with $L = 1$, $K = 1$, no privately owned firm, and $p^{*\rho}$ being normalized to one, Eq. (5) becomes

$$x_i^{*\rho} = w_i^\rho + \frac{x_i^\mu}{\sum_{i=1}^n x_i^\mu} \sum_{i=1}^n \left(x_i^{*\rho} - w_i^\rho\right), \tag{6}$$

i.e., individual $i$'s consumption of the privately owned good equals his endowment minus a fraction $x_i^\mu / \left(\sum_{i=1}^n x_i^\mu\right)$ of the inputs (costs) of the public sector. The fraction $x_i^\mu / \left(\sum_{i=1}^n x_i^\mu\right)$ is the ratio of $i$'s consumption of public sector output to the total output of the public sector. Roemer and Silvestre (1993) gave two alternative interpretations. One is that the proportional solution can be viewed as "efficiency with payment according to average cost" (i.e., payment equals average cost), and the other is that "efficiency without Marxian exploitation" (i.e., labor spent equals labor embodied (when regarding the private good input as labor)). Suh (1995) and Yoshihara (1999) considered the double implementation of the proportional solution for the simple case specified by (6).

Roemer and Silvestre (1993) proved the existence of a proportional solution for the general setting of the model, and they also proved that an allocation $(x^*, y^*)$ is a proportional solution if and only if it is a Walrasian

[6]When Roemer and Silvestre (1993) defined the proportional shares as well as the proportional solution, they introduced the notation of net outputs and inputs of the public sector $t^+$ and $t^-$ which may make the expressions of the proportional shares and proportional solution less intuitive. Here we give an explicit expression of these notions which are equivalent to the those defined by Roemer and Silvestre (1993) by noting that $\hat{x} - \hat{w} - \hat{y}_P = y_C + w_C$, $w_C^\rho = 0$, $w_i^\mu = 0$ for all $i \in N$, $t^+ = \hat{x}^\mu - \hat{y}_f^\mu$, and $t^- = \hat{x}^{*\rho} - \hat{w}^\rho - \hat{y}_P^\rho$ when the feasibility condition holds with equality.

allocation with endogenous profit shares for publicly owned sector production, i.e., if and only if it is a Walrasian allocation when profit shares from firm C are given by (4). This equivalence relationship will display an essential rule in our implementation results. We state it here as a lemma.

LEMMA 1. *An allocation* $(x^*, y_P^*, y_C^*)$ *is a proportional solution for an economy* $e$ *if and only if it is feasible and there exists a price vector* $p^* \in \mathbb{R}_+^H$ *such that*

(1) $y^*$ *maximizes* $p^* \cdot y_g$ *on* $\mathcal{Y}_g$ *for all* $g = 1, \ldots, F, C$;

(2) $p^* \cdot x_i^* = p^* \cdot w_i + \sum_{f=1}^F \theta_{if} p^* \cdot y_f^* + \theta_{iC}^* p^* \cdot (y_C^* + w_C)$ *for all* $i \in N$;

(3) *for all* $i \in N$, *there does not exist an* $x_i$ *such that* $x_i \, P_i \, x_i^*$ *and* $p^* \cdot x_i \leqq p^* \cdot w_i + \sum_{f=1}^F \theta_{if} p^* \cdot y_f^* + \theta_{iC}^* p^* \cdot (y_C^* + w_C)$.

Note that $p^* \cdot w_i = p^{*\rho} \cdot w_i^\rho$ since $w_i^\mu = 0$. We will use this fact for the remainder of the paper. From the budget equation specified in condition (2), one can see that profits made by the publicly owned firm are distributed among consumers according to $\theta_{iC}^*$ which is defined by (4), i.e., in proportion to the difference between consumers' consumption and privately owned firms' production input demand for public sector commodities. In this paper, we consider double implementation of proportional allocations by designing feasible and continuous mechanisms which doubly implement the above Walrasian allocations with the endogenous profit shares $\theta_{iC}^*$. To do so, we need make the following indispensable assumption.

ASSUMPTION 1 (Interiority of Preferences). For all $i \in N$, $x_i \, P_i \, x_i'$ for all $x_i \in \mathbb{R}_{++}^H$ and $x_i' \in \partial\mathbb{R}_+^H$, where $\partial\mathbb{R}_+^H$ is the boundary of $\mathbb{R}_+^H$.[7]

*Remark* 1. The family of Cobb-Douglas utility functions satisfies Assumption 1.

## 2.3. *Mechanism*

Let $M_i$ denote the $i$th agent's message domain. Its elements are written as $m_i$ and called messages. Let $M = \prod_{i=1}^n M_i$ denote the message space. Denote by $h : M \to \mathbb{R}_+^{nH+H(F+1)}$ the outcome function, or more explicitly, $h(m) = (X_1(m), \ldots, X_n(m), Y_1(m), \ldots, Y_F(m), Y_C(m))$. Then the mechanism consists of $\langle M, h \rangle$ which is defined on $E$.

A message $m^* = (m_1^*, \ldots, m_n^*) \in M$ is said to be a *Nash equilibrium* of the mechanism $\langle M, h \rangle$ for an economy $e$ if, for all $i \in N$ and $m_i \in M_i$,

$$X_i(m^*) \, R_i \, X_i(m_i, m_{-i}^*), \tag{7}$$

[7] $\partial D$ denotes the boundary of the set $D$.

where $(m_i, m^*_{-i}) = (m^*_1, \dots, m^*_{i-1}, m_i, m^*_{i+1}, \dots, m^*_n)$. $h(m^*)$ is then called a *Nash (equilibrium) allocation* of the mechanism for the economy $e$. Denote by $V_{M,h}(e)$ the set of all such Nash equilibria and by $N_{M,h}(e)$ the set of all such Nash (equilibrium) allocations.

A mechanism $\langle M, h\rangle$ is said to *Nash-implement* the proportional solution on $E$, if, for all $e \in E$, $N_{M,h}(e) = PR(e)$.

A *coalition* $S$ is a non-empty subset of $N$.

A message $m^* = (m^*_1, \dots, m^*_n) \in M$ is said to be a *strong Nash equilibrium* of the mechanism $\langle M, h\rangle$ for an economy $e$ if there does not exist any coalition $S$ and $m_S \in \prod_{i\in S} M_i$ such that for all $i \in S$,

$$X_i(m_S, m^*_{-S})\ P_i\ X_i(m^*). \tag{8}$$

$h(m^*)$ is then called a *strong Nash (equilibrium) allocation* of the mechanism for the economy $e$. Denote by $SV_{M,h}(e)$ the set of all such strong Nash equilibria and by $SN_{M,h}(e)$ the set of all such strong Nash (equilibrium) allocations.

A mechanism $\langle M, h\rangle$ is said to doubly *implement* the proportional solution on $E$, if, for all $e \in E$, $SN_{M,h}(e) = N_{M,h}(e) = PR(e)$.

A mechanism $\langle M, h\rangle$ is said to be *feasible* if, for all $m \in M$, (1) $X(m) \in \mathbb{R}^{nH}_+$, (2) $Y_g(m) \in \mathcal{Y}_g$ for $g = 1, F, C$, and (3) $\sum_{i=1}^n X_i(m) \leqq \sum_{i=1}^n w_i + w_C + \sum_{f=1}^F Y_f(m) + Y_C(m)$.

A mechanism $\langle M, h\rangle$ is said to be *continuous* if the outcome function $h$ is continuous on $M$.

## 3. IMPLEMENTATION OF THE PROPORTIONAL SOLUTION

### 3.1. *The Description of the Mechanism*

In the following we will present a feasible and continuous mechanism which doubly implements proportional allocations in Nash and strong Nash equilibria when preferences, endowments, and patterns of cooperations are unknown to the designer. It is well known by now that, to have feasible implementation, the designer has to know some information about individual endowments. That is, we have to require that an agent cannot overstate his own endowment although he can understate his own endowment. This requirement is necessary to guarantee the feasibility even at disequilibrium points. The intuition here is straightforward: if a mechanism allows agents to overstate their endowments, then it allows for unfeasible outcomes—it will sometimes attempt to allocate more than is possible, given the true aggregate endowment. Note that, when goods are physical goods, this requirement can be guaranteed by asking agents to *exhibit* their reported endowments to the designer. Since $w_C$ is publicly owned, we assume that $w_C$ is known to the designer.

The message space of the mechanism is defined as follows. For each $i \in N$, let the message domain of agent $i$ be of the form

$$M_i = (0, w_i^\rho] \times \Delta_{++}^{H-1} \times \mathbb{R}_{++}^{nH} \times \mathbb{R}^{(F+1)H} \times \mathcal{Y} \times \mathbb{R}_{++} \times (0, 1], \qquad (9)$$

where $\Delta_{++}^{H-1} = \{(p \in \mathbb{R}_{++}^H : \sum_{l=1}^H p^l = 1\}$, and $\mathcal{Y} = \prod_{f=1}^F \mathcal{Y}_f \times \mathcal{Y}_C$.

A generic element of $M_i$ is $m_i = (v_i, p_i, (z_{i1}, \ldots, z_{i,n}), (y_{i1}, \ldots, y_{iF}, y_{iC}), (t_{i1}, \ldots, t_{iF}, t_{iC}), \gamma_i, \eta_i)$ whose components have the following interpretations. The component $v_i$ denotes a profession of agent $i$'s endowment, and the inequality $0 < v_i \leqq w_i^\rho$ means that the agent cannot overstate his own endowment; on the other hand, the endowment can be understated, but the claimed endowment $v_i$ must be positive. The component $p_i$ is the price vector proposed by agent $i$ and is used as a price vector of agent $i-1$, where $i-1$ is read to be $n$ when $i = 1$. The component $z_i := (z_{i,1}, \ldots, z_{in})$ is consumption proposed by agent $i$ in which $z_{ij}$ is $j$'s consumption level of goods proposed by agent $i$. The component $y_i := (y_{i1}, \ldots, y_{iF}, y_{iC})$ is $i$'s proposed production plan of all firms, in which $y_{ig}$, $g = 1, \ldots, F, C$, is firm $g$'s production plan proposed by $i$. The component $t_i := (t_{i1}, \ldots, t_{iF}, t_{iC})$ is $i$'s reported points in production possibility sets of all firms, in which $t_{ig}$ is agent $i$'s reported point in the $g$th production possibility set. The component $\gamma_i$ is a shrinking index of agent $i$ used to shrink the consumption of other agents. The component $\eta_i$ is the penalty index of agent $i$ for avoiding some undesirable Nash equilibria.

*Remark* 2. It may be remarked that the dimension of the message space can be reduced in a number of ways. First, it will be seen that, by the construction of the proportional share function, each individual $i$'s proportional share of profit from the publicly owned firm is determined only by his neighbor $i+1$. It is not necessary for the mechanism designer to require each individual to announce goods consumptions for all other individuals. We define our mechanism as it is to keep exploration simple. Second, each individual only needs to announce $L-1$ goods consumption in group $\rho$ since the remaining one can be determined by the budget constraint, and thus the message space can be reduced by $n$. Third, when privately owned firms do not use public sector goods to produce private sector goods (i.e., $y_f^\mu = 0$ for all $f = 1, \ldots, F$), the proportional share becomes $\theta_{iC} = (p^\mu \cdot x_i^\mu)/(p^\mu \cdot \hat{x}^\mu)$, and thus the reporting $(t_{i1}, \ldots, t_{iF}, t_{iC})$ becomes unnecessary. Therefore, the message can be deducted by $nH(F+1)$. These remarks will become clear after we define the outcome function.

*Remark* 3. There are three reasons we require each individual $i$ to announce a proposed production plan $y_i$ which may not be in production possibility sets $\mathcal{Y}$ and a production plan $t_i$ in $\mathcal{Y}$. First, we want the mechanism to have the property that agent $i$ can reach any production plan within the feasible production set $B_y(m)$ defined below. At the same time, the component $y_f^{\mu}$ in his proportional share is determined by a feasible production plan in $B_y(m)$. Finally, $y_i^*$ is in $B_y(m^*)$ when $m^*$ is a Nash equilibrium so that $Y(m^*)$ is a relative interior point of $B_y(m^*)$ on $\mathcal{Y}$. As a result, the proportional shares have a form determined by Eq. (4) at Nash equilibrium (see Lemma 12 below).

Before we formally define the outcome function of the mechanism, we give a brief description and explain why the mechanism works. For each announced message $m \in M$, the designer first assigns a price function vector $p_i(m)$ to each individual $i$, in a way that is determined by the price vector announced by his neighbor, agent $i+1$. With this determined price function vector $p_i(m)$, the proportional share function of agent $i$ is determined by Eq. (4), but evaluated by $z_{i+1,\,i}^{\mu}$ and $t_{i+1,\,f}^{\mu}$, which are proposed by agent $i+1$. Thus, each individual takes his price vector and proportional shares as given and cannot change them by changing his own messages. We then define three feasible constrained choice sets. First, define the feasible constrained production set $B_y(m)$ by which goods can be produced with total resources available in the society. The outcome of production plan $Y(m)$ is chosen from $B_y(m)$ so that it is the closest to the average of the proposed production plans of firms by all agents. Then, to determine the preliminary outcome of consumption for agent $i$, construct two constrained budget sets $B_i^{\mu}(m)$ and $B_i^{\rho}(m)$ for public sector goods consumption and private sector goods consumption, respectively. First, determine the preliminary outcome for public sector goods consumption $x_i^{\mu}(m)$ such that it is the closest to his own proposed public sector goods consumption, $z_{ii}^{\mu}$. Then define $\bar{x}_i^{\rho}(m)$ as the closest point to his own proposed private sector goods consumption, $z_{ii}^{\rho}$. To give an incentive for all agents to announce the same price vector and consumption vector, as well as give an incentive for each agent to announce $y_i$ and $t_i$ as close as possible at equilibrium, the preliminary outcome for private sector goods consumption is determined by the product of $\bar{x}_i^{\rho}$ and a penalty multiplier (see Eq. (21) below). To obtain the feasible outcome consumption $X(m)$, we need to shrink the preliminary outcome consumption $x_i(m)$ in some way that will be specified below. We will show that a mechanism constructed in such a way will have the properties we desire. In addition, at equilibrium, all consumers maximize their preferences, all firms maximize their profits, all individuals take the prices of goods as given, and individuals' proportional shares are determined by Eq. (4). Thus the mechanism doubly implements the proportional solution.

Now we formally present the outcome function of the mechanism.

Define agent $i$'s price vector $p_i : M \to \mathbb{R}^H_{++}$ by

$$p_i(m) = p_{i+1}, \tag{10}$$

where $n+1$ is to be read as 1. Note that although $p_i(\cdot)$ is a function of the proposed price vector announced by agent $i+1$ only (so that every individual takes the prices of goods as given), for simplicity, we can write $p(\cdot)$ as a function of $m$ without loss of generality.

Define agent $i$'s (endogenous) proportional share function $\theta_{iC} : M \to \mathbb{R}_{++}$ by

$$\theta_{iC}(m) = \frac{p_i^{\mu}(m) \cdot z_{i+1,\,i}^{\mu} - \sum_{f=1}^{F} \theta_{if}\, p_i^{\mu}(m) \cdot t_{i+1,\,f}^{\mu}(m)}{p_i^{\mu}(m) \cdot \left[\sum_{j=1}^{n} z_{i+1,\,j}^{\mu} - \sum_{f=1}^{F} t_{i+1,\,f}^{\mu}\right]} \tag{11}$$

which is continuous because $\sum_{j=1}^{n} z_{i+1,\,j}^{\mu} - \sum_{f=1}^{F} t_{i+1,\,f}^{\mu}$ is positive and $p_i(\cdot)$ is continuous. Note that, by the construction of $p_{iC}(\cdot)$, $\theta_{iC}(\cdot)$ is independent of $m_i$ so that no individual can change his profit share by changing his messages.

To determine the level of production for each $g$, define the feasible production correspondence $B_y$: $M \to 2^{\mathbb{R}^{(F+1)H}}$ by

$$\begin{aligned} B_y(m) = \big\{ y \in \mathbb{R}^{(F+1)H} :\ & y_g \in \mathcal{Y}_g \ \forall g = 1, \ldots, F, C, \\ & \hat{v} + \hat{y}_P^{\rho} + y_C^{\rho} \geqq 0, \\ & \hat{y}_P^{\mu} + y_C^{\mu} + w_C^{\mu} \geqq 0, \\ & p_i(m) \cdot v_i + \sum_{f=1}^{F} \theta_{if}\, p_i(m) \cdot y_f(m) \\ & \quad + \theta_{iC}(m) p_i(m) \cdot [y_C(m) + w_C] \geqq 0 \big\} \end{aligned} \tag{12}$$

which is clearly nonempty compact convex (by the closedness and convexity of $\mathcal{Y}_g$ as well as total resources constraints) for all $m \in M$. We will show the following lemma in the Appendix.

LEMMA 2. *$B_y(\cdot)$ is continuous on $M$.*

Let $\tilde{y}_g = \frac{1}{n} \sum_{i=1}^{n} y_{ig}$ which is the average of the proposed production plans of firms, $\tilde{y}_P = (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_F)$, and $\tilde{y} = (\tilde{y}_P, \tilde{y}_C)$.

Define the outcome function for production plan $Y$: $M \to \mathbb{R}^{(F+1)H}$ by

$$Y(m) = \arg\min\{\| y - \tilde{y} \| : y \in B_y(m)\}, \tag{13}$$

which is the closest point to $\tilde{y}$. Here $\| \cdot \|$ is the Euclidian norm. Then, by Berge's Maximum Theorem (see Berge (1963, p.116)), we know that $Y(m)$ is an upper semicontinuous correspondence. Also, since $B_y(m)$ is closed and convex valued, it is also single-valued (see Mas-Colell (1985, p. 28)). Then $Y(m)$ is single-valued and continuous on $M$.

Define agent $i$'s proportional share function $\pi_{iC} : M \to \mathbb{R}_+$ by

$$\pi_{iC}(m) = \theta_{iC}(m) p_i(m) \cdot [Y_C(m) + w_C] \tag{14}$$

and the profit share function of agent $i$ from privately owned firms by

$$\pi_{iP}(m) = \sum_{f=1}^{F} \theta_{if} p_i(m) \cdot Y_f(m) \tag{15}$$

and the total profit share of agent $i$ from both privately and publicly owned firms

$$\pi_i(m) = \pi_{iP}(m) + \pi_{iC}(m), \tag{16}$$

which are all continuous.

To determine the level of consumption for each individual $i$, we first define the $\mu$-feasible consumption correspondence $B_i^\mu$: $M \to 2^{\mathbb{R}_+^K}$ by

$$\begin{aligned} B_i^\mu(m) = \{x_i^\mu \in \mathbb{R}_+^K\colon\ \hat{x}_i^\mu \leqq \hat{Y}_P^\mu(m) + Y_C^\mu(m) + w_C^\mu, \,\&\\ p_i^\mu(m) \cdot x_i^\mu \leqq p_i^\rho(m) \cdot v_i + \pi_i(m)\}, \end{aligned} \tag{17}$$

which is a correspondence with non-empty compact convex values. We will prove the following lemma in the Appendix.

LEMMA 3. *$B_i^\mu(\cdot)$ is continuous on $M$.*

Define $x_i^\mu$: $M \to \mathbb{R}_+^K$ by

$$x_i^\mu(m) = \arg\min\{\| x_i^\mu - z_{ii}^\mu \|: x_i^\mu \in B_i^\mu(m)\}, \tag{18}$$

which is the closest point to $z_{ii}$. Then, by the same reasoning, $x_i^\mu(m)$ is single-valued and continuous on $M$.

We then define the $\rho$-feasible consumption correspondence $B_i^\rho$: $M \to 2_+^{\mathbb{R}^L}$ by

$$\begin{aligned} B_i^\rho(m) = \{x_i^\rho \in \mathbb{R}_+^L\colon\ x_i^\rho \leqq \hat{v} + \hat{Y}_P^\rho(m) + Y_C^\rho(m), \,\&\\ p_i^\rho(m) \cdot x_i^\rho + p_i^\mu(m) \cdot x_i^\mu(m) \leqq p_i^\rho(m) \cdot v_i + \pi_i(m)\}, \end{aligned} \tag{19}$$

which is a correspondence with non-empty compact convex values. We will prove the following lemma in the Appendix.

LEMMA 4. *$B_i^\rho(\cdot)$ is continuous on $M$.*

Define $\bar{x}_i^\rho : M \to \mathbb{R}_+^L$ by

$$\bar{x}_i^\rho(m) = \arg\min\{\| \bar{x}_i^\rho - z_{ii}^\rho \|: \bar{x}_i^\rho \in B_i^\rho(m)\}, \tag{20}$$

which is the closest point to $z_{ii}^\rho$. Then, as above, we know that $\bar{x}_i^\rho(m)$ is single-valued and continuous on $M$.

Define $x_i^\rho$: $M \to \mathbb{R}_+^L$ by

$$x_i^\rho(m) = \frac{1}{1 + \eta_i(\|s_i - s_{i+1}\| + \|y_i - t_i\|)} \bar{x}_i^\rho(m), \tag{21}$$

where $s_i = (p_i, z_i, t_i)$.

Define the $\gamma$-correspondence $A$: $M \to 2^{\mathbb{R}_+}$ by

$$A(m) = \{\gamma \in \mathbb{R}_+ : \gamma\gamma_i \leq 1 \ \forall i \in N \& \gamma \sum_{i=1}^n \gamma_i x_i(m) \leqq \hat{w} + w_C + \hat{Y}_P(m) + Y_C(m)\}, \tag{22}$$

where $x_i(m) = (x_i^\rho(m), x_i^\mu(m))$.

Let $\bar{\gamma}(m)$ be the largest element of $A(m)$, i.e., $\bar{\gamma}(m) \in A(m)$ and $\bar{\gamma}(m) \geq \gamma$ for all $\gamma \in A(m)$.

Finally, define agent $i$'s outcome function for consumption goods $X_i$: $M \to \mathbb{R}_+^H$ by

$$X_i(m) = \bar{\gamma}(m)\gamma_i x_i(m), \tag{23}$$

which is agent $i$'s consumption resulting from the strategic configuration $m$.

Thus the outcome function $(X(m), Y(m))$ is continuous and feasible on $M$ since, by the construction of the mechanism, $(X(m), Y_P(m), Y_C(m)) \in \mathbb{R}_+^{nH} \times \mathbb{R}^{KH} \times \mathbb{R}^H$, $Y_g(m) \in \mathcal{Y}_g$ for all $g = 1, \ldots, F, C$, and

$$\hat{X}^\rho(m) \leqq \hat{w}^\rho + \hat{Y}_P^\rho(m) + Y_C^\rho(m)$$
$$\hat{X}^\mu(m) \leqq \hat{Y}_P^\mu(m) + Y_C^\mu(m) + w_C^\mu \tag{24}$$

for all $m \in M$.

*Remark* 4. Unlike the existing mechanisms such as those in Tian (1992) and Hong (1995), we have separately defined three feasible sets $B_y(m)$, $B_i^\mu(m)$, and $B_i^\rho(m)$ instead of just defining one feasible set in constructing the above mechanism. The reason for this is that we need the mechanism to be such that the outcome function for $Y(m^*)$ is a relative interior point of $B_y(m^*)$ on $\mathcal{Y}$ (see Lemma 9 below), $x^\mu(m^*)$ is an interior point of $B_i^\mu(m^*)$ so that $Y(m^*)$ equals the average of the proposed production plan $\tilde{y}^* = \frac{1}{n}\sum_{i=1}^n y_i^*$, and $x_i^\mu(m^*)$ equals his announced consumption proposal $z_{ii}^{*\mu}$ for all $i \in N$. We need these properties to determine the profit shares of agents from the publicly owned firm. This will become clear when we prove the implementation results.

*Remark* 5. Note that our mechanism works not only for three or more agents, but also for a two-agent world. While most existing mechanisms which implement market-type social choice correspondences (such as Walrasian, Lindahl, Ratio, or LCSE allocations) need to distinguish the case of two agents from that of three or more agents, this paper gives a unified mechanism which is irrespective of the number of agents.

*Remark* 6. If there are no publicly owned firms, i.e., $\mu = 0$, the mixed ownership economies reduce to private ownership economies, and the part of the mechanism for public sector goods becomes unnecessary. The simplified mechanism is therefore a new mechanism which doubly implements the Walrasian solution in private ownership economies.

### 3.2. *Implementation Result*

The remainder of this section is devoted to proving the following theorem.

THEOREM 1. *For the class of mixed ownership economic environments $E$ specified in Section* 2, *if the following assumptions are satisfied*:

(1) $n \geqq 2$;

(2) $w_C^{\mu} > 0$ *and* $w_i^{\rho} > 0$ *for all* $i \in N$;

(3) *For each* $i \in N$, *preference orderings*, $R_i$, *are strictly increasing*, *continuous*, *convex*, *and satisfy the Interiority Condition of Preferences*; *and*

(4) *For each* $g = 1, \ldots, F, C$, *the production set* $\mathcal{Y}_g$ *is nonempty*, *closed*, *convex*, *and* $0 \in \mathcal{Y}_g$.

*then the above-defined mechanism*, *which is continuous and feasible*, *doubly implements proportional allocations in Nash and strong Nash equilibria on $E$ when preferences*, *endowments*, *and patterns of cooperation are unknown to the designer.*[8]

*Proof.* The proof of Theorem 1 consists of the following three propositions which show the equivalence among Nash allocations, strong Nash allocations, and proportional allocations. Proposition 1 below proves that every Nash allocation is a proportional allocation. Proposition 2 below proves that every proportional allocation is a Nash allocation. Proposition 3 below proves that every Nash equilibrium allocation is a strong Nash equilibrium allocation.

To show these propositions, we first prove the following lemmas.

[8]Condition 2 can be weakened so that $\sum_{i \in N} w_i + w_C > 0$ if individual endowments $w_i$ are known to the designer.

LEMMA 5. *Suppose $x_i(m)P_ix_i$. Then agent i can choose a very large $\gamma_i$ such that $X_i(m)P_ix_i$.*

*Proof.* If agent $i$ declares a large enough $\gamma_i$, then $\bar{\gamma}(m)$ becomes very small (since $\bar{\gamma}(m)\gamma_i \leqq 1$) and thus almost nullifies the effect of other agents in $\bar{\gamma}(m)\sum_{i=1}^n \gamma_i x_i(m) \leqq (\hat{v},0) + w_C + \hat{Y}_P(m) + Y_C(m)$. Thus, $X_i(m) = \bar{\gamma}(m)\gamma_i x_i(m)$ can arbitrarily approach to $x_i(m)$ as agent $i$ wishes. From $x_i(m)P_ix_i$ and continuity of preferences, we have $X_i(m)P_ix_i$ if agent $i$ chooses a very large $\gamma_i$. Q.E.D.

LEMMA 6. *If $m^* \in V_{M,h}(e)$, then $X(m^*) \in \mathbb{R}^{nH}_{++}$.*

*Proof.* We argue by contradiction. Suppose $X(m^*) \in \partial\mathbb{R}^{nH}_{+}$. Then there is some $i \in N$ such that $X_i(m^*) \in \partial\mathbb{R}^{H}_{+}$. Since $p_i(m^*) > 0$, $v_i^* > 0$, and $\sum_{i=1}^n (v_i,0) + w_C > 0$, then there is some $x_i \in \mathbb{R}^H_{++}$ such that $p_i(m^*)\cdot x_i \leqq p_i(m^*)\cdot v_i^*$, $x_i < \sum_{i=1}^n (v_i,0) + w_C$, and $x_i\ P_i\ X_i(m^*)$ by interiority of preferences. Now suppose that agent $i$ chooses $y_i = -\sum_{j\neq i}^n y_j^*$, $z_{ii} = x_i$, $\gamma_i > \gamma_i^*$, and keeps other components of the message unchanged. Then $0 \in B_y(m_i, m_{-i}^*)$, $x_i^\mu \in B_i^\mu(m_i, m_{-i}^*)$, and $x_i^\rho \in B_i^\rho(m_i, m_{-i}^*)$. Thus $Y(m_i, m_{-i}^*) = 0$, and $x_i(m_i, m_{-i}^*) = (x_i^\rho(m_i, m_{-i}^*), x_i^\mu(m_i, m_{-i}^*)) = (1/(1+\eta_i^*(\|s_i - s_{i+1}^* + \|y_i - t_i^*\|))x_i^\rho, x_i^\mu) > 0$. Then, we have $x_i(m_i, m_{-i}^*)P_iX_i(m^*)$ by interiority of preferences. Therefore, by Lemma 5, $X_i(m_i, m_{-i}^*)\ P_i\ X_i(m^*)$ if agent $i$ chooses a very large $\gamma_i$. This contradicts $m^* \in V_{M,h}(e)$ and thus we must have $X_i(m^*) \in \mathbb{R}^H_{++}$ for all $i \in N$. Q.E.D.

LEMMA 7. *If $m^*$ is a Nash equilibrium, then $p_1^* = p_2^* = \cdots = p_n^*$, $z_1^* = z_2^* = \cdots = z_n^*$, and $y_1^* = y_2^* = \cdots = y_n^* = t_1^* = t_2^* = \cdots = t_n^*$. Consequently, $p_1(m^*) = p_2(m^*) = \cdots = p_n(m^*) := p(m^*)$.*

*Proof.* Suppose, by way of contradiction, that $y_i^* \neq t_i^*$ or $s_i^* \neq s_{i+1}^*$ for some $i \in N$. Then agent $i$ can choose a smaller $\eta_i < \eta_i^*$ in $[0,1]$ so that his consumption of the private sector goods in group $\rho$ becomes larger and he would be better off by monotonicity of preferences. Hence, no choice of $\eta_i$ could constitute part of a Nash equilibrium strategy when $y_i^* \neq t_i^*$ or $s_i^* \neq s_{i+1}^*$. Thus, we must have $s_1^* = s_2^* = \cdots = s_n^*$ and $y_i^* = t_i^*$ for all $i \in N$ at Nash equilibrium. Therefore, $p_1^* = p_2^* = \cdots = p_n^*$, $z_1^* = z_2^* = \cdots = z_n^*$, and $y_1^* = y_2^* = \cdots = y_n^* = t_1^* = t_2^* = \cdots = t_n^*$. Consequently, we have $p_1(m^*) = p_2(m^*) = \cdots = p_n(m^*) := p(m^*)$. Q.E.D.

LEMMA 8. *Suppose $X(m^*) \in \mathbb{R}^{nH}_{++}$ for some $m^* \in M$ and there is $x_i \in \mathbb{R}^H_{++}$ for some $i \in N$ such that $p(m^*)\cdot x_i \leqq p^\rho(m^*)\cdot v_i^{*\rho} + \pi_i(m^*)$ and $x_iP_iX_i(m^*)$. Then there is some $m_i \in M_i$ such that $X_i(m_i, m_{-i}^*)P_iX_i(m^*)$.*

*Proof.* Since $X(m^*) > 0$, $X_i(m^*) < \sum_{j\in N}(v_j^*, 0) + w_C + \sum_{f=1}^F Y_f(m^*) + Y_C(m^*)$. Let $x_{\lambda i} = \lambda x_i + (1-\lambda)X_i(m^*)$. Then, by convexity of preferences, we have $x_{\lambda i}\ P_i\ X_i(m^*)$ for any $0 < \lambda < 1$. Also $x_{\lambda i} \in \mathbb{R}^H_{++}$,

$p(m^*) \cdot x_{\lambda i} \leqq p^\rho(m^*) \cdot v_i^* + \pi_i(m^*)$, and $x_{\lambda i} < \sum_{j \in N}(v_j^*, 0) + w_C + \sum_{f=1}^F Y_f(m^*) + Y_C(m^*)$ when $\lambda$ is sufficiently close to 0. Now suppose agent $i$ chooses $z_{ii} = x_{\lambda i}$, $\gamma_i > \gamma_i^*$, $\eta_i < \eta_i^*$, and keeps other components of the message unchanged, then $Y(m^*) \in B_y(m_i, m_{-i}^*)$, $x_{\lambda i}^\mu \in B_i^\mu(m_i, m_{-i}^*)$, and $x_{\lambda i}^\rho \in B_i^\rho(m_i, m_{-i}^*)$. Thus we have $x_i^\mu(m_i, m_{-i}^*) = x_{\lambda i}^\mu$ and $x_i^\rho(m_i, m_{-i}^*) = 1/(1 + \eta_i[\|s_i - s_{i+1}^*\| + \|y_i^* - t_i^*\|])\bar{x}_i^\rho(m_i, m_{-i}^*) = 1/(1 + \eta_i[\|s_i - s_{i+1}^*\| + \|y_i^* - t_i^*\|])x_{\lambda i}^\rho$. Since $x_i^\rho(m_{-i}^*, m_i)$ can arbitrarily approach to $\bar{x}_i^\rho$ by choosing a sufficiently small $\eta_i$, from $x_i \ P_i \ X_i(m^*)$ and continuity of preferences, we have $x_i(m_i, m_{-i}^*) \ P_i \ X_i(m^*)$. Therefore, by Lemma 5, agent $i$ can choose a very large $\gamma_i$ such that $X_i(m_i, m_{-i}^*) \ P_i X_i(m^*)$. Q.E.D.

LEMMA 9. *If $m^*$ is a Nash equilibrium, then $v_i^* = w_i^\rho$ for all $i \in N$.*

*Proof.* Suppose, by way of contradiction, that $v_i^* \neq w_i^\rho$ for some $i \in N$. Then $p(m^*) \cdot X_i(m^*) \leqq p^\rho(m^*) \cdot v_i^* + \pi_i(m^*) < p^\rho(m^*) \cdot w_i^\rho + \pi_i(m^*)$, and thus there is some $x_i > X_i(m^*)$ such that $p(m^*) \cdot x_i \leqq p^\rho(m^*) \cdot w_i^\rho + \pi_i(m^*)$ and $x_i \ P_i \ X_i(m^*)$. Since $X(m^*) \in \mathbb{R}_{++}^{nH}$ by Lemma 6, there is some $m_i \in M_i$ such that $X_i(m_i, m_{-i}^*) \ P_i \ X_i(m^*)$ by Lemma 8. This contradicts $(X(m^*), Y(m^*)) \in N_{M,h}(e)$. Q.E.D.

LEMMA 10. *If $(X(m^*), Y(m^*)) \in N_{M,h}(e)$, then $p(m^*) \cdot X_i(m^*) = p^\rho(m^*) \cdot w_i^\rho + \pi_i(m^*)$. Consequently, the feasibility condition must hold with equality, i.e., $\hat{X}^\rho(m^*) = \hat{w}^\rho + \hat{Y}_P^\rho(m^*) + Y_C^\rho(m^*)$ and $\hat{X}^\mu(m^*) = \hat{Y}_P^\mu(m) + Y_C^\mu(m) + w_c$.*

*Proof.* Suppose, by way of contradiction, that $p(m^*) \cdot X_i(m^*) < p^\rho(m^*) \cdot w_i^\rho + \pi_i(m^*)$. Then, there is $x_i > X_i(m^*)$ such that $p(m^*) \cdot x_i \leqq p^\rho(m^*) \cdot w_i^\rho + \pi_i(m^*)$ and $x_i \ P_i \ X_i(m^*)$ by monotonicity of preferences. Since $X(m^*) \in \mathbb{R}_{++}^{nH}$ by Lemma 6, there is some $m_i \in M_i$ such that $X_i(m_i, m_{-i}^*) \ P_i \ X_i(m^*)$ by Lemma 8. This contradicts $(X(m^*), Y(m^*)) \in N_{M,h}(e)$. Consequently, the feasibility condition must hold with equality. Otherwise, there exists $i \in N$ such that $p(m^*) \cdot X_i(m^*) < p^\rho(m^*) \cdot w_i^\rho + \pi_i(m^*)$. Q.E.D.

LEMMA 11. *If $(X(m^*), Y(m^*)) \in N_{M,h}(e)$, then $\bar{\gamma}(m^*)\gamma_i^* = 1$ for all $i \in N$ and thus $X(m^*) = x(m^*)$.*

*Proof.* This is a direct corollary of Lemma 10. Suppose $\bar{\gamma}(m^*)\gamma_i^* < 1$ for some $i \in N$. Then $X_i(m^*) = \bar{\gamma}(m^*)\gamma_i^* x_i(m^*) < x_i(m^*)$, and therefore $p(m^*) \cdot X_i(m^*) < p^\rho(m^*) \cdot v_i^* + \pi_i(m^*)$. But this is impossible by Lemma 10. Q.E.D.

LEMMA 12. *If $(X(m^*), Y(m^*)) \in N_{M,h}(e)$, then $Y(m^*)$ is a relative interior point in $B_y(m^*)$ on $\mathcal{Y}$, $x_i^\mu(m^*) \in intB_i^\mu(m^*)$, and thus*

$Y(m^*) = \tilde{y}^* = \frac{1}{n}\sum_{i=1}^n y_i^* = y_1^* = \cdots = y_n^* = t_1^* = \cdots = t_n^*$, $x_i^\mu(m^*) = z_{ii}^{*\mu}$ *for all* $i \in N$. *Consequently,* $x^\mu(m^*) = z_1^{*\mu} = \cdots = z_n^{*\mu}$, *and thus*

$$\theta_{iC}(m^*) = \frac{p^\rho(m^*) \cdot X_i^\mu(m^*) - \sum_{f=1}^F \theta_{if}\, p^\rho(m^*) \cdot Y_f^\mu(m^*)}{p^\rho(m^*) \cdot [\sum_{j=1}^n X_j^{\mu*}(m^*) - \sum_{f=1}^F Y_f^\mu(m^*)]}$$

*and*

$$\sum_{i=1}^n \theta_{iC}(m^*) = 1.$$

*Proof.* We first prove $Y(m^*)$ is a relative interior point in $B_y(m^*)$ on $\mathcal{Y}$. Indeed, since $y_{ig}^* = t_{ig}^* \in \mathcal{Y}_g$ for all $g = 1, \ldots, F, C$ by Lemma 7 and $X(m^*) > 0$ by Lemma 6, we have $\hat{Y}(m^*) + \sum_{i=1}^n (v_i, 0) + w_C > 0$. Thus, by the definition of $Y(m)$, we must have $\tilde{y}^* \in B_y(m^*)$ which means that $\tilde{y}^*$ is a relative interior point in $B_y(m^*)$ on $\mathcal{Y}$, and therefore $Y(m^*) = \tilde{y}^* = \frac{1}{n}\sum_{i=1}^n y_i^* = y_1^* = \cdots = y_n^* = t_1^* = \cdots = t_n^*$.

Now we prove $x_i^\mu(m^*) \in intB_i^\mu(m^*)$. Suppose it is not true. Then $x_i^\mu(m^*) \in \partial B_i^\mu(m^*)$. Since $x_i(m^*) < \hat{Y}(m^*) + \sum_{i=1}^n (v_i, 0) + w_C$ by Lemma 10, we must have $p^\mu(m^*) \cdot x_i^\mu(m^*) = p^\rho(m^*) \cdot v_i^\rho + \pi_i(m^*)$ which implies that, by the construction of $x_i^\rho(m^*)$, $x_i^\rho(m^*) = 0$ for some $i \in N$. But this is impossible by Lemma 6. So $x_i^\mu(m^*) \in intB_i^\mu(m^*)$. Thus, $x_i^\mu(m^*) = z_{ii}^{*\mu}$ for all $i \in N$. Also, since $z_1^{*\mu} = \cdots = z_n^{*\mu}$ and $y_i^* = t_i^*$ for all $i \in N$ by Lemma 7, $z_1^{*\mu} = \cdots = z_n^{*\mu} = x^\mu(m^*) = X^\mu(m^*)$ (by Lemma 11) and $Y_f^\mu(m^*) = t_{if}^{*\mu}$ for all $i \in N$. Therefore, we have

$$\begin{aligned}\theta_{iC}(m^*) &= \frac{p^\mu(m^*) \cdot z_{i+1,i}^\mu - \sum_{f=1}^F \theta_{if}\, p^\mu(m^*) \cdot t_{i+1,f}^{\mu*}}{p^\mu(m^*) \cdot [\sum_{j=1}^n z_j^{*\mu} - \sum_{f=1}^F t_{i+1,f}^{\mu*}]} \\ &= \frac{p^\mu(m^*) \cdot X_i(m^*) - \sum_{f=1}^F \theta_{if}\, p^\mu(m^*) \cdot Y_f^\mu(m^*)}{p^\mu(m^*) \cdot [\sum_{j=1}^n X_j^{*\mu}(m^*) - \sum_{f=1}^F Y_f^\mu(m^*)]}. \qquad (25)\end{aligned}$$

And thus

$$\sum_{i=1}^n \theta_{iC}(m^*) = 1.$$

LEMMA 13. *If* $m^*$ *is a Nash equilibrium, then* $Y_g(m^*)$ *maximizes the profit of firm* $g$ *under* $p(m^*)$, *i.e.,* $p(m^*) \cdot Y_g(m^*) \geqq p(m^*) \cdot y_g$ *for all* $y_g \in \mathcal{Y}_g$ *and* $g = 1, \ldots, F, C$.

*Proof.* Suppose, by way of contradiction, that there is some $g$ and a production plan $y_g' \in \mathcal{Y}_g$ such that $p(m^*) \cdot Y_g(m^*) < p(m^*) \cdot y_g'$. Let $y_{g\lambda} = \lambda y_g' + (1 - \lambda) Y_g(m^*)$ with $0 < \lambda < 1$. Then $p(m^*) \cdot Y_g(m^*) < p(m^*) \cdot y_{g\lambda}$, and $y_{g\lambda} \in \mathcal{Y}_g$ by the convexity of $\mathcal{Y}_g$. Also, since

$X_i(m^*) < \sum_{j\in N}(v_j, 0) + w_C + \sum_{f=1}^{F} Y_f(m^*) + Y_C(m^*)$, we have $X_i(m^*) < \sum_{j\in N}(v_j, 0) + w_C + \sum_{l\neq g} Y_l(m^*) + y_{g\lambda}$ when $\lambda$ is sufficiently close to 0. Note that $\theta_{jC}(m^*) > 0$ for all $j \in N$. Since there is $i \in N$ such that $\theta_{ig} > 0$, let $\pi_{i\lambda}$ be the profit share of agent $i$ when $Y_g(m^*)$ is replaced by $y_{g\lambda}$ in $\pi_i(m^*)$. Then $\pi_{i\lambda} > \pi_i(m^*)$. Thus we have $p(m^*) \cdot X_i(m^*) \leqq p^\rho(m^*) \cdot w_i^\rho + \pi_i(m^*) < p^\rho(m^*) \cdot w_i^\rho + \pi_{i\lambda}$. Then there is $x_i > X_i(m^*)$ such that $p(m^*) \cdot x_i \leqq p^\rho(m^*) \cdot w_i^\rho + \pi_{i\lambda}$, $x_i \leqq \sum_{j\in N}(v_j, 0) + w_C + \sum_{l\neq g} Y_l(m^*) + y_{g\lambda}$, and $x_i \ P_i \ X_i(m^*)$ by monotonicity of preferences. Thus, if agent $i$ chooses $y_{ig} = ny_{\lambda g} - \sum_{j\neq i} y_{jg}^*$, $z_{ii} = x_i$, $\eta_i < \eta_i^*$, and keeps other components of the message unchanged, then $(y_{g\lambda}, Y_{-g}(m^*)) \in B_y(m_i, m_{-i}^*)$, $x_i^\mu \in B_i^\mu(m_i, m_{-i}^*)$, and $x_i^\rho \in B_i^\rho(m_i, m_{-i}^*)$. Thus we have $Y_g(m_i, m_{-i}^*) = y_{g\lambda}$, $x_i^\mu(m_i, m_{-i}^*) = x_i^\mu$ and $x_i^\rho(m_i, m_{-i}^*) = 1/(1 + \eta_i[\|s_i - s_{i+1}^*\| + \|y_i - t_i^*\|])\bar{x}_i^\rho(m_i, m_{-i}^*) = 1/(1 + \eta_i[\|s_i - s_{i+1}^*\| + \|y_i - t_i^*\|])x_i^\rho$. Since $x_i^\rho(m_{-i}^*, m_i)$ can arbitrarily approach to $\bar{x}_i^\rho$ by choosing a sufficiently small $\eta_i$. From $x_i \ P_i \ X_i(m^*)$ and continuity of preferences, we have $x_i(m_i, m_{-i}^*) \ P_i \ X_i(m^*)$. Since $\bar{\gamma}(m^*)\gamma_i^* = 1$ by Lemma 11, we have $X(m_i, m_{-i}^*) = x(m_i, m_{-i}^*)$ and thus $X_i(m_i, m_{-i}^*) \ P_i \ X_i(m^*)$. This contradicts $(X(m^*), Y(m^*)) \in N_{M,h}(e)$. Thus, $Y_g(m^*)$ must maximize the profit of the firm $g$ for all $g = 1, \ldots, F, C$. Q.E.D.

PROPOSITION 1. *If the mechanism defined above has a Nash equilibrium $m^*$, then the Nash allocation $(X(m^*), Y(m^*))$ is a proportional allocation with the efficiency price vector $p(m^*)$, i.e., $N_{M,h}(e) \subseteq PR(e)$.*

*Proof.* Let $m^*$ be a Nash equilibrium. We need to prove that $(X(m^*), Y(m^*))$ is a proportional solution with $p(m^*)$ as a price vector. By Lemma 1, we only need to show that it is a Walrasian allocation with endogenous profit shares determined by (4). Note that, by construction, the mechanism is feasible, and by Lemmas 2–10, $Y_g(m^*)$ maximizes the profit of firm $g$ for $g = 1, \ldots, F, C$, $p(m^*) > 0$, $\theta_{iC}(m^*) = (p^\rho(m^*) \cdot X_i^\mu(m^*) - \sum_{f=1}^{F} \theta_{if} p^\rho(m^*) \cdot Y_f^\mu(m^*))/(p^\rho(m^*) \cdot [\sum_{j=1}^{n} X_j^{\mu*}(m^*) - \sum_{f=1}^{F} Y_f^\mu(m^*)]) > 0$, $\sum_{i=1}^{n} \theta_{iC}(m^*) = 1$, and $p(m^*) \cdot X_i(m^*) = p^\rho(m^*) \cdot w_i^\rho + \pi_i(m^*)$ for all $i \in N$. So we only need to show that each individual is maximizing his/her preferences subject to his/her budget constraint.

Suppose, by way of contradiction, that there is some $x_i \in \mathbb{R}_+^H$ such that $x_i P_i X_i(m^*)$ and $p(m^*) \cdot x_i \leqq p^\rho(m^*) \cdot w_i^\rho + \pi_i(m^*)$. Since $X(m^*) \in \mathbb{R}_{++}^{nH}$ by Lemma 6, there is some $m_i \in M_i$ such that $X_i(m_i, m_{-i}^*) \ P_i \ X_i(m^*)$ by Lemma 8. This contradicts $(X(m^*), Y(m^*)) \in N_{M,h}(e)$. Thus, $(X(m^*), Y(m^*))$ satisfies all conditions of Lemma 1, and therefore it is a proportional allocation.

PROPOSITION 2. *If $(x^*, y^*)$ is a proportional allocation with $p^* \in \Delta_+^{H-1}$ as the price vector, then there is a Nash equilibrium $m^*$ such that $Y(m^*) = y^*$, $p(m^*) = p^*$, and $X_i(m^*) = x_i^*$ for $i \in N$, i.e., $PR(e) \subseteq N_{M,h}(e)$.*

*Proof.* We first note that $x^* \in \mathbb{R}^H_{++}$ by interiority of preferences. Also, by the strict monotonicity of preference orderings, the normalized price vector $p^*$ must be in $\Delta^{H-1}_{++}$. We need to show that there is a message $m^*$ such that $(x^*, y^*)$ is a Nash equilibrium allocation. For each $i \in N$, define $m_i^* = (v^*, p_i^*, (z_{i1}^*, \dots, z_{in}^*), (y_{i1}^*, \dots, y_{iF}^*, y_{iC}^*), (t_{i1}^*, \dots, t_{iF}^*, t_{iC}^*), \gamma_{i1}^*, \eta_i^*)$ by $v_i^* = w_i$, $p_i^* = p^*$, $(z_{i1}^*, \dots, z_{in}^*) = (x_1^*, x_2^*, \dots, x_n^*)$, $(y_{i1}^*, \dots, y_{iF}^*, y_{iC}^*) = (y_1^*, y_2^*, \dots, y_F^*, y_C^*)$, $(t_{i1}^*, \dots, t_{iF}^*, t_{iC}^*) = (y_1^*, y_2^*, \dots, y_F^*, y_C^*)$, $\gamma_i^* = 1$, and $\eta_i^* = 1$. Then, it can be easily verified that $Y(m^*) = y^*$, $p_i(m^*) = p^*$, $\theta_{iC}(m^*) = \theta_{iC}^*$, and $X_i(m^*) = x_i^*$ for all $i \in N$. Since $p(m_i, m_{-i}^*) = p_i(m^*)$, $\theta_{iC}(m_i, m_{-i}^*) = \theta_{iC}(m^*)$ for all $m_i \in M_i$, then for all $m_i \in M_i$,

$$\begin{aligned} p(m^*) \cdot X_i(m_i, m_{-i}^*) &\leqq p^\rho(m^*) \cdot w_i^\rho + \pi_i(m_i, m_{-i}^*) \\ &\leqq p^\rho(m^*) \cdot w_i^\rho + \pi_i(m^*) \end{aligned} \tag{26}$$

since $Y_g(m^*)$ maximizes the profit on $\mathcal{Y}_g$ for all $g = 1, \dots, F, C$ under the price vector $p(m^*)$ and $\theta_{iC}(m)$ is independent of $m_i$. Thus, $X_i(m_i, m_{-i}^*)$ satisfies the budget constraint for all $m_i \in M_i$. Thus, we must have $X_i(m^*) R_i X_i(m_i, m_{-i}^*)$, or it contradicts the fact that $(X(m^*), Y(m^*))$ is a proportional allocation. So $(X(m^*), Y(m^*))$ must be a Nash equilibrium allocation. Q.E.D.

PROPOSITION 3. *Every Nash equilibrium $m^*$ of the mechanism defined above is a strong Nash equilibrium, that is,* $N_{M,h}(e) \subseteq SN_{M,h}(e)$.

*Proof.* Let $m^*$ be a Nash equilibrium. By Proposition 1, we know that $(X(m^*), Y(m^*))$ is a proportional allocation with $p(m^*)$ as the price vector. Then $(X(m^*), Y(m^*))$ is Pareto optimal and thus the coalition $N$ cannot be improved upon by any $m \in M$. Now for any coalition $S$ with $\varnothing \neq S \neq N$, choose $i \in S$ such that $i + 1 \notin S$. Then no strategy played by $S$ can change $p(m)$ and $\theta_{iC}(m)$ since they are determined by $m_{i+1}$. Furthermore, because $(X(m^*), Y(m^*)) \in PR(e)$ and

$$\begin{aligned} p(m^*) \cdot X_i(m_S, m_{-S}^*) &\leqq p^\rho(m^*) \cdot w_i + \pi_i(m_S, m_{-S}^*) \\ &\leqq p^\rho(m^*) \cdot w_i^\rho + \pi_i(m^*), \end{aligned} \tag{27}$$

$X_i(m^*)$ is the maximal consumption in the budget set of $i$, and thus $S$ cannot improve upon $(X(m^*), Y(m^*))$. Q.E.D.

Since every strong Nash equilibrium is clearly a Nash equilibrium, by combining Propositions 1–3, we know that $SN_{M,h}(e) = N_{M,h}(e) = PR(e)$ for all $e \in E$ and thus the proof of Theorem 1 is completed. Q.E.D.

## 4. CONCLUDING REMARKS

In this paper we have considered the incentive aspect of the proportional solution for general mixed ownership economies which allow any number of private sector commodities and public sector commodities, any number of firms and individuals, and the coexistence of privately and publicly owned firms with general convex production possibility sets. We presented market-type mechanisms which doubly implement the proportional allocations when coalition patterns, preferences, and endowments are unknown to the designer. Two important reasons for preferring double implementation over Nash (social) implementation and strong Nash (social) implementation are: (1) The double implementation covers the case where agents in some coalitions may cooperate and in other coalitions may not, when such information is unknown to the designer. (2) This combining solution concept, which characterizes agents' strategic behavior, may bring about a state which takes advantage of both Nash (social) equilibrium and strong Nash (social) equilibrium, so that it may be easy to reach and hard to leave. The mechanisms constructed in the paper are well-behaved in the sense that they are feasible and continuous. In addition, unlike most mechanisms proposed in the literature, it gives a unified mechanism which is irrespective of the number of agents.

We now end the paper by mentioning some limitations of the paper. First, unlike the mechanisms proposed by Tian (1992) and Hong (1995), which Nash implement the Walrasian correspondence, the mechanism constructed in the paper does not meet the so-called non-wastefulness and best response properties which require that all resulting allocations be balanced (but not merely weakly balanced) and that each participant have a best response for every strategy profile of the other participants even for non-equilibrium messages (Jackson et al., 1994). The requirements of double implementation and the endogenous profit distributions make the case of the proportional solution more difficult than that of the Walrasian in constructing a feasible and continuous mechanism which meets the above properties. Whether or not there exists a non-wastefulness and best response mechanism which doubly implements the proportional allocations in Nash and strong Nash equilibrium is a hard question and remains to be solved.

Second, this paper assumes that all agents have complete information about economic environments. Similar to Bayesian implementation of rational expectations Walrasian or Lindahl equilibria studied by Wettstein (1990) and Tian (1996), one may consider (double) implementation of proportional allocations in Bayesian equilibrium (and strong Bayesian equilibrium) by using some of the techniques developed in this paper.

## APPENDIX

*Proof of Lemma 2.* $B_y(m)$ has closed graph by the closedness of $\mathcal{Y}_g$ $(j = 1, \ldots, F, C)$ and the continuity of outcome functions. Since the range space of the correspondence $B_y(\cdot)$ is bounded by the resource constraint, it is compact. Thus, $B_y(\cdot)$ is upper hemi-continuous on $M$. We only need to show that $B_y(m)$ is also lower hemi-continuous at every $m \in M$. Let $m \in M$, $y \in B_y(m)$, and let $\{m(k)\}$ be a sequence such that $m(k) \to m$, where $m(k) = (m_1(k), \ldots, m_n(k))$ and $m_i(k) = (v_i(k), p_i(k), z_i(k), y_i(k), t_i(k), \gamma_i(k), \eta_i(k))$. We want to prove that there is a sequence $\{y(k)\}$ such that $y(k) \to y$, and, for all $k$ larger than a certain integer, $y(k) \in B_y(m(k))$, i.e., $y_g(k) \in \mathscr{Y}_g$ for $g = 1, \ldots, F, C$, $\hat{v}(k) + \hat{y}_P^\rho(k) + \hat{y}_C^\rho(k) \geqq 0$, $w_C^\mu + \hat{y}_P^\mu(k) + \hat{y}_C^\mu(k) \geqq 0$, and $p_i(m(k)) \cdot v_i + \sum_{f=1}^F \theta_{if} p_i(m(k)) \cdot y_f(k) + \theta_{iC}(k) p_i(m(k)) \cdot [y_C(k) + w_C] \geqq 0$. We first prove that there is a sequence $\{y_1(k)\}$ such that $y_1(k) \to y$, and, for all $k$ larger than a certain integer, $y_{1g}(k) \in \mathscr{Y}_g$ for $g = 1, \ldots, F, C$, $\hat{v}(k) + \hat{y}_{1P}^\rho(k) + \hat{y}_{1C}^\rho(k) \geqq 0$ and $w_C^\mu + \hat{y}_{1P}^\mu(k) + \hat{y}_{1C}^\mu(k) \geqq 0$.

Let $\mathscr{L} = \{l : 1 \leqq l \leqq L \& \hat{v}^l + \hat{y}_P^{\rho l} + \hat{y}_C^{\rho l} = 0\}$. Two cases will be considered.

*Case* 1. $\mathscr{L} = \emptyset$, i.e., $\hat{v}^l + \hat{y}_P^{\rho l} + \hat{y}_C^{\rho l} > 0$ for all $l = 1, \ldots, L$. Then, for all $k$ larger than a certain integer $k'$, we have $\hat{v}(k) + \hat{y}_P^\rho + \hat{y}_C^\rho > 0$, and $w_C^\mu + \hat{y}_P^\mu + \hat{y}_C^\mu \geqq 0$. Let $y_1(k) = y$ for all $k > k'$ and $y_1(k) = 0$ for $k \leqq k'$. Then, $y_1(k) \to y$, and, for all $k$, $y_1(k) \in \mathscr{Y}_g$ for $g = 1, \ldots, F, C$, $\hat{v}(k) + \hat{y}_{1P}^\rho(k) + \hat{y}_{1C}^\rho(k) \geqq 0$, and $w_C^\mu + \hat{y}_{1P}^\mu(k) + \hat{y}_{1C}^\mu(k) \geqq 0$.

*Case* 2. $\mathscr{L} \neq \emptyset$. Then $\hat{v}^l + \hat{y}_P^{\rho l} + \hat{y}_C^{\rho l} = 0$ for all $l \in \mathscr{L}$. Note that since $\hat{v}^l > 0$, we must have $\hat{y}_P^{\rho l} + \hat{y}_C^{\rho l} < 0$ for all $l \in \mathscr{L}$. For each $k$ and each $l \in \mathscr{L}$, let

$$\lambda^l(k) = \begin{cases} \frac{-\hat{v}^l(k)}{\hat{y}_P^{\rho l} + \hat{y}_C^{\rho l}} & \text{if } \hat{v}^l(k) + \hat{y}_P^{\rho l} + \hat{y}_C^{\rho l} \leqq 0 \\ 1 & \text{otherwise} \end{cases}$$

Then, for all $l \in \mathscr{L}$, $0 \leqq \lambda^l(k) \leqq 1$, $\lambda^l(k) \to 1$, and $\hat{v}^l(k) + \lambda^l(k)[\hat{y}_P^{\rho l} + \hat{y}_C^{\rho l}] \geqq 0$.

Let $\lambda(k) = \min_{l \in \mathscr{L}}\{\lambda^l(k)\}$, and let $y_1(k) = \lambda(k) y$. Then $y_1(k) \to y$. Since $\mathscr{Y}_g$ is convex and $0 \in \mathscr{Y}_g$ as well as $y_g \in \mathscr{Y}_g$, we have $y_{1g}(k) \in \mathscr{Y}_g$ for all $g = 1, \ldots, F, C$ and all $k$. Also, for every good $l$ in group $\rho$, if $l \in \mathscr{L}$, we have $\hat{v}^l(k) + \hat{y}_{1P}^{\rho l}(k) + \hat{y}_{1C}^{\rho l}(k) = \hat{v}^l(k) + \lambda(k)[\hat{y}_P^{\rho l} + \hat{y}_C^{\rho l}] \geqq \hat{v}^l(k) + \lambda^l(k)[\hat{y}_P^{\rho l} + \hat{y}_C^{\rho l}] \geqq 0$ by noting that $0 \leqq \lambda^l(k) \leqq 1$ and $\hat{y}_P^{\rho l} + \hat{y}_C^{\rho l} < 0$. If $l \notin \mathscr{L}$, we then have $\hat{v}^l(k) + \hat{y}_{1P}^{\rho l}(k) + \hat{y}_{1C}^{\rho l}(k) = \hat{v}^l(k) + \lambda(k)[\hat{y}_P^{\rho l} + \hat{y}_C^{\rho l}] > 0$ for all $k$ larger than a certain integer $k'$ by noting $\lambda^l(k) \to 1$ and $\hat{v}^l + \hat{y}_P^{\rho l} + \hat{y}_C^{\rho l} > 0$.

For every good $l$ in group $\mu$, if $\hat{y}_P^{\mu l} + \hat{y}_C^{\mu l} < 0$, we have $w_C^{\mu l} + \hat{y}_{1P}^{\mu l}(k) + \hat{y}_{1C}^{\mu l}(k) = w_C^{\mu l} + \lambda(k)[\hat{y}_P^{\mu l} + \hat{y}_C^{\mu l}] \geqq w_C^{\mu l} + \hat{y}_P^{\mu l} + \hat{y}_C^{\mu l} \geqq 0$. If $\hat{y}_P^{\mu l} + \hat{y}_C^{\mu l} > 0$, we have $w_C^{\mu l} + \hat{y}_{1P}^{\mu l}(k) + \hat{y}_{1C}^{\mu l}(k) = w_C^{\mu l} + \lambda(k)[\hat{y}_P^{\mu l} + \hat{y}_C^{\mu l}] > 0$ for all $k$ larger than a certain integer $k''$ by noting $\lambda^l(k) \to 1$.

Therefore, for $k \geqq \max\{k', k''\}$, we have $\hat{v}(k) + \hat{y}^{\rho}_{1P}(k) + \hat{y}^{\rho}_{1C}(k) \geqq 0$, and $w^{\mu}_C + \hat{y}^{\mu}_{1P}(k) + \hat{y}^{\mu}_{1C}(k) \geqq 0$.

Now we show that there is a sequence $\{y_2(k)\}$ such that $y_2(k) \to y$, and, for all $k$ larger than a certain integer, $y_2(k) \in \mathcal{Y}_g$ for $g = 1, \ldots, F, C$, and $p_i(m(k)) \cdot v_i + \sum_{f=1}^F \theta_{if} p_i(m(k)) \cdot y_{2f}(k) + \theta_{iC}(k) p_i(m(k)) \cdot [y_{2C}(k) + w_C] \geqq 0$.

Let $N' = \{i \in N : p_i(m) \cdot v_i + \sum_{f=1}^F \theta_{if} p_i(m) \cdot y_f + \theta_{iC}(m) p_i(m) \cdot [y_C + w_C] = 0\}$. Two cases will be considered.

*Case* 1. $N' = \emptyset$, i.e., for all $i \in N$, $p_i(m) \cdot v_i + \sum_{f=1}^F \theta_{if} p_i(m) \cdot y_f + \theta_{iC}(m) p_i(m) \cdot [y_C + w_C] > 0$. Then, for all $k$ larger than a certain integer $k'$, we have $p_i(m(k)) \cdot v_i + \sum_{f=1}^F \theta_{if} p_i(m(k)) \cdot y_f + \theta_{iC}(m(k)) p_i(m(k)) \cdot [y_C) + w_C] > 0$ by the continuity of outcome functions. Let $y_2(k) = y$ for all $k > k'$ and $y_2(k) = 0$ for $k \leqq k'$. Then, $y_2(k) \to y$, and, for all $k$, $y_{2g}(k) \in \mathcal{Y}_g$ for $g = 1, \ldots, F, C$, and $p_i(m(k)) \cdot v_i + \sum_{f=1}^F \theta_{if} p_i(m(k)) \cdot y_{2f}(k) + \theta_{iC}(m(k)) p_i(m(k)) \cdot [y_{2C}(k) + w_C] > 0$.

*Case* 2. $N' \neq \emptyset$. Then $p_i(m) \cdot v_i + \sum_{f=1}^F \theta_{if} p_i(m) \cdot y_f + \theta_{iC}(m) p_i(m) \cdot [y_C + w_C] = 0$ for all $i \in N'$. Note that since $p_i(m) \cdot v_i > 0$, we must have $\sum_{f=1}^F \theta_{if} p_i(m) \cdot y_f + \theta_{iC}(m) p_i(m) \cdot [y_C + w_C] < 0$, and thus, by the continuity of $p_i(\cdot)$ and $\theta_{iC}(\cdot)$, $\sum_{f=1}^F \theta_{if} p_i(m(k)) \cdot y_f + \theta_{iC}(m(k)) p_i(m(k)) \cdot [y_C + w_C] < 0$ for all $k$ larger than a certain integer $k'$. For each $k \geqq k'$ and each $i \in N'$, let $\omega(m(k)) = \sum_{f=1}^F \theta_{if} p_i(m(k)) \cdot y_f + \theta_{iC}(m(k)) p_i(m(k)) \cdot [y_C + w_C]$, and let

$$\alpha_i(k) = \begin{cases} -\frac{p_i(m(k)) \cdot v_i(k)}{\omega(m(k))} & \text{if } p_i(m(k)) \cdot v_i(k) + \omega(m(k)) \leqq 0 \\ 1 & \text{otherwise} \end{cases}$$

Then, for all $i \in N'$, $0 \leqq \alpha_i(k) \leqq 1$ and $\alpha_i(k) \to 1$.

Let $\alpha(k) = \min_{i \in N'}\{\alpha_i(k)\}$, and let $y_2(k) = \alpha(k) y$. Then $y_2(k) \to y$, and by the convexity of $\mathcal{Y}_g$, $y_{2g}(k) \in \mathcal{Y}_g$ for all $g = 1, \ldots, F, C$ and all $k$. Now we claim that $y_2(k)$ also satisfies $p_i(m(k)) \cdot v_i(k) + \sum_{f=1}^F \theta_{if} p_i(m(k)) \cdot y_{2f}(k) + \theta_{iC}(m(k)) p_i(m(k)) \cdot [y_{2C}(k) + w_C] \geqq 0$ for all $k \geqq k'$. Indeed, for each $i \in N'$, by the definition of $y_2(k)$, $p_i(m(k)) \cdot v_i(k) + \sum_{f=1}^F \theta_{if} p_i(m(k)) \cdot y_{2f}(k) + \theta_{iC}(m(k)) p_i(m(k)) \cdot [y_{2C}(k) + w_C] \geqq 0$.

For all $i \in N \setminus N'$, since $p_i(m) \cdot v_i + \sum_{f=1}^F \theta_{if} p_i(m) \cdot y_f + \theta_{iC}(m) p_i(m) \cdot [y_C + w_C] > 0$, we have $p_i(m(k)) \cdot v_i(k) + \sum_{f=1}^F \theta_{if} p_i(m(k)) \cdot y_{2f}(k) + \theta_{iC}(m(k)) p_i(m(k)) \cdot [y_{2C}(k) + w_C] > 0$ for all $k$ larger than a certain integer $k''$ by the continuity of outcome functions and by $y_2(k) \to y$. Thus, for all $k \geqq \max\{k', k''\}$ and all $i \in N$, we have shown that $p_i(m(k)) \cdot v_i(k) + \sum_{f=1}^F \theta_{if} p_i(m(k)) \cdot y_{2f}(k) + \theta_{iC}(m(k)) p_i(m(k)) \cdot [y_{2C}(k) + w_C] \geqq 0$.

Finally, let $y(k) = \min(y_1(k), y_2(k))$. Then $y(k) \to y$ since $y_1(k) \to y$ and $y_2(k) \to y$. Also, when $k$ is sufficiently larger than a certain integer,

$y_g(k) \in \mathcal{Y}_g$ by noting that $y_{1g}(k) \in \mathcal{Y}_g$ and $y_{2g}(k) \in \mathcal{Y}_g$ for $g = 1, \ldots, F, C$, $\hat{v}(k) + \hat{y}_P^\rho(k) + \hat{y}_C^\rho(k) \geqq 0$, $w_C^\mu + \hat{y}_P^\mu(k) + \hat{y}_C^\mu(k) \geqq 0$, and $p_i(m(k)) \cdot v_i + \sum_{f=1}^F \theta_{if} p_i(m(k)) \cdot y_f(k) + \theta_{iC}(k) p_i(m(k)) \cdot [y_C(k) + w_C] \geqq 0$ since, by the same arguments as above, one can show that $\hat{v}(k) + \hat{y}_{2P}^\rho(k) + \hat{y}_{2C}^\rho(k) \geqq 0$, $w_C^\mu + \hat{y}_{2P}^\mu(k) + \hat{y}_{2C}^\mu(k) \geqq 0$, and $p_i(m(k)) \cdot v_i + \sum_{f=1}^F \theta_{if} p_i(m(k)) \cdot y_{1f}(k) + \theta_{iC}(k) p_i(m(k)) \cdot [y_{1C}(k) + w_C] \geqq 0$. Thus, $y(k) \in B_y(m(k))$ for all $k$ sufficiently larger than a certain integer. Therefore, the sequence $\{y(k)\}$ has all the desired properties. So $B_y(m)$ is lower hemi-continuous at every $m \in M$. Q.E.D.

*Proof of Lemma* 3. $B_i^\mu(m)$ is upper hemi-continuous for all $m \in M$ by the continuity of the outcome function and the compactness of the range space of the correspondence. We only need to show that $B_i^\mu(m)$ is also lower hemi-continuous at every $m \in M$. Let $m \in M$, $x_i^\mu \in B_i^\mu(m)$, and let $\{m(k)\}$ be a sequence such that $m(k) \to m$, where $m(k) = (m_1(k), \ldots, m_n(k))$ and $m_i(k) = (v_i(k), p_i(k), z_i(k), y_i(k), t_i(k), \gamma_i(k), \eta_i(k))$. We want to prove that there is a sequence $\{x_i^\mu(k)\}$ such that $x_i^\mu(k) \to x_i^\mu$, and, for all $k$ larger than a certain integer, $x_i^\mu(k) \in B_i^\mu(m(k))$, i.e., $x_i^\mu(k) \in \mathbb{R}_+^K$, $\hat{x}_i^\mu(k) \leqq \hat{Y}_P^\mu(m(k)) + Y_C^\mu(m(k)) + w_C^\mu$, and $p_i^\mu(m(k)) \cdot x_i^\mu(k) \leqq I(m(k))$, where $I(m(k)) = p_i^\rho(m(k)) \cdot v_i(k) + \pi_i(m(k))$. We first prove that there is a sequence $\{x_{1i}^\mu(k)\}$ such that $x_{1i}^\mu(k) \to x_i^\mu$, and, for all $k$ larger than a certain integer, $x_{1i}^\mu(k) \in \mathbb{R}_+^K$, and $p_i^\mu(m(k)) \cdot x_{1i}^\mu(k) \leqq I(m(k))$. Two cases will be considered.

*Case* 1. $p_i^\mu(m) \cdot x_{1i}^\mu < I(m)$. Then, for all $k$ larger than a certain integer $k'$, we have $p_i^\mu(m(k)) \cdot x_{1i}^\mu < I(m(k))$ by the continuity of $p^\mu(\cdot)$ and $I(\cdot)$. Let $x_{1i}^\mu(k) = x_i^\mu$ for all $k > k'$ and $x_{1i}^\mu(k) = 0$ for $k \leqq k'$. Then, $x_{1i}^\mu(k) \to x_i^\mu$, and, for all $k$, $x_{1i}^\mu(k) \in \mathbb{R}_+^K$, and $p_i^\mu(m(k)) \cdot x_{1i}^\mu(k) \leqq I(m(k))$.

*Case* 2. $p_i^\mu(m) \cdot x_{1i}^\mu = I(m)$. Note that, since $I(m) > 0$, we must have $p_i^\mu(m) \cdot x_{1i}^\mu > 0$, and thus, by the continuity of $p_i(\cdot)$ and $I(\cdot)$, $p_i^\mu(m(k)) \cdot x_{1i}^\mu > 0$ for all $k$ larger than a certain integer $k'$. For each $k \geqq k'$, define $x_{1i}^\mu(k)$ as follows:

$$x_{1i}^\mu(k) = \begin{cases} \frac{I(m(k))}{p_i^\mu(m(k)) \cdot x_i^\mu} x_i^\mu & \text{if } p_i^\mu(m(k)) \cdot x_{1i}^\mu \geqq I(m(k)) \\ x_i^\mu & \text{otherwise} \end{cases}$$

Then, by the construction of $x_{1i}^\mu(k)$, $x_{1i}^\mu(k) \leqq x_{1i}^\mu$, $x_{1i}^\mu(k) \to x_{1i}^\mu$, and $p_i^\mu(m(k)) \cdot x_{1i}^\mu(k) \leqq I(m(k))$ for all $k$.

We now show that there is a sequence $\{x_{2i}^\mu(k)\}$ such that $x_{2i}^\mu(k) \to x_i^\mu$, and, for all $k$ larger than a certain integer, $x_{2i}^\mu(k) \in \mathbb{R}_+^K$, $\hat{x}_{2i}^\mu(k) \leqq \hat{Y}_P^\mu(m(k)) + Y_C^\mu(m(k)) + w_C^\mu$. For each good $l$ in group $\mu$, if $\hat{Y}_P^{\mu l}(m(k)) + Y_C^{\mu l}(m(k)) + w_C^{\mu l} = 0$, we must have $x_i^{\mu l} = 0$ by the construction of $B_i^\mu(m)$. Let $x_{2i}^{\mu l}(k) = 0$

for all $k$. Then it is clear $x_{2i}^{\mu l}(k) \to x_i^{\mu l}$ and $\hat{x}_{2i}^{\mu l}(k) = 0 \leqq \hat{Y}_P^{\mu l}(m(k)) + Y_C^{\mu l}(m(k)) + w_C^{\mu l}$.

So we suppose $\hat{Y}_P^{\mu l}(m(k)) + Y_C^{\mu l}(m(k)) + w_C^{\mu l} > 0$. Two cases will be considered.

*Case i.* $\hat{x}_i^{\mu l} < \hat{Y}_P^{\mu l}(m) + Y_C^{\mu l}(m) + w_C^{\mu l}$. Then, by the continuity of $Y_P^{\mu l}(m)$ and $Y_C^{\mu l}(m)$, we have $\hat{x}_i^{\mu l} < \hat{Y}_P^{\mu l}(m(k)) + Y_C^{\mu l}(m(k)) + w_C^{\mu l}$ for all $k$ larger than a certain integer $k'$. Let $x_{2i}^{\mu l}(k) = x_i^{\mu l}$ for all $k > k'$ and $x_{2i}^{\mu l}(k) = 0$ for $k \leqq k'$. Then, $x_{2i}^{\mu l}(k) \to x_i^{\mu l}$, and, for all $k$, and $\hat{x}_{2i}^{\mu l}(k) \leqq \hat{Y}_P^{\mu l}(m(k)) + Y_C^{\mu l}(m(k)) + w_C^{\mu l}$.

*Case ii.* $\hat{x}_i^{\mu l} = \hat{Y}_P^{\mu l}(m) + Y_C^{\mu l}(m) + w_C^{\mu l}$. Define $x_{2i}^{\mu l}(k)$ as follows:

$$x_{2i}^{\mu l}(k) = \begin{cases} \hat{Y}_P^{\mu l}(m(k)) + Y_C^{\mu l}(m(k)) + w_C^{\mu l} & \text{if } x_{2i}^{\mu l} > \hat{Y}_P^{\mu l}(m(k)) \\ & \quad + Y_C^{\mu l}(m(k)) + w_C^{\mu l} \\ x_i^{\mu l} & \text{otherwise} \end{cases}$$

Then $x_{2i}^{\mu l}(k) \to x_i^{\mu l}$ and $\hat{x}_{2i}^{\mu l}(k) \leqq \hat{Y}_P^{\mu l}(m(k)) + Y_C^{\mu l}(m(k)) + w_C^{\mu l}$ by the construction of $x_{2i}^{\mu l}(k)$.

Thus, in both cases, there is a sequence $\{x_{2i}^{\mu l}(k)\}$ such that $x_{2i}^{\mu l}(k) \to x_i^{\mu l}(k)$, and, for all $k$ larger than a certain integer, and $\hat{x}_{2i}^{\mu l}(k) \leqq \hat{Y}_P^{\mu l}(m(k)) + Y_C^{\mu l}(m(k)) + w_C^{\mu l}$.

Finally, let $x_i^{\mu}(k) = \min(x_{1i}^{\mu}(k), x_{2i}^{\mu}(k))$. Then $x_i^{\mu}(k) \to x_i^{\mu}$ since $x_{1i}^{\mu}(k) \to x_i^{\mu}$ and $x_{2i}^{\mu}(k) \to x_i^{\mu}$. Also, for all $k$ sufficiently larger than a certain integer, $x_i^{\mu}(k) \in \mathbb{R}_+^K$, $\hat{x}_i^{\mu}(k) \leqq \hat{Y}_P^{\mu}(m(k)) + Y_C^{\mu}(m(k)) + w_C^{\mu}$, and $p_i^{\mu}(m * k)) \cdot x_i^{\mu}(k) \leqq I(m(k))$, thus $x_i^{\mu}(k) \in B_i^{\mu}(m(k))$. Therefore, the sequence $\{x_i^{\mu}(k)\}$ has all the desired properties. So $B_i^{\mu}(m)$ is lower hemi-continuous at every $m \in M$. Q.E.D.

*Proof of Lemma* 4. $B_i^{\rho}(m)$ is clearly upper hemi-continuous for all $m \in M$ by the continuity of the outcome function and the compactness of the range space of the correspondence. We only need to show that $B_i^{\rho}(m)$ is also lower hemi-continuous at every $m \in M$. Let $m \in M$, $x_i^{\rho} \in B_i^{\rho}(m)$, and let $\{m(k)\}$ be a sequence such that $m(k) \to m$, where $m(k) = (m_1(k), \ldots, m_n(k))$ and $m_i(k) = (v_i(k), p_i(k), z_i(k), y_i(k), t_i(k), \gamma_i(k), \eta_i(k))$. We want to prove that there is a sequence $\{x_i^{\rho}(k)\}$ such that $x_i^{\rho}(k) \to x_i^{\rho}$, and, for all $k$ larger than a certain integer, $x_i^{\rho}(k) \in B_i^{\rho}(m(k))$, i.e., $x_i^{\rho}(k) \in \mathbb{R}_+^L$, $\hat{x}_i^{\rho}(k) \leqq \hat{v}(k) + \hat{Y}_P^{\rho}(m(k)) + Y_C^{\rho}(m(k)) + w_C^{\rho}$, and $p_i^{\rho}(m(k)) \cdot x_i^{\rho}(k) + p_i^{\mu}(m(k)) \cdot x_i^{\mu}(m(k)) \leqq I(m(k))$, where $I(m(k)) = p_i^{\rho}(m(k)) \cdot v_i(k) + \pi_i(m(k))$. We first prove that there is a sequence $\{x_{1i}^{\rho}(k)\}$ such that $x_{1i}^{\rho}(k) \to x_i^{\rho}$, and, for all $k$ larger than a certain integer, $x_{1i}^{\rho}(k) \in \mathbb{R}_+^L$, and $p_i^{\rho}(m(k)) \cdot x_{1i}^{\rho}(k) + p_i^{\mu}(m(k)) \cdot x_i^{\mu}(m(k)) \leqq I(m(k))$. Two cases will be considered.

*Case* 1. $p_i^\rho(m) \cdot x_{1i}^\rho + p_i^\mu(m) \cdot x_i^\mu(m) < I(m)$. Then, for all $k$ larger than a certain integer $k'$, we have $p_i^\rho(m(k)) \cdot x_i^\rho + p_i^\mu(m(k)) \cdot x_i^\mu(m(k)) < I(m(k))$ by the continuity of $p^\rho(\cdot)$, $x_i^\mu(\cdot)$ and $I(\cdot)$. Let $x_{1i}^\rho(k) = x_i^\rho$ for all $k > k'$ and $x_{1i}^\rho(k) = 0$ for $k \leqq k'$. Then, $x_{1i}^\rho(k) \to x_i^\rho$, and, for all $k$, $x_{1i}^\rho(k) \in \mathbb{R}_+^K$, and $p_i^\rho(m(k)) \cdot x_{1i}^\rho(k) + p_i^\mu(m(k)) \cdot x_i^\mu(m(k)) \leqq I(m(k))$.

*Case* 2. $p_i^\rho(m) \cdot x_i^\rho + p_i^\mu(m) \cdot x_i^\mu(m) = I(m)$. If $I(m) - p_i^\mu(m) \cdot x_i^\mu(m) = 0$, then we must have $x_i^\rho = 0$. Let $x_{1i}^\rho(k) = 0$ for all $k$. Then it is clear $x_{1i}^\rho(k) \to x_i^\rho$ and $p_i^\rho(m(k)) \cdot x_{1i}^\rho(k) + p_i^\mu(m(k)) \cdot x_i^\mu(m(k)) \leqq I(m(k))$. So we suppose that $I(m) - p_i^\mu(m) \cdot x_i^\mu(m) > 0$. Then $p_i^\rho(m) \cdot x_i^\rho > 0$, and thus $p_i^\rho(m(k)) \cdot x_i^\rho > 0$ for all $k$ larger than a certain integer $k'$ by the continuity of $p_i(\cdot)$. For each $k \geqq k'$, define $x_{1i}^\rho(k)$ as follows:

$$x_{1i}^\rho(k) = \begin{cases} \frac{I(m(k)) - p_i^\mu(m(k)) \cdot x_i^\mu(m(k))}{p_i^\rho(m(k)) \cdot x_i^\rho} x_i^\rho & \text{if } p_i^\rho(m(k)) \cdot x_{1i}^\rho \geqq I(m(k)) \\ & \quad - p_i^\mu(m(k)) \cdot x_i^\mu(m(k)) \\ x_i^\rho & \text{otherwise} \end{cases}$$

Then, by the construction of $x_{1i}^\rho(k)$, $x_{1i}^\rho(k) \leqq x_{1i}^\rho$, $x_{1i}^\rho(k) \to x_{1i}^\rho$, and $p_i^\rho(m(k)) \cdot x_{1i}^\rho(k) \leqq I(m(k)) - p_i^\mu(m(k)) \cdot x_i^\mu(m(k))$ for all $k \geqq k'$.

We now show that there is a sequence $\{x_{2i}^\rho(k)\}$ such that $x_{2i}^\rho(k) \to x_i^\rho$, and, for all $k$ larger than a certain integer, $x_{2i}^\rho(k) \in \mathbb{R}_+^L$, $\hat{x}_{2i}^\rho(k) \leqq \hat{v}(k) + \hat{Y}_P^\rho(m(k)) + Y_C^\rho(m(k)) + w_C^\rho$. For each good $l$ in group $\rho$ , if $\hat{Y}_P^{\rho l}(m(k)) + Y_C^{\rho l}(m(k)) + w_C^{\rho l} = 0$, we must have $x_i^{\rho l} = 0$ by the construction of $B_i^\rho(m)$. Let $x_{2i}^{\rho l}(k) = 0$ for all $k$. Then it is clear $x_{2i}^{\rho l}(k) \to x_i^{\rho l}$ and $\hat{x}_{2i}^{\rho l}(k) = 0 \leqq \hat{v}^l(k) + \hat{Y}_P^{\rho l}(m(k)) + Y_C^{\rho l}(m(k)) + w_C^{\rho l}$.

So we suppose that $\hat{Y}_P^{\rho l}(m(k)) + Y_C^{\rho l}(m(k)) + w_C^{\rho l} > 0$. Two cases will be considered.

*Case i.* $\hat{x}_i^{\rho l} < \hat{v}(k) + \hat{Y}_P^{\rho l}(m) + Y_C^{\rho l}(m) + w_C^{\rho l}$. Then, by the continuity of $Y_P^{\rho l}(m)$ and $Y_C^{\rho l}(m)$, we have $\hat{x}_i^{\rho l} < \hat{v}(k) + \hat{Y}_P^{\rho l}(m(k)) + Y_C^{\rho l}(m(k)) + w_C^{\rho l}$ for all $k$ larger than a certain integer $k'$. Let $x_{2i}^{\rho l}(k) = x_i^{\rho l}$ for all $k > k'$ and $x_{2i}^{\rho l}(k) = 0$ for $k \leqq k'$. Then, $x_{2i}^{\rho l}(k) \to x_i^{\rho l}$, and, for all $k$, and $\hat{x}_{2i}^{\rho l}(k) \leqq \hat{v}^l(k) + \hat{Y}_P^{\rho l}(m(k)) + Y_C^{\rho l}(m(k)) + w_C^{\rho l}$.

*Case ii.* $\hat{x}_i^{\rho l} = \hat{v} + \hat{Y}_P^{\rho l}(m) + Y_C^{\rho l}(m) + w_C^{\rho l}$. Define $x_{2i}^{\rho l}(k)$ as follows:

$$x_{2i}^{\rho l}(k) = \begin{cases} \hat{v}(k) + \hat{Y}_P^{\rho l}(m(k)) & \text{if } x_{2i}^{\rho l} > \hat{v}(k) + \hat{Y}_P^{\rho l}(m(k)) \\ \quad + Y_C^{\rho l}(m(k)) + w_C^{\rho l} & \quad + Y_C^{\rho l}(m(k)) + w_C^{\rho l} \\ x_i^{\rho l} & \text{otherwise} \end{cases}$$

Then $x_{2i}^{\rho l}(k) \to x_i^{\rho l}$ and $\hat{x}_{2i}^{\rho l}(k) \leqq \hat{v}^l(k) + \hat{Y}_P^{\rho l}(m(k)) + Y_C^{\rho l}(m(k)) + w_C^{\rho l}$ by the construction of $x_{2i}^{\rho l}(k)$.

Thus, in both cases, there is a sequence $\{x_{2i}^{\rho}(k)\}$ such that $x_{2i}^{\rho}(k) \to x_i^{\rho}$, and, for all $k$ larger than a certain integer, and $\hat{x}_{2i}^{\rho}(k) \leqq \hat{v}(k) + \hat{Y}_P^{\rho}(m(k)) + Y_C^{\rho}(m(k)) + w_C^{\rho}$.

Finally, let $x_i^{\rho}(k) = \min(x_{1i}^{\rho}(k), x_{2i}^{\rho}(k))$. Then $x_i^{\rho}(k) \to x_i^{\rho}$ since $x_{1i}^{\rho}(k) \to x_i^{\rho}$ and $x_{2i}^{\rho}(k) \to x_i^{\rho}$. Also, for all $k$ larger than a certain integer, $x_i^{\rho}(k) \in \mathbb{R}_+^K$, $\hat{x}_i^{\rho}(k) \leqq \hat{v}(k) + \hat{Y}_P^{\rho}(m(k)) + Y_C^{\rho}(m(k)) + w_C^{\rho}$, and $p_i^{\rho}(m * k)) \cdot x_i^{\rho}(k) + p_i^{\mu}(m(k)) \cdot x_i^{\mu}(m(k)) \leqq I(m(k))$. Thus, $x_i^{\rho}(k) \in B_i^{\rho}(m_k)$ for all $k$ sufficiently larger than a certain integer. Therefore, the sequence $\{x_i^{\rho}(k)\}$ has all the desired properties. So $B_i^{\rho}(m)$ is lower hemi-continuous at every $m \in M$. Q.E.D.

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