Journal of Public Economics 76 (2000) 239–261
JOURNAL OF PUBLIC ECONOMICS
www.elsevier.nl/locate/econbase

Implementation of balanced linear cost share equilibrium solution in Nash and strong Nash equilibria

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Received 1 February 1998; received in revised form 1 February 1999; accepted 1 March 1999

Abstract

This paper considers the incentive aspect of the Balanced Linear Cost Share Equilibrium (BLCSE), which yields an endogenous theory of profit distribution for public goods economies with convex production technologies. We do so by presenting an incentive compatible mechanism which doubly implements the BLCSE solution in Nash and strong Nash equilibria so that Nash allocations and strong Nash allocations coincide with BLCSE allocations. The mechanism presented here allows not only preferences and initial endowments, but also coalition patterns to be privately observed. In addition, it works not only for three or more agents, but also for two-agent economies.

Keywords: Mechanism design; Incentives; Double implementation; Balanced linear cost share equilibrium; Public goods economies

JEL classification: C72; D61; D78

1. Introduction

This paper considers the problem of doubly implementing Balanced Linear Cost Share Equilibrium (BLCSE) allocations, which is a solution concept introduced by Mas-Colell and Silvestre (1989), in Nash and strong Nash equilibria by a feasible and continuous mechanism when coalition patterns, preferences, and endowments

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PII: S0047-2727(99)00041-9

are unknown to the designer. The notions of various cost share equilibria have been introduced in the literature, including the Ratio equilibrium notion of Kaneko (1977), the Generalized Ratio equilibrium notions of Diamantaras and Wilkie (1994), and Tian and Li (1994a, 1994b), as well as the more general solution notions of Linear Cost Share Equilibrium (LCSE) and BLCSE of Mas-Colell and Silvestre (1989). All of these cost-share solutions have desired properties that the conventional Lindahl equilibrium principle does not share: (1) They yield Pareto efficient allocations even in the presence of some types of increasing returns production technologies; (2) they do not need to take profit shares as exogenously given, and (3) the core equivalence holds for cost share equilibria under regularity conditions (see Kaneko, 1977 and Weber and Wiesmeth, 1991). The various cost share equilibrium notions are not radically different from the Lindahl equilibrium solution. They actually coincide with the Lindahl equilibrium notion for the case of convex economies with constant returns to scale.

Like the Lindahl mechanism, these various cost share equilibrium solutions are not incentive-compatible either. Tian and Li (1994a) and Tian (1994) presented incentive-compatible mechanisms which implement the (Generalized) Ratio Equilibrium allocations and LCSE allocations by using Nash equilibrium as a solution concept to describe individuals' self-interested behavior. Nash equilibrium is a strictly noncooperative notion and is only concerned with single individual deviations from which no one can be improved by unilateral deviation from a prescribed strategy profile. No cooperation among agents is allowed. As a result, although a Nash equilibrium may be easy to reach, it may not be stable in the sense that there may exist a group of agents which can be improved by forming a coalition. Thus it is natural to adopt strong Nash equilibrium to allow all possible cooperation (coalitions) among agents. Although strong Nash equilibrium may result in a more stable equilibrium outcome, it requires more information about the communication network and other agents' characteristics in order to eliminate those outcomes that can be upset by coalitionary action. To have a solution concept combining the properties of Nash and strong Nash equilibria, it is desirable to construct a mechanism which doubly implements a social choice rule by Nash and strong Nash equilibria so that its equilibrium outcomes are not only easy to reach, but also hard to leave. Also, by double implementation, it can cover the situation where agents in some coalitions will cooperate and in some other coalitions will not, and thus the designer does not need to know which coalitions are permissible. Consequently, it allows the possibility for agents to manipulate coalition patterns. Recently, Corchon and Wilkie (1996) provided an incentive-compatible mechanism which doubly implements Ratio equilibrium allocations in Nash and strong Nash equilibria, and Tian (1997) provided a feasible and continuous mechanism which doubly implements LCSE allocations in Nash and strong Nash equilibria.

However, the notion of Ratio Equilibrium allows each firm to produce only a single public good, and it therefore cannot be applied to economies where a firm produces more than one public good. While the Generalized Ratio Equilibrium solution allows production of multiple goods, it only permits the presence of by-products (joint production). On the other hand, the notion of LCSE allows the general case of multiple good production, but it has a serious multiple equilibrium problem. For the strict-convex-technology case, Mas-Colell and Silvestre (1989) showed that the LCSE allocations are in one-to-one correspondence with the Lindahl equilibrium allocations. The correspondence is established by varying the profit share parameters which characterize the Lindahl equilibrium allocations. Since profit share for an n-person and K public goods economy can be any element in an $n-1$ dimensional unit simplex, the set of LCSE allocations can have as many points as there are in $\mathbb{R}^{n-1}$. In fact, as Mas-Colell and Silvestre (1989) indicated, the linear cost share equilibrium concept does not yield an endogenous theory of profit distribution. This equilibrium concept leaves $n-1$ degrees of freedom which correspond implicitly to profit shares. To arrive at an endogenous determination of the latter, Mas-Collel and Silvestre (1989) introduced the BLCSE solution principle, which closes the degrees of freedom in such a manner that the individual payments for public goods be in accordance with individual benefits. In fact, for a convex and differentiable production technology, they showed that the BLCSE allocations can be regarded as Lindahl allocations with endogenous profit shares proportional to the consumption of public goods. Roemer and Silvestre (1993) called such a solution the proportional solution. However, they neglected the issues of asymmetric information and incentive compatibility of the solution principle. When the number of agents is small, as with the other solution principles, the BLCSE mechanism is not incentive-compatible. So one needs to design incentive-compatible mechanisms which implement the BLCSE allocations under some solution concepts of self-interested behavior of individuals such as the Nash and strong Nash equilibrium strategy.

In this paper we investigate the incentive aspect of the BLCSE solution for public goods economies. We propose an incentive compatible mechanism whose Nash allocations and strong Nash allocations coincide with Linear Cost Share equilibrium allocations. The mechanism presented here allows not only preferences and initial endowments, but also coalition patterns to be privately observed. In addition, unlike most mechanisms proposed in the literature, our mechanism works not only for three or more agents, but also for two-agent economies, and thus it is a unified mechanism which is irrespective of the number of agents. In addition, the mechanism is well-behaved in the sense that it is a market-type mechanism, feasible, continuous, and has a message space of finite dimension.

The logic of the mechanism can be briefly described as follows. Each participant is asked to report the following messages to the designer: a level of endowment which may be understated but cannot be overstated; a list of the personalized price vectors; a desired level of public goods production; a desired level of public goods consumption; a proposed contribution (tax) used for producing the public goods; a shrinking index which is used to shrink the private good consumption of other agents for obtaining a feasible allocation; and a penalty index when the announced personalized price vectors and proposed allocation for public goods by the individual deviates from his neighbor. According to the messages reported by all the individuals, the designer first assigns a price vector and a proportional share to each individual i, which are determined by the price vector and the contribution announced by his neighbor $i+1$. Thus, each individual takes his price vector and proportional shares as given and cannot change them by changing his own messages. The designer then identifies two feasible constrained choice sets based on the messages announced by the individuals. One is the feasible constrained production set by which public goods can be produced with total resources available in the society. The other is the feasible public goods consumption set. The level of public goods production will be chosen from the feasible production set so that it is the closest to the average of the proposed production of public goods by all agents, and the level of public goods consumption will be chosen from the feasible consumption set so that it is the closest to the average of the proposed consumption of public goods. A preliminary private good consumption is determined by the budget equation. To give incentives for all agents to announce the same price and quantity of the public goods, the preliminary private good consumption is discounted by the penalty index. To obtain the feasible allocation, the final private good consumption will be determined by shrinking the discounted preliminary private good consumption in a certain method. The mechanism constructed in such a way has all the desired properties and we will show that it doubly implements the BLCSE allocations so that Nash allocations and strong Nash allocations coincide with BLCSE allocations.

Thus, from the above description of the mechanism, one can see that the mechanism is realistic to some extent and may be used in the real world for the efficient allocation of public goods, which may provide a partial response to the criticism that most mechanisms in the implementation literature are highly unrealistic. The implementation literature has taken two primary directions since Hurwicz (1979) formalized a general model to deal with incentive problems. One direction is to characterize what various institutions can achieve using incentive compatible mechanisms with various solution concepts of individual behavior. However, due to the general nature of the social choice rules under consideration in this body of work, the implementing mechanisms turn out to be complex, highly unrealistic, and impossible for a real player to use. Characterization results show what is possible for the implementation of a social choice rule (correspondence), but not what is realistic. The mechanisms for characterization are highly discontinuous (so they are not robust with respect to some misspecifications) and have message spaces of infinite dimension. The second direction is towards ‘‘better’’ mechanism design, i.e., designing mechanisms which implement specific and respected social choice rules such as efficient allocations, individually rational allocations, Walrasian allocations, Lindahl allocations, etc., and which have desirable properties such as continuity, feasibility, and lower dimensionality.

Seminal work on ‘‘better’’ mechanism design for Nash implementation was done by Groves and Ledyard (1977). This was followed by Hurwicz (1979), Schmeidler (1980), Walker (1981), Hurwicz et al. (1995), Tian (1989, 1991, 1994), Li et al. (1995), among many others, for Nash implementation and by Peleg (1996a,b), and Corchon and Wilkie (1996) for double implementation in Nash and strong Nash equilibria. While the literature on characterization results may be complete, a lot of work for better mechanism design of various social choice rules still remains.

The remainder of the paper is organized as follows. Section 2 sets forth the framework of a public goods model. Section 3 presents a mechanism which has the desirable properties mentioned above. It may be remarked that, since the BLCSE endogenously determines profit shares, and such endogenously determined profit shares depend on both prices and quantities at equilibrium, the problem of implementing BLCSE allocations is a difficult but worthwhile task. The techniques used in implementing other market-like equilibrium solutions such as Lindahl or Ratio allocations may not be applicable, and thus new techniques need to be developed. Section 4 proves that the mechanism doubly implements Balanced Linear Cost Share Equilibrium allocations in Nash and strong Nash equilibria. Concluding remarks are offered in Section 5.

2. Framework

2.1. Economic environments

We will study a model with $n \geqq 2$ agents, K public goods, and one private good, x being private (as a numeraire) and y public.[1] Denote by $N = \{1,2,\ldots,n\}$ the set of agents. The single private good x can, and probably should, be thought of as a Hicksian composite commodity or money, and public goods y can be thought of as K public projects and are producible from the private good. The technology is given to us as a single cost function[2], $C:\mathbb{R}^K_+ \to \mathbb{R}_+$, which is strictly increasing, convex, continuous, and satisfies $C(0) = 0$.

Each agent's characteristic is denoted by $e_i = (\mathring{w}_i, R_i)$, where $\mathring{w}_i > 0$ is the true initial endowment of the private good and R_i is the preference ordering defined on $\mathbb{R}^{1+K}_+$. Let P_i denote the asymmetric part of R_i (i.e., $a\ P_i\ b$ if and only if $a\ R_i\ b$, but not $b\ R_i\ a$). We assume that preference ordering R_i is strictly monotonically

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[1] As usual, vector inequalities are defined as follows: Let $a, b \in \mathbb{R}^m$. Then $a \geqq b$ means $a_s \geqq b_s$ for all $s = 1, \ldots, m$; $a \geq b$ means $a \geqq b$ but $a \neq b$; $a > b$ means $a_s > b_s$ for all $s = 1, \ldots, m$.

[2] This is actually an input requirement function. In this model, there is only one private good which is a numeraire, so that we can interpret $C(y)$ as a cost function.

increasing $\mathbb{R}^{1+K}_{++}$, continuous, and convex[3]. For the feasible implementation of BLCSE allocations, we also need to make the following indispensable assumption.

Assumption 1 (Interiority of Preferences). *For all* $i \in N, (x_i, y)\ P_i\ (x'_i, y')$ *for all* $(x_i, y) \in \mathbb{R}^{1+K}_{++}$ *and* $(x'_i, y') \in \partial\mathbb{R}^{1+K}_{+}$, *where* $\partial\mathbb{R}^{1+K}_{+}$ *is the boundary of* $\mathbb{R}^{1+K}_{+}$.

Remark 1. *The family of Cobb-Douglas utility functions satisfies Assumption* 1. *An economy is the full vector* $e = (e_1, \ldots, e_n, C(y))$ *and the set of all such economies is denoted by E. An allocation of the economy e is a vector* $(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^K_+$.

An allocation is *feasible* if

$$\sum_{i=1}^{n} x_i + C(y) \leqq \sum_{i=1}^{n} \mathring{w}_i. \tag{1}$$

A *coalition S* is a non-empty subset of *N*.

A feasible allocation (x, y) can be improved upon by $S \subset N$ if there exists an allocation (x', y') such that

(i) $\sum_{i \in S} x'_i + C(y') \leqq \sum_{i \in S} \mathring{w}_i$,

(ii) $(x'_i, y') P_i (x_i, y)$ *for all* $i \in S$.

A feasible allocation (x, y) is in the core of e if there does not exist any coalition S that can improve upon (x, y).

An allocation (x, y) is *Pareto-optimal* with respect to the preference profile $R = (R_1, \ldots, R_n)$ if it cannot be improved upon by N.

An allocation (x, y) is *individually rational* with respect to the preference profile $R = (R_1, \ldots, R_n)$ if it cannot be improved upon by every single individual i.

2.2. Balanced linear cost share equilibrium

To define a Balanced Linear Cost Share Equilibrium, let us first define a Linear Cost Share Equilibrium. Both were introduced in Mas-Colell and Silvestre (1989).

Let $\Delta^{n-1}_+ = \{t \in \mathbb{R}^n_+ : \Sigma^n_{i=1} t_i = 1\}$ be the $n-1$ dimensional unit simplex.

An allocation (x^*, y^*) is a *Linear Cost Share Equilibrium (LCSE) allocation* for an economy e if it is feasible and there are $(a^*_1, \ldots, a^*_n) \in \mathbb{R}^{nK}$ with $\Sigma^n_{i=1} a^*_i = 0$ and $(b^*_1, \ldots, b^*_n) \in \Delta^{n-1}_+$ such that

1. $x^*_i + a^*_i \cdot y^* + b^*_i C(y^*) \leqq \mathring{w}_i$ for all $i \in N$;

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[3] R_i is convex if for bundles a, b, c with $0 < \lambda \leqq 1$ and $c = \lambda a + (1-\lambda)b$, the relation $a\ P_i\ b$ implies $c\ P_i\ b$.

2. for all $i \in N$, there does not exist (x_i,y) such that $(x_i,y)\ P_i\ (x_i^*,y^*)$ and $x_i + a_i^* \cdot y + b_i^* C(y) \leqq \mathring{w}_i$.

Denote by $LCSE(e)$ the set of all such allocations. Mas-Colell and Silvestre (1989) showed that every LCSE allocation is Pareto optimal even in the increasing returns case. Further, they showed that if one reinterprets the commodity space, the equilibrium concept can be applied to economies with purely private goods or with externalities. Thus, the equilibrium concept can also be viewed as optimality-guaranteeing equilibrium concepts.

Remark 2. *An LCSE allocation does not always result in individually rational allocations. Wilkie* (1990) *provided such a counter-example. Therefore, an LCSE allocation may not be in the core. However, every interior LCSE allocation belongs to the core of the economy and is individually rational* (*cf. Weber and Wiesmeth,* 1991). *Since every LCSE allocation is an interior allocation by the interiority of preferences, all LCSE allocations under consideration in the paper are individually rational.*

Given the profit share vector $\theta \in \Delta^{n-1}$, an allocation (x^*,y^*) is a *θ-Lindahl equilibrium allocation* for an economy e if it is feasible and there are personalized price vectors, $q_i^* \in \mathbb{R}_+^K$, one for each, such that

1. y^* maximizes profits $p^* \cdot y - C(y)$;
2. $x_i^* + q_i^* \cdot y^* = \mathring{w}_i + \theta_i[p^* \cdot y^* - C(y^*)]$ for all $i \in N$;
3. for all $i \in N$, there does not exist (x_i,y) such that $(x_i,y)\ P_i\ (x_i^*,y^*)$ and $x_i + q_i^* \cdot y \leqq \mathring{w}_i + \theta_i[q^* \cdot y^* - C(y^*)]$;
4. $\Sigma_{i=1}^n q_i^* = p^*$.

Denote by $L(e;\theta)$ the set of all such allocations.

Note that in the constant returns case, the Linear Cost Share equilibrium allocation reduces to the Lindahl equilibrium allocation. For the strict convex production technology case, Mas-Colell and Silvestre (1989) also showed that the LCSE allocations are in one-to-one correspondence with the Lindahl equilibrium allocations so that $LCSE(e) = \cup_{\theta \in \Delta^{n-1}} L(e;\theta)$. Since θ can be any element in Δ^{n-1}, the set of LCSE allocations is very large. As Mas-Colell and Silvestre (1989) indicated, the linear cost share equilibrium concept does not yield an endogenous theory of profit distribution. This equilibrium concept leaves $n-1$ degrees of freedom which correspond implicitly to profit shares. In order to arrive at an endogenous determination of the latter, Mas-Collel and Silvestre (1989) introduced the concept of BLCSE which closes the degrees of freedom in such a manner that the individual payments for public goods are in accordance with individual benefits.

An allocation (x^*,y^*) is a *Balanced Linear Cost Share Equilibrium (BLCSE) allocation* for an economy e if

1. it is an LCSE allocation with $(a_1^*,\ldots,a_n^*) \in \mathbb{R}^{nK}$ and $(b_1^*,\ldots,b_n^*) \in \Delta_+^{n-1}$ as a linear cost share system.
2. $a_i^* \cdot y^* = 0$ for all $i \in N$.

Denote by $BLCSE(e)$ the set of all such allocations which is non-empty in the economic environments under consideration. Note that in the case of a single public good, a linear cost share equilibrium is balanced if and only if $a_i = 0$, or equivalently if and only if it is a Ratio Equilibrium in the sense of Kaneko (1977). An interpretation of the balanced linear cost share equilibrium can be easily given. Since $a_i \cdot y$ represents a transfer from (or to) individual i, the BLCSE principle implies that the net transfers of every individual is zero at equilibrium, i.e., $a_i \cdot y^* = 0$.

In this paper, we will use the following lemma to consider double implementation of BLCSE allocations. This lemma is due to Mas-Collel and Silvestre (1989), Proposition 6) which gives an equivalence between BLCSE allocations and Lindahl allocations with endogenous profit shares which are proportional to consumption of public goods. Roemer and Silvestre (1993) called such an allocation the proportional solution. They have extended the solution concept to a general setting of private good economies.

Lemma 1. *Suppose that C is convex and differentiable. Then an allocation* (x^*,y^*) *is a BLCSE allocation for an economy e if and only if it is feasible and there are personalized price vectors,* $q_i^* \in \mathbb{R}_+^K$, *one for each, such that*

1. y^* *maximizes profits* $p^* \cdot y - C(y)$;
2. *for all* $i \in N$, *there does not exist* (x_i,y) *such that* $(x_i,y)\ P_i\ (x_i^*,y^*)$ *and* $x_i + q_i^* \cdot y \leqq \mathring{w}_i + \frac{q_i^* \cdot y^*}{p^* \cdot y^*}[p^* \cdot y^* - C(y^*)]$;
3. $x_i^* + \frac{q_i^* \cdot y^*}{p^* \cdot y^*} C(y^*) = \mathring{w}_i$ *for all* $i \in N$;
4. $\Sigma_{i=1}^n q_i^* = p^*$.

Thus, when C is convex and differentiable, the above lemma implies that an allocation is a BLCSE allocation if and only if it is a Lindahl allocation with profit shares which are proportional to the consumption of public goods. Mas-Collel and Silvestre (1989), Remark 7) indicated that the 'if part' of the lemma remains valid even if C is not differentiable.

2.3. Mechanism

Let M_i denote the i-th agent's message domain. Its elements are written as m_i and called messages. Let $M = \prod_{i=1}^n M_i$ denote the message space. Denote by

$h:M \to \mathbb{R}_{+}^{n+K}$ the outcome function, or more explicitly, $h_i(m) = (X_i(m), Y(m))$. Then the mechanism consists of $\langle M,h \rangle$ which is defined on E.

A message $m^* = (m_1^*, \ldots, m_n^*) \in M$ is said to be a *Nash equilibrium* of the mechanism $\langle M,h \rangle$ for an economy e if, for each $i \in N$ and $m_i \in M_i$,

$$h_i(m^*) \; R_i \; h_i(m_i, m_{-i}^*), \tag{2}$$

where $(m_i, m_{-i}^*) = (m_1^*, \ldots, m_{i-1}^*, m_i, m_{i+1}^*, \ldots, m_n^*)$. $h(m^*)$ is then called a *Nash* (*equilibrium*) *allocation* of the mechanism for the economy e. Denote by $V_{M,h}(e)$ the set of all such Nash equilibria and by $N_{M,h}(e)$ the set of all such Nash (equilibrium) allocations.

A mechanism $\langle M,h \rangle$ is said to *weakly Nash-implement* BLCSE allocations on E, if, for all $e \in E$, $\emptyset \neq N_{M,h}(e) \subset BLCSE(e)$.

A mechanism $\langle M,h \rangle$ is said to *Nash-implement* BLCSE allocations on E, if, for all $e \in E$, $\emptyset \neq N_{M,h}(e) = BLCSE(e)$.

A message $m^* = (m_1^*, \ldots, m_n^*) \in M$ is said to be a *strong Nash equilibrium* of the mechanism $\langle M,h \rangle$ for an economy e if there does not exist any coalition S and $m_S \in \prod_{i \in S} M_i$ such that for all $i \in S$,

$$h_i(m_S, m_{-S}^*) \; P_i \; h_i(m^*). \tag{3}$$

$h(m^*)$ is then called a *strong Nash* (*equilibrium*) *allocation* of the mechanism for the economy e. Denote by $SV_{M,h}(e)$ the set of all such strong Nash equilibria and by $SN_{M,h}(e)$ the set of all such strong Nash (equilibrium) allocations.

The mechanism $\langle M,h \rangle$ is said to *doubly implement* BLCSE allocations on E, if, for all $e \in E$, $\emptyset \neq SN_{M,h}(e) = N_{M,h}(e) = BLCSE(e)$.

The mechanism $\langle M,h \rangle$ is said to *weakly doubly implement* BLCSE allocations on E, if, for all $e \in E$, $\emptyset \neq SN_{M,h}(e) = N_{M,h}(e) \subset BLCSE(e)$.

3. A feasible and continuous mechanism

In the following, we will present a feasible and continuous mechanism which doubly implements BLCSE allocations in Nash and strong Nash equilibrium. The message space of the mechanism is defined as follows.

For each $i \in N$, his/her message domain is of the form

$$M_i = (0, \mathring{w}_i] \times \mathbb{R}_{++}^{nK} \times \mathbb{R}^K \times \mathbb{R}^K \times \mathbb{R}_{++}^K \times \mathbb{R}_{++} \times (0,1]. \tag{4}$$

A generic element of M_i is $m_i = (w_i, q_i, z_i, y_i, t_i, \gamma_i, \eta_i)$ whose components have the following interpretations. The component w_i denotes a profession of agent i's endowment, the inequality $0 < w_i \leqq \mathring{w}_i$ means that the agent cannot overstate his own endowment; on the other hand, the endowment can be understated, but the claimed endowment w_i must be positive which is necessary to guarantee the feasibility even at disequilibrium points.[4] The component $q_i := (q_{i,1}, \ldots, q_{in})$ is a list of the personalized price vectors proposed by agent i. The component z_i is the public goods production proposed by agent i. The component y_i is the public goods consumption proposed by agent i. The component t_i is the tax contribution proposed by agent i. The component γ_i is a shrinking index of agent i used to shrink the private good consumption of other agents. The component η_i is the penalty index of agent i when the announced personalized price vectors and proposed allocation for public goods by agent i deviates from his neighbor, agent $i+1$.

For each $i \in N$, define the personalized price function by

$$q_i(m) = q_{i+1,i}, \tag{5}$$

and an aggregate price function for public goods by

$$p_i(m) = \sum_{j=1}^{n} q_{i+1,j}. \tag{6}$$

where $n+1$ is to be read as 1. Note that even though $q_i(m)$ and $p_i(m)$ are functions of the component, q_{i+1}, announced by agent $i+1$ for agent i, we can write them as functions of m without loss of generality.

Define agent i's (endogenous) proportional share function $\theta_i : M \to \mathbb{R}_{++}$ by

$$\theta_i(m) = \frac{q_i(m) \cdot t_{i+1}}{p_i(m) \cdot t_{i+1}}. \tag{7}$$

Note that, by definition, $q_i(m)$, $p_i(m)$ and $\theta_i(m)$ are independent of m_i. The reason for having this independence is that we want the mechanism to have the property that any agent i cannot enlarge his budget set by changing m_i, but he can reach any consumption bundle (x_i, y) he wishes in the budget set by changing message m_i. We will show that $p_1(m^*) = p_2(m^*) = \cdots = p_n(m^*) := p(m^*)$ and $\sum_{i=1}^{n} \theta_i(m^*) = 1$ at every Nash equilibrium $m^* \in V_{M,h}(e)$.

Define a constrained public goods production correspondence $B_z : M \to 2^{\mathbb{R}^K_+}$ by

$$B_z(m) = \left\{ z \in \mathbb{R}^K_+ : C(z) \leqq \sum_{i=1}^{n} w_i \& w_i/2 + \theta_i(m)[p_i(m) \cdot z - C(z)] \geqq 0 \ \forall\ i \in N \right\}, \tag{8}$$

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[4] The intuition here is straightforward: if a mechanism allows agents to overstate their endowments, then it allows for infeasible outcomes – it will sometimes attempt to allocate more than is possible, given the true aggregate endowment. Note that, when goods are physical goods, this requirement can be guaranteed by asking agents to *exhibit* their reported endowments to the designer. However, when goods are not physical goods, announced endowments may be unverifiable. This problem is not a limitation of the mechanism constructed in the paper since any feasible mechanism will have this limitation as long as endowments are unknown to the designer.

which is clearly nonempty, compact, and convex (by the convexity of $C(\cdot)$) for all $m \in M$. We will show the following lemma in the Appendix A.

Lemma 2. $B_z(\cdot)$ *is continuous on* M.

Let $\tilde{z} = 1/n\ \Sigma_{i=1}^{n}\ z_i$, which is the average of proposed production of public goods.

Define the outcome function for production of public goods Z: $M \to B_z$ by

$$Z(m) = \operatorname{argmin}\{\|z - \tilde{z}\| : z \in B_z(m)\}, \tag{9}$$

which is the closest point to $\tilde{z}$. By Berge's Maximum Theorem (see Berge, 1963, p.116), we know that $Z(m)$ is an upper semi-continuous correspondence. Also, since $B_z(m)$ is closed and convex valued, it is also single-valued (see Mas-Colell, 1985, p. 28). Then $Z(m)$ is single-valued and continuous on M.

Define the proportional profit sharing function $\pi_i : M \to \mathbb{R}$ by

$$\pi_i(m) = \theta_i(m)[p_i(m) \cdot Z(m) - C(Z(m))] \tag{10}$$

and proportional cost sharing function $g_i : M \to \mathbb{R}_+$ by

$$g_i(m) = \theta_i(m)C(Z(m)), \tag{11}$$

which are both continuous.

Define a feasible public goods consumption correspondence B_y: $M \to 2^{\mathbb{R}_+^K}$ by

$$\begin{aligned} B_y(m) = \{y \in \mathbb{R}_+^K : y \leqq Z(m) \,\&\, w_i - q_i(m) \cdot y + \theta_i(m)[p_i(m) \cdot Z(m) - C(Z(m))] \\ \geqq 0\ \forall\ i \in N\}, \end{aligned} \tag{12}$$

which is a correspondence with non-empty, compact, and convex values. We will prove the following lemma in Appendix A.

Lemma 3. $B_y(\cdot)$ *is continuous on* M.

Let $\tilde{y} = 1/n\ \Sigma_{i=1}^{n}\ y_i$, which is the average of proposed consumption of public goods.

Define the outcome function for public goods consumption Y: $M \to B_y$ by

$$Y(m) = \operatorname{argmin}\{\|y - \tilde{y}\| : y \in B_y(m)\}, \tag{13}$$

which is the closest point to $\tilde{y}$. As above, we know that $Y(m)$ is single-valued and continuous on M.

Define the preliminary private good consumption x_i: $M \to \mathbb{R}_+$ by

$$x_i(m) = \frac{w_i - q_i(m) \cdot Y(m) + \theta_i(m)[p_i(m) \cdot Z(m) - C(Z(m))]}{1 + \eta_i[\|q_i - q_{i+1}\| + \|t_i - t_{i+1}\| + \|z_i - y_i\| + \|y_i - t_i\|]}. \tag{14}$$

Define the γ-correspondence A: $M \to 2^{\mathbb{R}_+}$ by

$$A(m) = \left\{ \gamma \in \mathbb{R}_+ : \gamma\gamma_i \leq 1 \ \forall i \in N \ \& \ \gamma \sum_{i=1}^{n} \gamma_i x_i(m) + C(Z(m)) \leqq \sum_{i=1}^{n} w_i \right\}. \tag{15}$$

Let $\bar{\gamma}(m)$ be the largest element of A, i.e., $\bar{\gamma}(m) \in A(m)$, $\bar{\gamma}(m) \geq \gamma$ for all $\gamma \in A(m)$.

Finally, define the outcome function for private good consumption $X(m)$: $M \to \mathbb{R}_+$ by

$$X_i(m) = \bar{\gamma}(m)\gamma_i x_i(m), \tag{16}$$

which is agent i's consumption resulting from the strategic configuration m.

Thus the outcome function is continuous and feasible on M since, by the construction of the mechanism, $(X(m),Y(m)) \in \mathbb{R}_+^{n+K}$ and

$$\sum_{i=1}^{n} X_i(m) + C(Y(m)) \leqq \sum_{i=1}^{n} \mathring{w}_i \tag{17}$$

for all $m \in M$.

Remark 3. *Note that our mechanism works not only for three or more agents, but also for a two-agent world. While most mechanisms which implement market-type social choice correspondences* (*such as Walrasian, Lindahl, Ratio, or LCSE allocations*) *in the existing literature need to distinguish the case of two agents from that of three or more agents, this paper gives a unified mechanism which holds irrespective of the number of agents.*

4. Double Implementation

The remainder of this paper is devoted to proving Theorems 1 and 2 below.

Theorem 1. *For the class of economic environments E with one private and K public goods, if the following assumptions are satisfied*:

1. $n \geqq 2$;
2. *preference orderings* R_i *are strictly increasing on* $\mathbb{R}_{++}^{1+K}$, *continuous, convex, and satisfy the Interiority Condition of Preferences*; *and*
3. *the cost function* $C: \mathbb{R}_+^K \to \mathbb{R}_+$ *is strictly increasing, convex, differentiable, and* $C(0) = 0$,

then there exists a continuous and feasible mechanism which doubly implements BLCSE allocations in Nash and strong Nash equilibria on E.

Proof. The proof of Theorem 1 consists of the following three propositions which show the equivalence among Nash allocations, strong Nash allocations, and BLCSE allocations. Proposition 1 below proves that every Nash allocation is a BLCSE allocation. Proposition 2 below proves that every BLCSE allocation is a Nash allocation. Proposition 3 below proves that every Nash equilibrium allocation is a strong Nash equilibrium allocation.

To show these propositions, we first prove the following lemmas.

Lemma 4. *If* m^* *is a Nash equilibrium, then* $q_1^* = q_2^* = \cdots = q_n^*$, $t_1^* = t_2^* = \ldots = t_n^* = y_1^* = y_2^* = \cdots = y_n^* = z_1^* = z_2^* = \cdots = z_n^*$. *Consequently,* $p_1(m^*) = p_2(m^*) = \cdots = p_n(m^*) = p(m^*) = \Sigma_{i=1}^n\ q_i(m^*)$, *and* $\Sigma_{i=1}^n\ \theta_i(m^*) = 1$.

Proof. Suppose, by way of contradiction, that $q_i^* \neq q_{i+1}^*$, $t_i^* \neq t_{i+1}^*$, $z_i^* \neq y_i^*$, or $y_i^* \neq t_i^*$ for some $i \in N$. Then agent i can choose a smaller $\eta_i < \eta_i^*$ in (0,1] so that his consumption of the private good becomes larger and thus he would be better off by monotonicity of preferences. Hence, no choice of η_i could constitute part of a Nash equilibrium strategy when $q_i^* \neq q_{i+1}^*$, $t_i^* \neq t_{i+1}^*$, $z_i^* \neq z_{i+1}^*$, or $y_i^* \neq t_i^*$. Thus, we must have $q_1^* = q_2^* = \cdots = q_n^*$, $t_1^* = t_2^* = \cdots = t_n^* = y_1^* = y_2^* = \cdots = y_n^* = z_1^* = z_2^* = \cdots = z_n^* = \tilde{y}^*$. Consequently, we have $p_1(m^*) = p_2(m^*) = \cdots = p_n(m^*) = p(m^*) = \Sigma_{i=1}^n\ q_i(m^*)$, and $\Sigma_{i=1}^n\ \theta_i(m^*) = \Sigma_{i=1}^n\ (q_i(m^*) \cdot \tilde{y}^*)/(p(m^*) \cdot \tilde{y}^*) = (p(m^*) \cdot \tilde{y}^*)/(p(m^*) \cdot \tilde{y}^*) = 1$. □

Lemma 5. *Suppose* $(x_i(m),Y(m))\ P_i\ (x_i,y)$ *for* $i \in N$. *Then agent* i *can choose a very large* γ_i *such that* $(X_i(m),Y(m))\ P_i\ (x_i,y)$.

Proof. If agent i declares a large enough γ_i, then $\bar{\gamma}(m)$ becomes very small (since $\bar{\gamma}(m)\gamma_i \leqq 1$) and thus almost nullifies the effect of other agents in $\gamma\ \Sigma_{i=1}^n\ \gamma_i x_i(m) + C(Z(m)) \leqq\ \Sigma_{i=1}^n\ w_i$. Thus, $X_i(m) = \bar{\gamma}(m)\gamma_i x_i(m)$ can arbitrarily approach $x_i(m)$ as agent i wishes. From the hypothesis that $(x_i(m),Y(m))\ P_i\ (x_i,y)$ and continuity of preferences, we have $(X_i(m),Y(m))\ P_i\ (x_i,y)$ if agent i chooses a very large γ_i. □

Lemma 6. *If* $(X(m^*),Y(m^*)) \in N_{M,h}(e)$, *then* $(X(m^*),Y(m^*)) \in \mathbb{R}_{++}^{n+K}$.

Proof. We argue by contradiction. Suppose $(X(m^*),Y(m^*)) \in \partial\mathbb{R}_+^{n+K}$. Then $Y(m^*) \in \partial\mathbb{R}_+^K$ or $X_i(m^*) = 0$ for some $i \in N$. Since $w_j^* > 0$ for all $i \in N$ and $C(y)$ is continuous with $C(0) = 0$, then there is some $(x_i,y) \in \mathbb{R}_{++}^{1+K}$ such that $x_i + q_i(m^*) \cdot y = w_i^* + \theta_i(m^*)[p(m^*) \cdot y - C(y)]$, $w_j^*/2 + \theta_j(m^*)[p(m^*) \cdot y - C(y)] \geqq 0$ for all $j \in N$, $w_j^* - q_j(m^*) \cdot y + \theta_j(m^*)[p(m^*) \cdot y - C(y)] \geqq 0$ for all $j \neq i$, $x_i + C(y) <\ \Sigma_{j=1}^n\ w_j$, and $(x_i,y)\ P_i\ (X_i(m^*),Y(m^*))$ by interiority of preferences. Now suppose that agent i chooses $y_i = ny - \Sigma_{j\neq i}^n\ y_j^*$, $z_i = ny - \Sigma_{j\neq i}^n\ z_j^*$, $\gamma_i > \gamma_i^*$, and keeps other components of the message unchanged. Then, $y \in B_z(m_i,m_{-i}^*)$, $y \in B_y(m_i,m_{-i}^*)$. Thus, $Y(m_i,m_{-i}^*) = Z(m_i,m_{-i}^*) = y$, and $x_i(m_i,m_{-i}^*) = (1/(1 + \eta_i^*[\|z_i -$

$y_i\| + \|y_i - t^*_{-i}\|]))x_i > 0$. Then, we have $(x_i(m_i,m^*_{-i}),Y(m_i,m^*_{-i}))\ P_i\ (X_i(m^*),Y(m^*))$ by interiority of preferences. Therefore, by Lemma 4, $(X_i(m_i,m^*_{-i}),Y(m_i,m^*_{-i}))\ P_i\ (X_i(m^*),Y(m^*)$ if agent i chooses a very large γ_i. This contradicts the hypothesis that $(X(m^*),Y(m^*)) \in N_{M,h}(e)$. So we must have $(X(m^*),Y(m^*)) \in \mathbb{R}^{n+K}_{++}$. □

Lemma 7. *Suppose* $(X(m^*),Y(m^*)) \in \mathbb{R}^{n+K}_{++}$ *for some* $m^* \in M$ *and there is* $(x_i,y) \in \mathbb{R}^{1+K}_{+}$ *for some* $i \in N$ *such that* $x_i + q_i(m^*)\cdot y \leqq w^*_i + \pi_i(m^*)$ *and* $(x_i,y)\ P_i\ (X_i(m^*),Y(m^*))$. *Then there is some* $m_i \in M_i$ *such that* $(X_i(m_i,m^*_{-i}),Y(m_i,m^*_{-i}))\ P_i\ (X_i(m^*),Y(m^*))$.

Proof. Since $(X(m^*),Y(m^*)) > 0$, then $X_i(m^*) + C(Y(m^*)) < \Sigma_{j\in N}\ w^*_j$, and $q_j(m^*)\cdot y < w^*_j + \pi_j(m^*)$ for all $j \in N$. Let $x_{\lambda i} = \lambda x_i + (1-\lambda)X_i(m^*)$ and $y_\lambda = \lambda y + (1-\lambda)Y(m^*)$. Then, by convexity of preferences, we have $(x_{\lambda i},y_\lambda)\ P_i\ (X_i(m^*),Y(m^*))$ for any $0 < \lambda < 1$. Also $(x_{\lambda i},y_\lambda) \in \mathbb{R}^{1+K}_{+}$, $x_{\lambda i} + q_i(m^*)\cdot y_\lambda \leqq w^*_i + \pi_i(m^*)$, $w^*_j - q_j(m^*)\cdot y_\lambda + \pi_j(m^*) \geqq 0$ for all $j \neq i$, and $x_{\lambda i} + C(y_\lambda) < \Sigma_{j\in N}\ w^*_j$ when λ is sufficiently close to 0. Now suppose agent i chooses $y_i = ny_\lambda - \Sigma^n_{j\neq i}\ y^*_j$, $\gamma_i > \gamma^*_i$, $\eta_i < \eta^*_i$, and keeps other components of the message unchanged, then $y_\lambda \in B_y(m_i,m^*_{-i})$. Thus we have $Y(m_i,m^*_{-i}) = y_\lambda$, and $x_i(m_i,m^*_{-i}) = (1/(1 + \eta_i[\|q^*_i - q^*_{i+1}\| + \|t^*_i - t^*_{i+1}\| + \|z^*_i - y_i\| + \|y_i - t^*_i\|]))x_{\lambda i} > 0$. Since $x_i(m^*_{-i},m_i)$ can arbitrarily approach $x_{\lambda i}$ by choosing a sufficiently small η_i, from $(x_{\lambda i},y_\lambda)\ P_i\ (X_i(m^*),Y(m^*))$ and continuity of preferences, we have $(x_i(m_i,m^*_{-i}),Y(m_i,m^*_{-i}))\ P_i\ (X_i(m^*),Y(m^*))$. Therefore, by Lemma 4, agent i can choose a very large γ_i such that $(X_i(m_i,m^*_{-i}),Y(m_i,m^*_{-i}))\ P_i\ (X_i(m^*),Y(m^*))$. □

Lemma 8. *If* m^* *is a Nash equilibrium, then* $w^*_i = \mathring{w}_i$ *for all* $i \in N$.

Proof. Suppose, by way of contradiction, that $w^*_i \neq \mathring{w}_i$ for some $i \in N$. Then $X_i(m^*) + q(m^*)\cdot Y(m^*) \leqq w^*_i + \pi_i(m^*) < \mathring{w}_i + \pi_i(m^*)$, and thus there is $(x_i,y) \in \mathbb{R}^{1+K}_{+}$ such that $x_i + q_i(m^*)\cdot y \leqq \mathring{w}^*_i + \pi_i(m^*)$ and $(x_i,y)\ P_i\ (X_i(m^*),Y(m^*))$. Since $(X(m^*),Y(m^*)) \in \mathbb{R}^{n+K}_{++}$ by Lemma 4, there is some $m_i \in M_i$ such that $(X_i(m_i,m^*_{-i}),Y(m_i,m^*_{-i}))\ P_i\ (X_i(m^*),Y(m^*))$ by Lemma 4. This contradicts $(X(m^*),Y(m^*)) \in N_{M,h}(e)$. □

Lemma 9. *If* $(X(m^*),Y(m^*)) \in N_{M,h}(e)$, *then* $X_i(m^*) + q_i(m^*)\cdot Y(m^*) = \mathring{w}_i + \pi_i(m^*)$ *for all* $i \in N$.

Proof. Suppose, by way of contradiction, that $X_i(m^*) + q_i(m^*)\cdot Y(m^*) < \mathring{w}_i + \pi_i(m^*)$ for some $i \in N$. Then, there is some $(x_i,y) \in \mathbb{R}^{1+K}_{+}$ such that $x_i + q_i(m^*)\cdot y \leqq \mathring{w}^*_i + \pi_i(m^*)$ and $(x_i,y)\ P_i\ (X_i(m^*),Y(m^*))$ by monotonicity of preferences. Since $(X(m^*),Y(m^*)) \in \mathbb{R}^{n+K}_{++}$ by Lemma 4, there is some $m_i \in M_i$ such that $(X_i(m_i,m^*_{-i}),Y(m_i,m^*_{-i}))\ P_i\ (X_i(m^*),Y(m^*))$ by Lemma 4. This contradicts $(X(m^*),Y(m^*)) \in N_{M,h}(e)$. □

Lemma 10. *If* $(X(m^*),Y(m^*)) \in N_{M,h}(e)$, *then* $\bar{\gamma}(m^*)\gamma_i^* = 1$ *for all* $i \in N$ *and thus* $X(m^*) = x(m^*)$.

Proof. This is a direct corollary of Lemma 4. Suppose $\bar{\gamma}(m^*)\gamma_i^* < 1$ for some $i \in N$. Then $X_i(m^*) = \bar{\gamma}(m^*)\gamma_i^* x_i(m^*) < x_i(m^*)$, and therefore $X_i(m^*) + q_i(m^*) \cdot Y(m^*) < \mathring{w}_i + \pi_i(m^*)$. But this is impossible by Lemma 4. □

Lemma 11. *If* $(X(m^*),Y(m^*)) \in N_{M,h}(e)$, *then* $Z(m^*) \in$ *int* $B_z(m^*)$ *and* $Y(m^*) \in$ *int* $B_y(m^*)$. *Consequently,* $Z(m^*) = Y(m^*) = \tilde{y}^* = y_1^* = \cdots = y_n^* = \tilde{z} = z_1^* = \cdots = z_n^*$, *the budget equation becomes*

$$X_i(m^*) = w_i^* - \frac{q_i(m^*) \cdot Y(m^*)}{p(m^*) \cdot Y(m^*)} C(Y(m^*)) \tag{18}$$

for all $i \in N$, *and thus the feasibility condition must hold with equality, i.e.,* $\Sigma_{i=1}^n X_i(m^*) + C(Y(m^*)) = \Sigma_{i=1}^n \mathring{w}_i$.

Proof. Suppose, by way of contradiction, that $Y(m^*) \in \partial B_y(m^*)$. Then either $Y^k(m^*) = 0$ for some k or $0 = w_i + \pi_i(m^*) - q_i(m^*) \cdot Y(m^*) = X_i(m^*)$ for some $i \in N$. But both cases are impossible by Lemma 4. So $Y(m^*) \in$ int $B_y(m^*)$. Therefore, $Y(m^*) = \tilde{y}^* = y_1^* = \cdots = y_n^*$. We also must have $Z(m^*) \in$ int $B_z(m^*)$. Otherwise, either $Z^k(m^*) = 0$ for some k or $C(Z(m^*)) = \Sigma_{i=1}^n \mathring{w}_i$, and thus $Y^k(m^*) = 0$ or $0 = w_i + \pi_i(m^*) - q_i(m^*) \cdot Y(m^*) = X_i(m^*)$ for some $i \in N$. But both cases are impossible by Lemma 4. So $Z(m^*) \in$ int$B_z(m^*)$. Therefore, $Z(m^*) = \tilde{z}^* = z_1^* = \cdots = z_n^* = y_1^* = \cdots = y_n^* = \tilde{y}^* = Y(m^*)$, and thus

$$\begin{aligned} X_i(m) &= [w_i^* - q_i(m^*) \cdot Y(m^*) + \pi_i(m^*)] \quad \text{(by Lemma 4)} \\ &= w_i^* \\ &\quad - \frac{q_i(m^*) \cdot Y(m^*)}{p(m^*) \cdot Y(m^*)} C(Y(m^*)) \quad \text{(by noting that } Y(m^*) = Z(m^*) = \tilde{y}(m^*)\text{)} \end{aligned} \tag{19}$$

for all $i \in N$. Finally, summing the above equation over individuals and applying Lemma 4, we have $\Sigma_{i=1}^n X_i(m^*) + C(Y(m^*)) = \Sigma_{i=1}^n \mathring{w}_i$. □

Lemma 12. *If* m^* *is a Nash equilibrium, then* $Y(m^*)$ *is the profit maximizing level of output of public goods under* $p(m^*)$, *i.e.,* $p(m^*) \cdot Y(m^*) - C(Y(m^*)) \geqq p(m^*) \cdot z - C(z)$ *for all* $z \in \mathbb{R}_+^K$.

Proof. Suppose, by way of contradiction, that there is some $z \in \mathbb{R}_+$ such that $p(m^*) \cdot Y(m^*) - C(Y(m^*)) < p(m^*) \cdot z - C(z)$. Let $z_\lambda = \lambda z + (1-\lambda)Y(m^*)$ with $0 < \lambda < 1$. Then $p(m^*) \cdot Y(m^*) - C(Y(m^*)) < p(m^*) \cdot z_\lambda - C(z_\lambda)$ by the convexity of $C(\cdot)$. Also, since $X_i(m^*) + C(Y(m^*)) < \Sigma_{j \in N} w_j^*$, we have $X_i(m^*) + C(z_\lambda) < \Sigma_{j \in N}$

w_j^* for all $i \in N$ when λ is sufficiently close to 0 by the continuity of $C(\cdot)$. Note that $\theta_j(m^*) > 0$ for all $j \in N$. Let $\pi_{j\lambda}$ be the profit share of agent j when $Y(m^*)$ is replaced by z_λ in $\pi_j(m^*)$. Then $\pi_{j\lambda} > \pi_j(m^*)$. Thus we have $w_i^*/2 + \pi_i(m^*) < w_i^*/2 + \pi_{i\lambda}$ and $X_i(m^*) + q_i(m^*) \cdot Y(m^*) \leqq w_i^* + \pi_i(m^*) < w_i^* + \pi_{i\lambda}$.

Then there is $x_i > X_i(m^*)$ such that $x_i + q_i(m^*) \cdot Y(m^*) \leqq w_i^* + \pi_{i\lambda}$, $x_i + C(z_\lambda)) \leqq \Sigma_{j \in N} w_j^*$, and $(x_i, Y(m^*))\ P_i\ (X_i(m^*), Y(m^*))$ by monotonicity of preferences. Thus, if agent i chooses $z_i = nz_\lambda - \Sigma_{j \neq i} z_j^*$, $\eta_i < \eta_i^*$, and keeps other components of the message unchanged, then $z_\lambda \in B_z(m_i, m_{-i}^*)$, $Y(m^*) \in B_y(m_i, m_{-i}^*)$. Thus we have $Z(m_i, m_{-i}^*) = z_\lambda$, $Y(m_i, m_{-i}^*) = Y(m^*)$, and $x_i(m_i, m_{-i}^*) = [1/(1 + \eta_i \|z_i - y_i^*\|)][w_i^* + \pi_{i\lambda} - q_i(m^*) \cdot Y(m^*)] \geq [(1/(1 + \eta_i \|z_i - y_i^*\|)]x_i$. Since $x_i(m_{-i}^*, m_i)$ can arbitrarily approach $[1/(1 + \eta_i \|z_i - y_i^*\|)][w_i^* + \pi_{i\lambda} - q_i(m^*) \cdot y(m^*)]$ by choosing a sufficiently small η_i. From $(x_i, Y(m^*))\ P_i\ (X_i(m^*), Y(m^*))$ and continuity as well as monotonicity of preferences, we have $(x_i(m_i, m_{-i}^*), Y(m^*))\ P_i\ (X_i(m^*), Y(m^*))$. Since $\bar{\gamma}(m^*)\gamma_i^* = 1$ by Lemma 10, we have $X_i(m_i, m_{-i}^*) = x_i(m_i, m_{-i}^*)$ and thus $(X_i(m_i, m_{-i}^*), Y(m^*))\ P_i\ (X_i(m^*), Y(m^*))$. This contradicts $(X(m^*), Y(m^*)) \in N_{M,h}(e)$. Thus, $Y(m^*)$ must be the profit maximizing level of public goods. □

Proposition 1. *If the mechanism defined above has a Nash equilibrium m^*, then the Nash allocation $(X(m^*), Y(m^*))$ is a BLCSE allocation, i.e., $N_{M,h}(e) \subseteqq BLCSE(e)$.*

Proof. Let m^* be a Nash equilibrium. By Lemma 1, we only need to prove that $(X(m^*), Y(m^*))$ is a proportional allocation with $(q_1(m^*), \ldots, q_n(m^*))$ as a personalized price system. Note that, by construction, the mechanism is feasible, also by Lemmas 5–12, $Y(m^*)$ is the profit maximizing level of output, $q_i(m^*) > 0$, and $X_i(m^*) + (q_i(m^*) \cdot Y(m^*)/p(m^*) \cdot Y(m^*))C(Y(m^*)) = \mathring{w}_i$ or equivalently $X_i(m^*) + q_i(m^*) \cdot Y(m^*) = \mathring{w}_i + \pi_i(m^*)$ for all $i \in N$. So we only need to show that each individual is maximizing his/her preferences subject to his/her budget constraint $X_i(m^*) + q_i(m^*) \cdot y \leqq \mathring{w}_i + \pi_i(m^*)$.

Suppose, by way of contradiction, that there is some $(x_i, y) \in \mathbb{R}_+^{1+K}$ such that $x_i + q_i(m^*) \cdot y \leqq \mathring{w}_i + \pi_i(m^*)$ and $(x_i, y)\ P_i\ (X_i(m^*), Y(m^*))$. Since $(X(m^*), Y(m^*)) \in \mathbb{R}_{++}^{n+K}$ by Lemma 4, there is some $m_i \in M_i$ such that $(X_i(m_i, m_{-i}^*), Y(m_i, m_{-i}^*))\ P_i\ (X_i(m^*), Y(m^*))$ by Lemma 4. This contradicts $(X(m^*), Y(m^*)) \in N_{M,h}(e)$. Thus, $(X(m^*), Y(m^*))$ satisfies all the conditions of Lemma 1, and therefore it is a proportional allocation and thus is a BLCSE allocation. □

Proposition 2. *If (x^*, y^*) is a BLCSE allocation, then there is a Nash equilibrium m^* such that $Y(m^*) = y^*$ and $X_i(m^*) = x_i^*$ for all $i \in N$, i.e., $BLCSE(e) \subseteqq N_{M,h}(e)$.*

Proof. We first note that $(x^*, y^*) \in \mathbb{R}_{++}^{n+K}$ by interiority of preferences. Also, since $C(\cdot)$ is differentiable and convex, by Lemma 1, we know it is a proportional allocation with $(q_1^*,\ldots,q_n^*)$ as the personalized price vector system. We need to show that there is a message m^* such that (x^*,y^*) is a Nash equilibrium allocation, and $q_i(m^*) = q_i^*$ for all $i \in N$. For each $i \in N$, define $m_i^* = (w_i^*,q_i^*,z_i^*,y_i^*,t_i^*,\gamma_i^*,\eta_i^*)$ by $w_i^* = \mathring{w}_i$, $q_{ij}^* = q_j^*$ for $j = 1,\ldots,n$, $z_i^* = y^*$, $y_i^* = y^*$, $t_i^* = y^*$, $\gamma_i^* = 1$, and $\eta_i^* = 1$. Then, it can be easily verified that $Y(m^*) = y^* = Z(m^*)$, $q_i(m^*) = q_i^*$, $X_i(m^*) = x_i^*$ for all $i \in N$. Furthermore, $p(m^*) = \Sigma_{i=1}^n q_i(m^*) = \Sigma_{i=1}^n q_i^*$. Notice that $q_i(m_i,m_{-i}^*) = q_i(m^*)$, $\theta_i(m_i,m_{-i}^*) = \theta_i(m^*)$ for all $m_i \in M_i$, $(X(m_i,m_{-i}^*),Y(m_i,m_{-i}^*)) \in \mathbb{R}_+^{1+K}$. Then, for all $m_i \in M_i$, we have

$$\begin{aligned} X_i(m_i,m_{-i}^*) + q_i(m^*)\cdot Y(m_i,m_{-i}^*) &\leqq \mathring{w}_i + \pi_i(m_i,m_{-i}^*) \\ &\leqq \mathring{w}_i + \pi_i(m^*) \end{aligned} \tag{20}$$

since $Y(m^*)$ is the profit maximizing level of public goods under the price vector $p(m^*)$ and $\theta_i(m)$ is independent of m_i. Thus, $(X_i(m_i,m_{-i}^*),Y(m_i,m_{-i}^*))$ satisfies the budget constraint for all $m_i \in M_i$. Therefore, we must have $(X_i(m^*),Y(m^*))\ R_i\ (X_i(m_i,m_i^*),Y(m_i,m_{-i}^*))$, or it contradicts the fact that $(X(m^*),Y(m^*))$ is a proportional allocation. So $(X(m^*),Y(m^*)$ is a Nash equilibrium allocation. □

Proposition 3. *Every Nash equilibrium m^* of the mechanism defined above is a strong Nash equilibrium. That is, $N_{M,h}(e) \subseteqq SN_{M,h}(e)$.*

Proof. Let m^* be a Nash equilibrium. By Proposition 1, we know that $(X(m^*),Y(m^*))$ is a BLCSE allocation and thus it is a proportional allocation with $(q_1(m^*),\ldots,q_n(m^*))$ as the personalized price vector system by Lemma 1. Then $(X(m^*),Y(m^*))$ is Pareto optimal and thus the coalition N cannot be improved upon by any $m \in M$. Now for any coalition S with $\emptyset \neq S \neq N$, choose $i \in S$ such that $i + 1 \neq \in S$. Then no strategy played by S can change $q_i(m)$, $p_i(m)$, and $\theta_i(m)$ since they are determined by m_{i+1}. Also, $\pi_i(m_S,m_{-S}^*) \leqq \pi_i(m^*)$ for all $m_S \in M_S$. Furthermore, because $(X(m^*),Y(m^*)) \in BLCSE(e)$ and $\{(x_i,y) \in \mathbb{R}_+^{1+K}\colon x_i \leq (1/1 + \|q_i - q_{i+1}^*\| + \|t_i - t_{i+1}^*\| + \|z_i - y_i\| + \|y_i - t_i\|)\quad [\mathring{w}_i + \pi_i(m^*) - q_i(m^*)\cdot y]\} \subset \{(x_i,y) \in \mathbb{R}_+^{1+K}\colon x_i \leq \mathring{w}_i + \pi_i(m^*) - q_i(m^*)\cdot y\}$, it is P_i-maximal in the budget set of i, and thus S cannot improve upon $(X(m^*),Y(m^*))$. □

Since every strong Nash equilibrium is clearly a Nash equilibrium, by combining Propositions 1–3, we know that $N_{M,h}(e) = BLCSE(e)$ for all $e \in E$ and thus the proof of Theorem 1 is completed. □

Theorem 1 requires that $C(\cdot)$ be differentiable. When the differentiability condition is not satisfied, a BLCSE allocation may not be a proportional allocation. However, as Mas-Collel and Silvestre (1989), Remark 7) pointed out, as long as $C(\cdot)$ is convex, a proportional allocation is a BLCSE allocation even if C is not differentiable. Thus, by the fact that Propositions 1–3 actually show the equivalence among Nash allocations, strong Nash allocations, and proportional allocations, we have the following implementation result without assuming the differentiability of the cost function.

Theorem 2. *For the class of economic environments E with one private and K public goods, if the following assumptions are satisfied*:

1. $n \geqq 2$;
2. *preference orderings* R_i *are strictly increasing, continuous, convex, and satisfy the Interiority Condition of Preferences*; *and*
3. *the cost function* $C:\mathbb{R}^K_+ \to \mathbb{R}_+$ *is strictly increasing, convex, continuous, and* $C(0) = 0$,

then there exists a continuous and feasible mechanism which doubly implements the proportional allocations and consequently weakly doubly implements BLCSE allocations in Nash and strong Nash equilibria on E.

5. Concluding remarks

In this paper, we presented a market-type mechanism which doubly implements the Balanced Linear Cost Share equilibrium allocations when coalition patterns, preferences and endowments are unknown to the designer. The mechanism is well-behaved in the sense that it is feasible and continuous. The advantages of double implementation over Nash implementation and strong Nash implementation are that it covers the case where agents in some coalitions may cooperate and in some other coalitions may not when such information is unknown to the designer. The combining solution concept may also give a state which takes advantage of both Nash equilibrium and strong Nash equilibrium so that it may be easy to reach and hard to leave.

It may be remarked that the implementation result presented here deals only with the case where the cost function is known to the designer. By using similar techniques given in Hong (1995) and Tian (1997), one may be able to give a mechanism which doubly implements the BLCSE solution when the cost function is also unknown to the designer. However, the resulting mechanism may have to significantly increase the dimension of the message space.

Acknowledgements

I wish to thank Professor Thomas Piketty and an anonymous referee for very helpful comments and suggestions. Financial support from the Private Enterprise Research Center, the Lewis Faculty Fellowship, the Bush Program in the

Economics of Public Policy, and the Program to Enhance Scholarly and Creative Activities at Texas A&M University is gratefully acknowledged.

Appendix A

Proof of Lemma 2. $B_z(m)$ is clearly upper hemi-continuous for all $m \in M$ by the continuity of $p_i(\cdot)$, $\theta_i(\cdot)$, and $C(\cdot)$. We only need to show that $B_z(m)$ is also lower hemi-continuous at every $m \in M$. Let $m \in M$, $z \in B_z(m)$, and let $\{m_k\}$ be a sequence such that $m_k \rightarrow m$, where $m_k = (m_1^k, \dots, m_n^k)$ and $m_i^k = (w_i^k, q_i^k, z_i^k, y_i^k, t_i^k, \gamma_i^k, \eta_i^k)$. We want to prove that there is a sequence $\{z_k\}$ such that $z_k \rightarrow z$, and, for all k, $z_k \in B_z(m_k)$, i.e., $z_k \in \mathbb{R}_+^K$, $C(z_k) \leqq \Sigma_{i\in N} w_i^k$, and $w_i^k/2 + \theta_i(m_k)[p_i(m_k) \cdot z_k - C(z_k)] \geqq 0$ for all $i \in N$. We first prove that there is a sequence $\{\bar{z}_k\}$ such that $\bar{z}_k \rightarrow z$, and, for all k, $\bar{z}_k \in \mathbb{R}_+^K$ and $C(\bar{z}_k) \leqq \Sigma_{i\in N} w_i^k$. Two cases will be considered.

Case i. $C(z) < \Sigma_{i\in N} w_i$. Then, for all k larger than a certain integer k', we have $C(z) < \Sigma_{i\in N} w_i^k$. Let $\bar{z}_k = z$ for all $k > k'$ and $\bar{z}_k = 0$ for $k \leqq k'$. Then, $\bar{z}_k \rightarrow z$, and, for all k, $\bar{z}_k \in \mathbb{R}_+^K$ and $C(\bar{z}_k) \leqq \Sigma_{i\in N} w_i^k$.

Case ii. $C(z) = \Sigma_{i\in N} w_i$. Define $\bar{z}_k$ as follows:

$$\bar{z}_k = \begin{cases} \dfrac{\sum\limits_{i\in N} w_i^k}{C(z)} z & \text{if } \dfrac{\sum\limits_{i\in N} w_i^k}{C(z)} \leqq 1. \\ z & \text{otherwise} \end{cases}$$

Then $\bar{z}_k \leqq z$. Also, since $(\Sigma_{i\in N} w_i^k / C(z)) \rightarrow (\Sigma_{i\in N} w_i / C(z) = 1)$, we have $\bar{z}_k \rightarrow z$. We now claim that $\bar{z}_k$ satisfies $C(\bar{z}_k) \leqq \Sigma_{i\in N} w_i^k$. Indeed, if $C(z) \geqq \Sigma_{i\in N} w_i^k$, i.e., $(\Sigma_{i\in N} w_i^k / C(z) \leqq 1)$, then $\bar{z}_k = (\Sigma_{i\in N} w_i^k / C(z))z$ and thus $C(\bar{z}_k) \leqq (\Sigma_{i\in N} w_i^k / C(z))C(z) = \Sigma_{i\in N} w_i^k$ by the convexity of $C(\cdot)$ and $C(0) = 0$.[5] If $C(z) < \Sigma_{i\in N} w_i^k$, i.e., $(\Sigma_{i\in N} w_i^k / C(z)) > 1$, then $\bar{z}_k = z$ and thus $C(\bar{z}_k) = C(z) < \Sigma_{i\in N} w_i^k$.

Thus, in both cases, there is a sequence $\{\bar{z}_k\}$ such that $\bar{z}_k \rightarrow z$, and, for all k, $C(\bar{z}_k) \leqq \Sigma_{i\in N} w_i^k$.

We now show that there is a sequence $\{\hat{z}_k\}$ such that $\hat{z}_k \rightarrow z$, and, for all k, $\hat{z}_k \in \mathbb{R}_+^K$ and $w_i^k/2 + \theta_i(m_k)[p_i(m_k) \cdot \hat{z}_k - C(\hat{z}_k)] \geqq 0$ for all $i \in N$.

Let $N' = \{i \in N : w_i/2 + \theta_i(m)[p_i(m) \cdot z - C(z)] = 0\}$. Again, two cases will be considered.

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[5] This is because, by the convexity of $C(\cdot)$ and $C(0) = 0$, $C(\lambda z) = C((1-\lambda)0 + \lambda z) \leqq (1-\lambda)C(0) + \lambda C(z) = \lambda C(z)$ for all $\lambda \leqq 1$.

Case 1. $N' = \emptyset$, i.e., $w_i/2 + \theta_i(m)[p_i(m)\cdot z - C(z)] > 0$ for all $i \in N$. Then, by the continuity of $\theta_i(\cdot)$ and $p_i(\cdot)$, for all k larger than a certain integer k', we have $w_i^k/2 + \theta_i(m_k)[p_i(m_k)\cdot z - C(z)] > 0$. Let $\hat{z}_k = z$ for all $k > k'$ and $\hat{z}_k = 0$ for $k \leqq k'$. Then, $\hat{z}_k \to z$, and, for all k, $\bar{z}_k \in \mathbb{R}_+^K$ and $w_i^k/2 + \theta_i(m_k)[p_i(m_k)\cdot \hat{z}_k - C(\hat{z}_k)] \geqq 0$ for all $i \in N$.

Case 2. $N' \neq \emptyset$. Then $w_i/2 + \theta_i(m)[p_i(m)\cdot z - C(z)] = 0$ for all $i \in N'$. Note that, since $w_i > 0$ and $\theta_i(\cdot) > 0$, we must have $C(z) - p_i(m)\cdot z > 0$, and thus, by the continuity of $p_i(\cdot)$, $f(z) \equiv C(z) - p_i(m_k)\cdot z > 0$ for all k larger than a certain integer k'. For each $k \geqq k'$ and $i \in N'$, let $\lambda_{ik} = w_i^k/2\theta_i(m_k)[C(z) - p_i(m_k)\cdot z]$, let

$$z_{ik} = \begin{cases} \lambda_{ik} z & \text{if } \dfrac{w_i^k}{2\theta_i(m_k)[C(z) - p_i(m_k)\cdot z]} \leqq 1, \\ z & \text{otherwise} \end{cases}$$

and let $\hat{z}_k = \min_{i \in N'}\{z_{ik}\}$. Then $\hat{z}_k \leqq z_{ik} \leqq z$. Also, since $\lambda_{ik} = w_i^k/2\theta_i(m_k)[C(z) - p_{iu}(mk_k)\cdot z] \to w_i/(1\theta_i(m)[C(z) - p_i(m)\cdot z]) = 1$ for all $i \in N'$, we have $\hat{z}_k \to z$ and $f(z_{ik}) > 0$ for all k larger than a certain integer k''. Now we claim that $\hat{z}_k$ also satisfies $(w_i^k/2 + \theta_i(m_k)[p_i(m_k)\cdot \hat{z}_k - C(\hat{z}_k)]) \geqq 0$ for all $i \in N$ and $k \geqq \max\{k', k''\}$. Indeed, for each $i \in N'$, if $(w_i^k/2\theta_i(m_k)[C(z) - p_i(m_k)\cdot z]) \leqq 1$, then $\hat{z}_k \leqq z_{ik} = \lambda_{ik} z$, and thus we have $2\theta_i(m_k)[C(\hat{z}_k) - p_i(m_k)\cdot \hat{z}_k] \leqq 2\theta_i(m_k)[C(\lambda_{ik} z) - p_i(m_k)\cdot \lambda_{ik} z] \leqq (w_i^k/2\theta_i(m_k)[C(z) - p_i(m_k)\cdot z])2\theta_i(m_k)[C(z) - p_i(m_k)\cdot z] = w_i^k$ by noting that $f(\hat{z}_k) \leqq f(z_{ik}) = f(\lambda_{ik} z) \leqq \lambda_{ik} f(z)$.[6] Consequently, we have $w_i^k/2 + \theta_i(m_k)[p_i(m_k)\cdot \hat{z}_k - C(\hat{z}_k)] \geqq 0$.

Now, for each $i \in N'$, if $(w_i^k/2\theta_i(m_k)[C(z) - p_i(m_k)\cdot z]) > 1$, i.e., $2\theta_i(m_k)[C(z) - p_i(m_k)\cdot z] < w_i^k$, then $\hat{z}_k \leqq z_{ik} = z$, and thus $2\theta_i(m_k)[C(\hat{z}_k) - p_i(m_k)\cdot \hat{z}_k] \leqq 2\theta_i(m_k)[C(z) - p_i(m_k)\cdot z] < w_i^k$, as above, by the convexity of $f(\cdot)$, $f(0) = 0$, and $0 < f(\hat{z}_k) \leqq f(z)$. Consequently, we have $w_i^k/2 + \theta_i(m_k)[p_i(m_k)\cdot \hat{z}_k - C(\hat{z}_k)] > 0$.

For all $i \in N \backslash N'$, since $w_i^k/2 + \theta_i(m_k)[p_i(m_k)\cdot z - C(z)] > 0$, we have $w_i^k/2 + \theta_i(m_k)[p_i(m_k)\cdot \hat{z}_k - C(\hat{z}_k)] > 0$ for all k larger than a certain integer k''' by the continuity of $C(\cdot)$. Thus, for all $k \geqq \max\{k', k'', k'''\}$ and $i \in N$, we have $w_i^k/2 + \theta_i(m_k)[p_i(m_k)\cdot \hat{z}_k - C(\hat{z}_k)] \geqq 0$.

Finally, let $z_k = \min(\bar{z}_k, \hat{z}_k)$. Then $z_k \to z$ since $\bar{z}_k \to z$ and $\hat{z}_k \to z$. Also, for every k larger than a certain integer k, we have $z_k \geqq 0$, $C(z_k) \leqq C(\bar{z}_k) \leqq \Sigma_{i \in N} w_i^k$ by the monotonicity of $C(\cdot)$, and, as above, by the convexity of $f(\cdot)$, $f(0) = 0$, and $0 < f(z_k) \leqq f(\hat{z}_k)$, we have $2\theta_i(m_k)[C(z_k) - p_i(m_k)\cdot z_k] \leqq 2\theta_i(m_k)[C(\hat{z}_k) - p_i(m_k)\cdot \hat{z}_k] \leqq w_i^k$ which implies that $w_i^k/2 + \theta_i(m_k)[p_i(m_k)\cdot z_k - C(z_k)] \geqq 0$ for all $i \in N$. Let $z_k = \min\{\bar{z}_k, \hat{z}_k\}$ for all $k > k$ and $z_k = 0$ for $k \leqq k$. Then, $z_k \to z$, and $z_k \in B_z(m_k)$ for

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[6] This is because $f(z)$ is convex with $f(0) = 0$ by the convexity of $C(\cdot)$ and $C(0) = 0$, and thus $f(\lambda_{ik} z) \leqq \lambda_{ik} f(z)$. So the second inequality holds. To see the first inequality also holds, writing $\hat{z}_k = \lambda z_{ik}$ for some $\lambda \leqq 1$ (because $\hat{z}_k \leqq z_{ik}$ as well as $\hat{z}_k$ and z_{ik} are both proportional to z, such a λ exists), we have $f(\hat{z}_k) = f(\lambda z_{ik}) \leqq \lambda f(z_{ik}) \leqq f(z_{ik})$ by noting that $f(z_{ik}) > 0$ for $k \geqq k'''$.

all k. Therefore, the sequence $\{z_k\}$ has all the desired properties. So $B_z(m)$ is lower hemi-continuous at every $m \in M$. $\square$

Proof of Lemma 3. $B_y(m)$ is clearly upper hemi-continuous for all $m \in M$ by the continuity of $Z(\cdot)$ and $\pi_i(\cdot)$. We only need to show that $B_y(m)$ is also lower hemi-continuous at every $m \in M$. Let $m \in M$, $y \in B_y(m)$, and let $\{m_k\}$ be a sequence such that $m_k \to m$, where $m_k = (m_1^k, \dots, m_n^k)$ and $m_i^k = (w_i^k, q_i^k, z_i^k, y_i^k, t_i^k, \gamma_i^k, \eta_i^k)$. We want to prove that there is a sequence $\{y_k\}$ such that $y_k \to y$, and, for all k, $y_k \in B_y(m_k)$, i.e., $y_k \in \mathbb{R}_+^K$, $y_k \leqq Z(m_k)$, and $w_i^k + \pi_i(m_k) \geqq q_i(m_k) \cdot y_k$ for all $i \in N$. We first prove that there is a sequence $\{\hat{y}_k\}$ such that $\hat{y}_k \to y$, and, for all k, $\hat{y}_k \in \mathbb{R}_+^K$ and $w_i^k + \pi_i(m_k) \geqq q_i(m_k) \cdot \hat{y}_k$ for all $i \in N$.

Let $N' = \{i \in N : q_i(m) \cdot y = w_i + \pi_i(m)\}$. Two cases will be considered.

Case 1. $N' = \emptyset$, i.e., $q_i(m) \cdot y < w_i + \pi_i(m)$ for all $i \in N$. Then, for all k larger than a certain integer k', we have $q_i(m_k) \cdot y < w_i^k + \pi_i(m_k)$. Let $\hat{y}_k = y$ for all $k > k'$ and $\hat{y}_k = 0$ for $k \leqq k'$. Then, $\hat{y}_k \to y$, and, for all k, $y_k \in \mathbb{R}_+^K$ and $q_i(m_k) \cdot y_k \leqq w_i^k + \pi_i(m_k)$ for all $i \in N$.

Case 2. $N' \neq \emptyset$. Then $q_i(m) \cdot y = w_i + \pi_i(m)$ for all $i \in N'$. Note that since $w_i + \pi_i(m) > 0$, we must have $q_i(m) \cdot y > 0$ and thus, by the continuity of $q_i(\cdot)$, $q_i(m_k) \cdot y > 0$ for all k larger than a certain integer k'. For each $k \geqq k'$, let $a_{ik} = (w_i^k + \pi_i(m_k)/q_i(m_k) \cdot y)y$ for $i \in N'$, let $N'(k) = \{i \in N' : q_i(m_k) \cdot y \geqq w_i^k + \pi_i(m_k)\}$, let $a_k = \min_{i \in N'(k)}\{a_{ik}\}$, and define $\hat{y}_k$ as follows:

$$\hat{y}_k = \begin{cases} a_k & \text{if } N'(k) \neq \emptyset \\ y & \text{otherwise} \end{cases}.$$

Then $\hat{y}_k \leqq y$ by noting that $(w_i^k + \pi_i(m_k)/q_i(m_k) \cdot y) \leqq 1$ and $a_{ik} = (w_i^k + \pi_{ik}/q_i(m_k) \cdot y)y \leqq y$ for $i \in N'(k)$. Also, since $(w_i^k + \pi_i(m_k)/q_i(m_k) \cdot y) \to (w_i + \pi_i(m)/q_i(m) \cdot y) = 1$ for all $i \in N'$, we have $a_{ik} \to y$ for all $i \in N'$ and thus $\hat{y}_k \to y$. Now we claim that $\hat{y}_k$ also satisfies allindividuals' budget sets. Two subcases are needed to consider: Subcase 1. $N'(k) \neq \emptyset$. Then $\hat{y}_k = a_k \leqq y$, and thus we have

$$q_i(m_k) \cdot \hat{y}_k \leqq q_i(m_k) \cdot a_{it} = w_i^k + \pi_i(m_k)$$

for any $i \in N'(k)$, and

$$q_i(m_k) \cdot \hat{y}_k \leqq q_i(m_k) \cdot y < w_i^k + \pi_i(m_k)$$

for any $i \in N' \backslash N'(k)$. Subcase 2. $N'(k) = \emptyset$. Then $\hat{y}_k = y$ and thus $q_i(m_k) \cdot y < w_i^k + \pi_i(m_k)$ for all $i \in N'$.

For all $i \in N \backslash N'$, since $q_i(m) \cdot y < w_i + \pi_i(m)$, we have $q_i(m_k) \cdot \hat{y}_k \leqq q_i(m_k) \cdot y < w_i^k + \pi_i(m_k)$ for all k larger than a certain integer k''. Thus, for all $k \geqq \max\{k', k''\}$, we have $q_i(m_k) \cdot \hat{y}_k \leqq w_i^k + \pi_i(m_k)$ for all $i \in N$.

We now show that there is a sequence $\{\bar{y}_k\}$ such that $\bar{y}_k \rightarrow y$, and, for all k, $\bar{y}_k \in \mathbb{R}^K_+$ and $y_k \leqq Z(m_k)$. If $Z(m) = 0$, we must have $y = 0$ because $y \in B_y(m)$. Let $\bar{y}_k = y = 0$ for all k. Then, the sequence $\{\bar{y}_k\}$ has all the desired properties. So we only need to consider the case where $Z(m) > 0$. For any component y^l of y $(1 \leqq l \leqq K)$, two cases will be considered.

Case i. $y^l < Z^l(m)$. Then, by the continuity of $Z^l(\cdot)$, we have $y^l < Z^l(m_k)$ for all k larger than a certain integer k'. Let $\bar{y}^l_k = y^l$ for all $k > k'$ and $\bar{y}^l_k = 0$ for $k \leqq k'$. Then, $\bar{y}^l_k \rightarrow y^l$, and, for all k, $\bar{y}^l_k \geqq 0$ and $\bar{y}^l_k \leqq Z^l(m_k)$.

Case ii. $y^l = Z^l(m)$. Define $\bar{y}^l_k$ as follows:

$$\bar{y}^l_k = \begin{cases} Z^l(m_k) & \text{if } y^l \geqq Z(m_k) \\ y^l & \text{otherwise} \end{cases}.$$

Then $\bar{y}^l_k \rightarrow y^l$ and $y^l_k \leqq Z^l(m_k)$ by the definition of $\bar{y}^l_k$.

Thus, in both cases, there is a sequence $\{\bar{y}_k\}$ such that $\bar{y}_k \rightarrow y$, and, for all k, $\bar{y}_k \leqq Z(m_k)$.

Finally, let $y_k = \min(\bar{y}_k, \hat{y}_k)$. Then $y_k \rightarrow y$ since $\bar{y}_k \rightarrow y$ and $\hat{y}_k \rightarrow y$. Also, for every k, $y_k \geqq 0$, $y_k \leqq Z(m_k)$ and $w^k_i + \pi_i(m_k) \geqq q_i(m_k) \cdot y_k$ for all $i \in N$ because $y_k \leqq \bar{y}_k$ and $y_k \leqq \hat{y}_k$. Thus, $y_k \in B_y(m_k)$ for all k. Therefore, the sequence $\{y_k\}$ has all the desired properties. So $B_y(m)$ is lower hemi-continuous at every $m \in M$. □

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