Equilibria in First Price Auctions with Participation Costs[*]

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Abstract

This paper characterizes equilibria of first price auctions with participation costs in the independent private values environment. We focus on the cutoff strategies in which each bidder participates and submits a bid if his value is greater than or equal to a critical value. It is shown that, when bidders are homogenous, there always exists a unique symmetric equilibrium, and further, there is no other equilibrium when valuation distribution functions are concave. However, when distribution functions are elastic at the symmetric equilibrium, there exists an asymmetric equilibrium. We find similar results when bidders are heterogenous.

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1 Introduction

Auction is an efficient way to increase the competition among buyers and, in turn, increases the efficiency of allocating scarce resources in the presence of private information. However, they are generally not freely implemented. In many situations, a pre-bid cost is required to permit bidders to attend an auction, and sometimes the cost can be very high. As Mills (1993) pointed

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out, the bidding cost incurred by a typical bidder in a government procurement auction often runs into the millions of dollars. This paper studies (Bayesian-Nash) equilibria of sealed-bid first price auctions with bidder participation costs in the independent private values environment.

1.1 Motivation

The fundamental structure of the first price auction with participation costs is one through which an indivisible object is allocated to one of many potential buyers, and in order to participate in the auction, buyers must incur some costs[1]. After the cost is incurred, a bidder can submit a bid. The bidder who submits the highest bid wins the object and pays his own bid.

There are many sources for participation costs. For instance, sellers may require that those who submit bids have a certain minimum amount of bidding funds which may compel some bidders to borrow; bidders themselves may have transportation costs to go to an auction place; or they need spend some money to learn the rules of the auction and how to submit bids. Bidders even have opportunity costs to attend an auction.

When participation costs are present, bidders' behavior may change. If a bidder's expected revenue in participating in an auction is less than the participation cost, he will choose not to participate in the auction. Even if he decides to participate in the auction, since the number of bidders who submit bids is endogenous, his bidding behavior may not be the same as it would be in the standard auction without participation costs. The number of bidders can affect the strategic behavior among the bidders greatly (cf. McAfee and MaMillan (1987) , Harstad, Kagel and Levin (1990), and Levin and Smith (1996)). For example, in first price auctions, bidders shade more of their valuations as fewer bidders submit bids in the auctions.

There are some studies on the information acquisition in auctions. A bidder may want to learn how he/she and the others value the item, and thus he/she may incur a cost in information acquisition about their valuations[2]. A main difference between participation costs and information acquisition costs is that information acquisition costs are avoidable while participation costs are not. If a bidder does not want to collect information about her own or others' valuations, she does not incur any cost, but she can still submit bids. Some researchers, such as McAfee and McMillan (1987), Harstad (1990) and Levin and Smith (1994), combine the idea of participation costs and the idea of information acquisition costs. Compete and Jehiel (2007)

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[1] Related terminology includes participation cost, participation fee, entry cost or opportunity costs of participating in the auction. See Laffont and Green (1984), Samuelson (1985), McAfee and McMillan (1987), etc.

[2] Persico (2000) studied the incentives of information acquisition in auctions. He found that bidders have more incentives for information acquisition in first price auctions than in second price auctions.

investigate the advantage of using dynamic auctions in the presence of information acquisition cost only. However, information acquisition costs and participation costs can both be regarded as sunk costs after the bidders submit bids.

1.2 Related Literature

The studies of participation costs in auctions so far have mainly focused on the second price auction due to its simplicity of bidding behavior in the interim information acquisition setting.[3] In second price auctions (Vickrey, 1961), bidders cannot do better than bidding their valuations when they find participating optimal. Much of the existing literature investigates equilibria of second price auctions with participation costs. Laffont and Green (1984) study the second price auction with participation costs in a general framework where bidders' valuations and participation costs are both private information and establish the existence of symmetric equilibrium with uniform distribution. Gal, Landsberger and Nemirovski (2007) study equilibria in a two dimensional framework with more general distributions, focusing on symmetric equilibrium only. Campbell (1998) and Tan and Yilankaya (2006) study equilibria and their properties of second price auctions in an economic environment with equal participation costs when bidders' values are private information. Cao and Tian (2008) investigate equilibria in second price auctions when bidders may have differentiated participation costs. They introduced the notions of monotonic equilibrium and neg-monotonic equilibrium. Kaplan and Sela (2006) consider a private entry model in second price auctions in which they assume all bidders' valuations are common knowledge while participation costs are private information.

Studies of first price auctions in the presence of participation costs, however, have received little attention, although they are used more often in practice[4], like the auctions for tendering, particularly for government contracts and auctions for mining leases. The difficulty partly lies in the fact that in first price auctions, bidding strategies are not so explicit, as compared with the strategies in second price auctions. Bidders in first price auctions no longer bid their true valuations. The degree of shading relies heavily on who others enter the auction and the information inferred from the entrance behavior of those bidders. The effect of the information inferred on the bidding strategy of first price auctions is greater than that on second price auctions.

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[3] There are also some work in the ex ante information acquisition setting, in which bidders make the entrance decisions before they know their valuations (cf. McAfee and McMillan (1987), Engelbrecht-Wiggans (1993), Levin and Smith (1994) and Chakraborty and Kosmopolov (2001)).

[4] Samuelson (1985) studies the entrance equilibrium of first price competitive procurement auctions and related welfare problem, focusing on the symmetric cutoff threshold.

Moreover, when bidders use different thresholds to enter an auction, the valuation distributions updated from their entrance behavior are different so that there may be no explicit bidding function and some bidders may use mixed strategies. As such, it is more technically difficult to solve the cutoff strategy since it is determined by the expected revenue of participating in the auction at the thresholds, which in turn depends on the more complicated bidding functions of bidders who submit bids.

There are some studies on equilibrium behavior in economic environments with different valuation distributions which can be used to study the equilibria of first price auctions with participation costs. Kaplan and Zamir (2000, 2007) discuss the properties of bidding functions when valuations are uniformly distributed with different supports. Martinez-Pardina (2006) study the first price auction in which bidders' valuations are common knowledge. They show that in equilibrium bidders whose valuations are common knowledge randomize their bids.

1.3 The Results of the Paper

In this paper, we investigate Bayesian-Nash equilibria of sealed-bid first price auctions in the independent private values environment with participation costs. We assume bidders know their valuations and participation costs before they make their decisions. Participation costs are assumed to be the same across all the bidders.

When bidders are homogenous, there is a unique symmetric equilibrium. We show that there is no other equilibrium when valuation distribution functions are concave. However, when valuation distribution functions are elastic at the symmetric equilibrium, there always exists an asymmetric equilibrium. It may be remarked that, when a distribution function is strictly convex, it is elastic everywhere, specifically at the symmetric equilibrium, and therefore there exists an asymmetric equilibrium. Moreover, when bidders are in two different groups, the cutoffs used by one group can always be different from those used by the other group.

The existence of asymmetric equilibria has important consequences for the strategic behavior of bidders and the efficiency of the auction mechanism. When an auction has a participation cost, a bidder would expect less bidders to submit their bids. When symmetric equilibrium is unique, every bidder has to follow the symmetric cutoff and has no other choices. However, when asymmetric equilibria exist, bidders may choose an equilibrium they are more desirable. In this case, some bidders may form a collusion to cooperate at the entrance stage by choosing a smaller cutoff point that may decrease the probability that other bidders enter the auction, and consequently, may reduce the competition in the bidding stage. An asymmetric equilibrium may become more desirable when an auction can run repeatedly. Also, an asymmetric equilibrium may be ex-post inefficient. The item being auctioned is not necessarily allocated to the bidder with the highest valuation.

We also consider the existence of equilibria in an economy with heterogenous bidders in sense that the distribution functions are different. Specifically, we consider the case where one distribution (called a weak bidder) is first order dominated by another (called a strong bidder). We concentrate on equilibria that the bidders in the same group use the same threshold. We show that there is always an equilibrium in which the strong bidders are more likely to enter the auction by using a smaller cutoff point for valuations. When the distribution functions are concave, the equilibrium is unique. However, when the distribution functions for the weak bidders are strictly convex, and the participation costs are sufficiently large, there exists an equilibrium in which weak bidders are more likely to enter the auction.

The remainder of the paper is structured as follows. Section 2 presents a general setting of economic environment. Section 3 studies the existence and uniqueness of equilibria for homogenous bidders. Section 4 studies equilibria for heterogenous bidders. Concluding remarks are provided in Section 5. All the proofs are presented in the appendix.

2 Economic Environment

We consider an independent private values economic environment with one seller and $n \geq 2$ risk-neutral buyers (bidders). The seller is also risk-neutral and has an indivisible object to sell to one of the buyers. The seller values the object as 0. Each buyer i's valuation for the object is v_i ($i = 1, 2, ..., n$), which is private information to the other bidders. It is assumed that v_i is independently distributed with a cumulative distribution function $F_i(\cdot)$ that has continuously differentiable density $f_i(\cdot) > 0$ everywhere with support $[0, 1]$.

The auction format is the sealed-bid first price auction. The bidder with the highest bid wins the auction and pays the price equal to his bid. His payoff is equal to the difference between his valuation and the price. The other bidders have zero payoff from submitting a bid. If the highest bid is submitted by more than one bidder, there is a tie which is broken by a fair lottery.

There is a participation cost, common to all bidders, denoted by $c \in (0, 1)$. Bidders must incur c in order to submit bids. It is assumed that each bidder knows his own valuation and who will participate, but does not know the others' valuations so that we are in the interim information setting. Specifically, the timing of the game is as follows:

- Nature draws a valuation v_i for each bidder i and tells the bidder only what his own valuation is.

- Bidder i decides whether or not to submit a bid. If he chooses to submit a bid, he pays the participation cost c which is not refundable, otherwise the game ends for him.

- All the bidders who pay the participation costs observe whose others also participate in the auction and submit bids. The item is awarded to the bidder who submits the highest bid and pays his own bid. If more than one bidder submit the highest bid, the allocation is determined by a fair lottery.

The individual action set for any bidder can be characterized as $No \cup [0, 1]$, where "No" denotes not submitting a bid. Bidder i incurs the participation cost c if and only if his action is different from "No." While it is always a weakly dominant strategy to bid one's true valuation in second price auctions, this is not true for first price auctions. In first price auctions, a bidder may submit a bid that may not be his true valuation. Nevertheless, given the strategies of all other bidders, a bidder's expected revenue from participating in the auction is a non-decreasing function of his valuation. Thus bidders use the cutoff strategy[5], i.e., a bidder submits a bid if and only if his valuation is greater than or equal to a cutoff point and does not enter otherwise[6].

An equilibrium strategy whether to participate is then given by a profile of the bidders' cutoff points, which are a vector of the minimum valuations for each bidder i to cover the cost. Let $v^* = (v_1^*, \ldots, v_n^*)$ denote the profile of bidders' cutoff points and $S_i(v^*)$ denote the set of bidders who also participate in the auction beside bidder i. The bidding decision function $b_i(\cdot)$ of each bidder is characterized by

$$b_i(v_i, v^*, S_i(v^*)) = \begin{cases} \lambda_i(v_i, v^*, S_i(v^*)) & \text{if } 1 \geq v_i \geq v_i^* \\ No & \text{if } v_i < v_i^*, \end{cases}$$

where $\lambda_i(v_i, v^*, S_i(v^*))$ is a contingent bidding function when bidder i participates in the auction. Note that, if bidder i enters the auction while all the others do not enter, bidder i will bid zero. If some other bidders also participate in the auction, the bid depends on the cutoff points and the valuation distributions of all others. For notational simplicity, we use $b_i(v_i, v^*)$ to denote $b_i(v_i, v^*, S_i(v^*))$ and $\lambda_i(v_i, v^*)$ to denote $\lambda_i(v_i, v^*, S_i(v^*))$ in the remainder of the paper.

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[5] In Lu and Sun (2007), they show that for any auction mechanism with participation costs, the participating and nonparticipating types of any bidder are divided by a nondecreasing and equicontinuous shutdown curve. Thus in our framework, when participation cost is given, the participating and nonparticipating types of any bidder can be divided by a cutoff value and the threshold form is the only form of equilibria.

[6] In Milgrom and Weber (1982), the term of "screening level" is used instead of using "cutoff point."

For the game described above, each bidder's action is to choose a cutoff and decide how to bid when he participates. Thus, a (Bayesian-Nash) equilibrium of the game is composed of bidders' cutoff strategies, together with participants' bidding strategies.

Formally, we have the following definitions:

Definition 1 An equilibrium

$$(v^*, \mathbf{b}(v_i, v^*)) = ((v_1^*, b_1(v_i, v^*)), \ldots, (v_n^*, b_n(v_i, v^*))) \in \mathbb{R}_+^{2n}$$

is a profile of cutoff points together with optimal bidding functions such that each bidder i's action is optimal, given others' strategies.

Note that, once the cutoff points are given, for those bidders who participate in the auction, the game is reduced to the standard first price auction and the optimal bidding functions are uniquely determined (see Maskin and Riley (2003)). As such, it is sufficient for us to focus on cutoff points $v^* = (v_1^*, \ldots, v_n^*) \in \mathbb{R}_+^n$ to describe the equilibrium. All of our results about uniqueness or multiplicity of equilibria, should be interpreted in terms of cutoffs, accordingly.

As usual, when bidders' distribution functions are the same; i.e., $F_1(\cdot) = F_2(\cdot) = \ldots = F_n(\cdot) = F(\cdot)$, we introduce the definition of symmetric and asymmetric equilibria focusing on the cutoffs:

Definition 2 For the economic environment with the same distribution functions, an equilibrium $v^* = (v_1^*, \ldots, v_n^*) \in \mathbb{R}_+^n$ is a symmetric (resp. asymmetric) equilibrium if the bidders have the same cutoff points; i.e., $v_1^* = v_2^* = \ldots = v_n^*$, (resp. different cutoff points). Denote the symmetric equilibrium by $v_s = (v^s, \ldots, v^s)$.

Remark 1 It may be worth to mention the following remarks on the cutoff points:

(1) $v_i^* > 1$ means that bidder i will never participate in the auction, no matter what his valuation is. This is the case where the bidder's revenue from participating in the auction is less than c even when $v_i = 1$.

(2) When $v_i^* < v_i \leq 1$, bidder i will enter the auction and submit a bid $\lambda_i(v_i, v^*)$. When $v_i = v_i^*$, bidder i is indifferent between participating in the auction and holding out. For discussion convenience, we assume he enters the auction. When $v_i < v_i^*$, bidder i does not participate in the auction.

(3) $v_i^* \geq c$.

(4) As shown in Cao and Tian (2007), $v_i^* \leq 1$ for at least one bidder i.

Note that, once a bidder enters the auction, he can observe who have also entered the auction and thus can update his belief about others' valuation distributions. If we observe that bidder i participates in the auction, it can be inferred that bidder i's value is bigger than or equal to v_i^*. Then, by Bayes's rule, bidder i's value is distributed on $[v_i^*, 1]$ with

$$Pr(\xi \leq v | v \geq v_i^*) = \frac{Pr(v_i^* \leq \xi < v)}{Pr(\xi \geq v_i^*)} = \frac{F_i(v) - F_i(v_i^*)}{1 - F_i(v_i^*)}.$$

The corresponding density function is given by $\frac{f_i(v)}{1-F_i(v_i^*)}$.

3 Homogeneous Bidders

In this section we analyze the case in which bidders' valuations are drawn from the same distribution function; i.e., $F_i(.) = F(.)$ for all i. We first study the symmetric equilibrium. For any two bidders who use the same cutoff point v^s, the supports of their updated valuation distributions have the same lower bound when they both participate in the auction. Then the minimal bids they submit should be equal. Thus when $v_i = v^s$, bidder i can only win the item when all others do not participate. In equilibrium we have

$$c = v^s F(v^s)^{n-1}.$$

Since $\rho(v) = vF(v)^{n-1} - c$ is an increasing function of v with $\rho(0) < 0$ and $\rho(1) > 0$, there exists a unique symmetric equilibrium. To illustrate how the bidders submit bids when they face different number of other bidders who enter the auction, consider the following example:

Example 1 Suppose $F(v)$ is uniform on $[0, 1]$. Then by $v^s F(v^s)^{n-1} = c$ we have $v^s = \sqrt[n]{c}$. Then when $v_i \geq \sqrt[n]{c}$, bidding function for i is $\lambda_i(v_i, v^*) = v_i - \frac{v_i - \sqrt[n]{c}}{1+S_i(v^*)}$ if $S_i(v^*) \in \{1, 2, \cdots, n-1\}$ and zero if $S_i(v^*) = 0$. Otherwise, bidder i will not participate in the auction. Hence, the unique symmetric equilibrium is $(\sqrt[n]{c}, \sqrt[n]{c} \cdots, \sqrt[n]{c})$ and the bidding function is given by

$$b_i(v_i, v^*) = \begin{cases} \lambda_i(v_i, v^*) & 1 \geq v_i \geq \sqrt[n]{c} \\ No & v_i < \sqrt[n]{c}, \end{cases}$$

where

$$\lambda_i(v_i, v^*) = \begin{cases} 0 & \text{if } S_i(v^*) = 0 \\ v_i - \frac{v_i - \sqrt[n]{c}}{1+S_i(v^*)} & \text{if } S_i(v^*) \in \{1, 2, \cdots, n-1\}. \end{cases}$$

Now we consider the existence of asymmetric equilibria. Suppose there are only two different cutoff points used by the bidders. Bidders $i = 1, \ldots, m$ use v_1^* and bidders $j = m+1, \ldots, n$ use v_2^* as the cutoff point. Without loss of generality, we assume $v_1^* < v_2^*$. By Remark 1, we must have $v_1^* \leq 1$. Thus we divide the bidders into two types or groups. Bidders in type 1 use v_1^* and bidders in type 2 use v_2^* as their cutoffs separately.

When bidder i in group 1 participates in the auction, his updated valuation is distributed on $[v_1^*, 1]$ with cumulative distribution function $G_i(v) = \frac{F(v)-F(v_1^*)}{1-F(v_1^*)}$, and when bidder j in group 2 participates in the auction, her updated valuation is distributed on $[v_2^*, 1]$ with cumulative distribution function $G_2(v) = \frac{F(v)-F(v_2^*)}{1-F(v_2^*)}$. The two distributions have the same upper bounds and different lower bounds. Thus if both types of bidders participate in the auction, the bidders are involved in an asymmetric first price auction in the sense that they have valuation distributions with different supports. To get the expected revenue at the cutoffs, we need to know how the bidders bid when there are both types of bidders participating in the auction.

Assume that a bidder with zero probability of winning bids his true value when he participates[7]. Then, by Maskin and Riley (2003), there is a unique optimal bidding strategy, which is characterized in the following lemma:

Lemma 1 *Suppose* k_1 *bidders in type 1 whose values are distributed on the interval* $[v_1^*, 1]$ *with cumulative distribution function* $G_1(v) = \frac{F(v)-F(v_1^*)}{1-F(v_1^*)}$ *and* k_2 *bidders in type 2 whose values are distributed on the interval* $[v_2^*, 1]$ *with cumulative distribution function* $G_2(v) = \frac{F(v)-F(v_2^*)}{1-F(v_2^*)}$ *participate in the auction, where* $v_1^* < v_2^*$. *Let* $\underline{b} = \max \arg\max_b (F(b) - F(v_1^*))^{k_1} (F(b) - F(v_2^*))^{k_2-1} (v_2^* - b)$. *The optimal inverse bidding functions* $v_1(b)$ *and* $v_2(b)$ *are uniquely determined by*

(1) $v_1(b) = b$ *for* $v_1^* \leq b \leq \underline{b}$*;*

(2) for $\underline{b} < b \leq \bar{b}$*, the inverse bidding functions are determined by the following differential equation system:*

$$\begin{cases} \frac{k_1 f(v_1(b)) v_1'(b)}{F(v_1(b))-F(v_1^*)} + \frac{(k_2-1) f(v_2(b)) v_2'(b)}{F(v_2(b))-F(v_2^*)} = \frac{1}{v_2(b)-b} \\ \frac{(k_1-1) f(v_1(b)) v_1'(b)}{F(v_1(b))-F(v_1^*)} + \frac{k_2 f(v_2(b)) v_2'(b)}{F(v_2(b))-F(v_2^*)} = \frac{1}{v_1(b)-b}. \end{cases}$$

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[7] Without this assumption a bidder with value v_i, who in optimum has zero probability of winning, can sometimes bid more than his value. However, this bidding strategy can be eliminated by a trembling-hand argument. Once a bidder bids above his value, he may have a positive probability to win the object which gives him a negative revenue. For a bidder, bidding below his value when he has zero probability of winning can also be supported in an optimal bidding strategy. However the allocation is the same as the optimal bidding strategy where he bids his value. For simplicity, we eliminate it.

with boundary conditions $v_2(\underline{b}) = v_2^*$, $v_1(\underline{b}) = \underline{b}$ *and* $v_1(\bar{b}) = v_2(\bar{b}) = 1$.

By Lemma 1, bidders in type 2 have an advantage in distribution so that they can benefit from the auction. Indeed, for a bidder in type 2 with any value on her support, she has a positive probability to win the auction. However, bidders in type 1, when $v_1 \in [v_1^*, \underline{b})$, have no chance to win the auction when any bidder in type 2 also submits a bid. From the above lemma, when there are two bidders using the same cutoff participating in the auction, the bidder with the value at the cutoff has zero expected revenue from the auction.

Remark 2 When there are k bidders in type 1 and one bidder in type 2 participate in the auction, the lower bound of the bid submitted by bidders in type 2 is $\max \arg\max_b (F(b) - F(v_1^*))^k (v_2^* - b)$.

Bidder $i \in \{1, \ldots, m\}$ with $v_i = v_1^*$ can only win the object when none of the others enter the auction. He bids zero when he is the only participant. Indeed, if another bidder $i' \in \{1, 2, \ldots, i-1, i+1, \ldots, m\}$ participates, we have $v_{i'} \geq v_i = v_1^*$, so $\lambda_{i'}(v_{i'}, v^*) \geq \lambda_i(v_i, v^*)$. Then bidder i gains zero revenue from the participation. When any bidder $j = m+1, \ldots, n$ also enters the auction, we have $v_j \geq v_2^* > v_1^* = v_1$, bidder i will lose the auction for sure.

Thus, at equilibrium we then have

$$c = v_1^* F(v_1^*)^{m-1} F(v_2^*)^{n-m}.$$

For bidder $j \in \{m+1, ..., n\}$ with $v_j = v_2^*$, she can bid zero and has revenue v_2^* when none of others enters the auction. If other bidders in type 2 enter the auction, she will lose the bid. If $k \leq m$ bidders in type 1 enter the auction, the optimal bid $\underline{b}_k$ for bid j is decided by

$$\underline{b}_k = \max \arg\max_b (F(b) - F(v_1^*))^k (v_2^* - b).$$

The first order condition for $\underline{b}_k$ gives

$$\underline{b}_k + \frac{F(\underline{b}_k) - F(v_1^*)}{k f(\underline{b}_k)} = v_2^*.$$

$\underline{b}_k$ is chosen with probability $C_m^k F(v_1^*)^{m-k}(1 - F(v_1^*))^k$. C_m^k is the combination number for choosing k candidates from the n items that are available and $C_m^k = \frac{m!}{k!(m-k)!}$. Thus in equilibrium we have

$$c \geq v_2^* F(v_1^*)^m F(v_2^*)^{n-m-1} + F(v_2^*)^{n-m-1} \sum_{k=1}^{m} C_m^k F(v_1^*)^{m-k} (F(\underline{b}_k) - F(v_1^*))^k (v_2^* - \underline{b}_k),$$

where the first part is the expected revenue when none of others enters the auction, which happens with probability $F(v_1^*)^m F(v_2^*)^{n-m-1}$; the second part is the expected revenue when no bidders in type 2 enters the auction and there are exactly $k \leq m$ bidders in the auction, which happens with probability $F(v_2^*)^{n-m-1}F(v_1^*)^{m-k}$. The inequality holds whenever bidders in type 2 never participate in the auction, i.e., $v_2^* > 1$.

Summarizing our discussion, we have the following proposition.

Proposition 1 *In an economic environment with n homogeneous bidders,*

(1) there is a unique symmetric equilibrium in which all bidders use the same cutoff point v^s that is determined by $v^s F(v^s)^{n-1} = c$;

(2) if $F(\cdot)$ is elastic at v^s, i.e., $F(v^s) < v^s f(v^s)$, then, for any $m \in \{1, 2, ..., n-1\}$, there exists an asymmetric equilibrium where m bidders use the cutoff point v_1^* and the others use the other cutoff point v_2^* that satisfy

$$\begin{aligned} c &= v_1^* F(v_1^*)^{m-1} F(v_2^*)^{n-m}, \\ c &\geq v_2^* F(v_1^*)^m F(v_2^*)^{n-m-1} + F(v_2^*)^{n-m-1} \sum_{k=1}^{m} C_m^k F(v_1^*)^{m-k} (F(\underline{b}_k) - F(v_1^*))^k (v_2^* - \underline{b}_k), \end{aligned}$$

with equality whenever $v_2^* \leq 1$ and $v_1^* < v^s < v_2^*$, where

$$\underline{b}_k = \max \arg \max_b (F(b) - F(v_1^*))^k (v_2^* - b);$$

(3) if $F(\cdot)$ is concave, there exists no asymmetric equilibrium.

In words, when $F(\cdot)$ is elastic at the symmetric equilibrium, if the bidders are randomly divided into two groups, there is an equilibrium where all bidders within one group use the same cutoff that is different from the cutoff used by bidders in the other group. One implication of this result is that some bidders can coordinate by choosing a smaller cutoff threshold so that they can reduce the probability that the others enter the auction which, in turn, can reduce the competition among the bidders who participate in the auction. However, when $F(\cdot)$ is concave, there is no such equilibrium.

Remark 3 When $F(.)$ is strictly convex, it is elastic at any point on its support, specifically at v^s, and therefore there exists an asymmetric equilibrium. For instance, when $c = 0.1$, $F(v) = v^2$ and $n = 2$, there is a symmetric equilibrium (0.466, 0.466) and an asymmetric equilibrium $(0.141, 0.842)$.

The intuition for the existence of asymmetric equilibria when $F(.)$ is strictly convex is the following. When bidders in type 1 use a smaller cutoff v_1^* to enter the auction, the expected payoff for any bidder j in type 2 with a lower value to enter the auction is smaller even when he wins the auction. This is true because bidder j's expected payment to the seller, which is equal to the expected value of the highest valuation of bidders in type 1, is higher when $F(.)$ is strictly convex. In this case, bidders in type 1 would stay out of the auction by using a larger cutoff point, and thus we have an asymmetric equilibrium. When $F(.)$ is concave, the above argument cannot be applied. Now the expected payment of bidder j to the seller when he wins is smaller since other bidders tend to have smaller valuations. Bidder j with lower value may also benefit from participating in the auction which can prevents bidders in type 1 from entering the auction with a smaller cutoff value.

Remark 4 Further remarks can be given as follows:

(1) When the lower bound of support is positive, $F(.)$ may be elastic at v^s even if $F(\cdot)$ is concave on its support. Thus we may have an asymmetric equilibrium.

(2) Since $v_i^* \geq c$, the condition that $F(\cdot)$ is concave can be weakened to $F(v) \geq vf(v)$ for all $v \in [c, 1]$.

(3) As $c \to 0$, one can check that both symmetric equilibrium and asymmetric equilibrium (if it exists) go to zero. Thus when $c = 0$, all bidders participate in the auction.

(4) Similar to Cao and Tian (2008a, 2008b), one can study equilibrium properties when bidders may have differentiated participation costs or when both values and participation costs are private information.

It may be remarked that asymmetric equilibria inevitably lead to inefficient allocation in first price auctions with participation costs. Indeed, like second price auctions with participation costs, they are not efficient because the object may be not allocated to the bidder with the highest valuation when bidders uses different cutoff points. However, unlike second price auctions with participation costs, they are not even weakly efficient (Miralles (2006)). The bidder who wins the object may not have the highest valuation among those who participate. To see this, suppose $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$) are the equilibrium bidding functions for any two bidders who use different cutoff points. Suppose $\lambda_1(v) < \lambda_2(v)$. By the continuity of the functions, $\lambda_1(v + \epsilon) < \lambda_2(v - \epsilon)$ when ϵ is sufficiently small. Thus bidder 2 will win the object even though he has a lower valuation. In conclusion, we have the following proposition:

Proposition 2 *The first price auctions with participation costs are not efficient and are not even weakly efficient at asymmetric equilibrium.*

In reality, we can expect that when the auction can run repeatedly, bidders may use asymmetric equilibria at earlier periods while using the symmetric equilibrium at later periods. We now investigate the welfare effect of participation costs on sellers when it is just one shot game, focusing on the symmetric equilibrium.[8] When bidders use the same threshold and participate in the auction, the optimal bidding function is unique, which is symmetric and monotonic increasing, given by

$$\lambda(v_i, v_s) = v_i - \frac{\int_{v^s}^{v_i} (F(y) - F(v^s))^{k-1} dy}{(F(v_i) - F(v^s))^{k-1}}$$

when there are $k \geq 2$ participants. Thus the seller's expected revenue from the auction is

$$R = \sum_{k=2}^{n} C_n^k F(v_s)^{n-k} \int_{v^s}^{1} (v_i - \frac{\int_{v^s}^{v_i} (F(y) - F(v^s))^{k-1} dy}{(F(v_i) - F(v^s))^{k-1}}) k(F(v_i) - F(v^s))^{k-1} f(v_i) dv_i,$$

and consequently, we have

$$R = n(n-1) \int_{v^s}^{1} (1 - F(x)) x f(x) F(x)^{n-2} dx \tag{1}$$

with integration by parts and changing the order of the integration in the double integrals[9].

There are several effects of increasing the magnitude of participation costs. First, as c increases, the probability to have k participants decreases. Secondly, participants bid more regressively. The reason is that to win the auction, a player has to bid the expected value of the highest among his opponents with values between v^s and 1. Lower participation reduces the expected revenue while more regressive bidding increases it. One might conjecture that there exists an optimal participation costs that will maximize the seller's expected revenue. However from the equation above, this is not true, which leads to the following result:

Proposition 3 *At the symmetric equilibrium, the seller's expected revenue decreases as the participation cost c increases.*

One implication of the above proposition is that in reality, the seller may give the potential bidders some subsidy to encourage them to participate in the auction to increase the expected revenue.

Remark 5 When participation costs are part of the seller's revenue, like the entry fee, the above conclusion no longer holds. In this case, the seller's expected revenue is

$$n(n-1) \int_{v^s}^{1} (1 - F(x)) x f(x) F(x)^{n-2} dx + nc(1 - F(v^s)),$$

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[8] The welfare analysis for the case of asymmetric equilibrium is much more complicated. Letting the bidders know the number of other bidders who submit bids may have different welfare implications for the sellers. We leave the welfare analysis at the asymmetric equilibrium for future research.

[9] See details in the Appendix.

which is equivalent to

$$n(n-1)\int_{v^s}^{1}(1-F(x))xf(x)F(x)^{n-2}dx+nv^sF(v^s)^{n-1}(1-F(v^s)).$$

First order condition $v^sf(v^s)=1-F(v^s)$ determines the optimal entry fee from the perspective of seller.

Remark 6 Menezes and Monteiro (2000) consider first price auctions with participation costs. However, they adopt a different specification on information structure. A bidder does not know who others are in the auction when he is to submit a bid. Besides, they only focus on the symmetric equilibrium in which all bidders use the same cutoff point (which is equal to v^s) and submit bids via the same bidding function. They mainly focus on comparing the revenue from first price auctions and second price auctions and investigate the effect of the number of potential bidders on seller's revenue. Within their framework, when a bidder decides to participate in the auction, he will bid as if all others are in the auction since he cannot observe any other's entrance behavior and the bidding function is given by

$$\lambda^*(v_i,v_s)=\frac{\int_{v^s}^{v_i}(n-1)yF(y)^{n-2}f(y)dy}{F(v)^{n-1}}$$

when $v\geq v^s$, and consequently the expected revenue is given by

$$\widetilde{R}=\int_{v^s}^{1}\lambda^*(v_i,v_s)nF^{n-1}(x)f(x)dx,$$

which can be shown to be equivalent to (1). Thus at symmetric equilibrium, letting the bidders observe or not observe who else participates will give the seller the same expected revenue.

4 Heterogenous Bidders

Now consider the case where we have n_1 strong bidders with value distribution $F_1(\cdot)$ and n_2 weak bidders with value distribution $F_2(\cdot)$. The total number of bidders is $n=n_1+n_2$. We concentrate on type-symmetric equilibrium in which all strong (resp., weak) bidders use the same cutoff point.

We first assume, provisionally, that the cutoff points v_1^* and v_2^* satisfy $v_1^*<v_2^*$. Then for a strong bidder i with $v_i=v_1^*$, he can only get the object when all the others do not participate in the auction. (If any strong bidder i' enters the auction, he must have a value greater than v_1^* and thus bids higher than bidder i; or if any weak bidder j enters, then it must be the case that $v_j\geq v_2^*>v_1^*$. As seen in the previous section, bidder i will lose the item for sure.) Thus, at equilibrium we have

$$c=v_1^*F_1(v_1^*)^{n_1-1}F_2(v_2^*)^{n_2}.$$

For a weak bidder j with $v_j = v_2^*$, we have the following three cases:

Case 1: All the other bidders do not enter the auction. Then bidder j bids zero and gains a surplus of v_2^*. The probability of this event is $F_1(v_1^*)^{n_1} F_2(v_2^*)^{n_2-1}$. In this case the expected revenue for bidder j is $v_2^* F_1(v_1^*)^{n_1} F_2(v_2^*)^{n_2-1}$.

Case 2: At least another weak bidder enters. Then bidder j will lose the auction, deriving zero revenue from participating.

Case 3: None of the other weak bidders enters and there are exactly $k \in \{1, 2, ..., n_1\}$ strong bidders participating in the auction. In this case bidder j with value v_2^* will submit a bid

$$\underline{b}_k = \max \arg \max_b [(F_1(b) - F_1(v_1^*))^k (v_2^* - b)].$$

The first order condition for $\underline{b}_k$ gives

$$\underline{b}_k + \frac{F_1(\underline{b}_k) - F_1(v_1^*)}{k f_1(\underline{b}_k)} = v_2^*.$$

The probability of this event is $C_{n_1}^k F_1(v_1^*)^{n_1-k}(1 - F_1(v_1^*))^k$. The expected revenue in this case is $C_{n_1}^k F_1(v_1^*)^{n_1-k} F_2(v_2^*)^{n_2-1}(F_1(\underline{b}_k) - F_1(v_1^*))^k (v_2^* - \underline{b}_k)$.

Then at equilibrium we have

$$c \geq v_2^* F_1(v_1^*)^{n_1} F_2(v_2^*)^{n_2-1} + \sum_{k=1}^{n_1} C_{n_1}^k F_1(v_1^*)^{n_1-k} F_2(v_2^*)^{n_2-1}(F_1(\underline{b}_k) - F_1(v_1^*))^k (v_2^* - \underline{b}_k).$$

Proposition 4 *When $F_1(v) < F_2(v)$ for all $v \in (0, 1)$, there always exists a type-symmetric equilibrium in which $v_1^* < v_2^*$. Further, the type-symmetric equilibrium $v_1^* < v_2^*$ is unique when both distributions are concave.*

Similarly for the case where $v_1^* \geq v_2^*$, tt equilibrium we have

$$c = v_2^* F_2(v_2^*)^{n_2-1} F_1(v_1^*)^{n_1},$$

and

$$c \geq v_1^* F_2(v_2^*)^{n_2} F_1(v_1^*)^{n_1-1} + \sum_{k=1}^{n_2} C_{n_2}^k F_2(v_2^*)^{n_2-k} F_1(v_1^*)^{n_1-1}(F_2(\underline{b}_k) - F_2(v_2^*))^k (v_1^* - \underline{b}_k),$$

where the first part on the right side of the inequality is the expected revenue when none of the others (no matter whether they are strong or weak bidders) participates in the auction. The second part is the expected revenue when at least one weak bidder participates and no other strong bidders participates.

Proposition 5 *In the heterogenous economy involving any number of bidders,*

(1) if $F_2(\cdot)$ is concave, there is no type-symmetric equilibrium with $v_2^* \leq v_1^*$;

(2) if $F_2(\cdot)$ is strictly convex, there exists $c^* < 1$ such that there exists a type-symmetric equilibrium with $v_2^* \leq v_1^*$ for all $c > c^*$.

This result indicates that, when the participation cost is sufficiently large, strong bidders may choose a higher cutoff point. The intuition behind this is that, when c is sufficiently large and the weak bidder is more likely to have higher valuation, the expected revenue of the strong bidder from entering the auction is low. Strong bidder' advantage in valuations is attenuated by the weak bidder' value distribution and a higher participation cost.

5 Conclusion

This paper investigates (Bayesian-Nash) equilibria of sealed-bid first price auctions with participation costs. We focus on equilibria in cutoff strategies. Once a bidder participates in the auction, the bidding strategy depends on the valuation distributions and cutoff points of other bidders.

When bidders are ex-ante homogeneous with the same valuation distribution, there always exists a unique symmetric equilibrium in which all bidders use the same cutoff to enter the auction and there may also exist an asymmetric equilibrium. In particular, there is no asymmetric equilibrium when $F(\cdot)$ is concave, and there exists an asymmetric equilibrium when $F(\cdot)$ is elastic at the symmetric equilibrium. When bidders can be ranked by their valuation distributions, we find that bidders with higher probability to have higher valuations are more likely to enter the auction. However the opposite can be obtained when the participation cost is sufficiently large and weak bidders's valuation distributions are strictly convex.

In the presence of participation costs, not all bidders will participate in the auction and the seller's expected revenue decreases as the participation costs increase. Then, it may be profitable for the sellers to subsidize the buyers to encourage them participating in the auction. How to implement this should be a potentially interesting question which will be left for future research.

Appendix: Proofs

Proof of Lemma 1:

Proof. Denote the inverses of the bidding function as $v_1(b)$ with support $[\underline{b}_1, \overline{b}_1]$ and $v_2(b)$ with support $[\underline{b}_2, \overline{b}_2]$. Let $(\underline{b}, \overline{b}]$ be the range in which a bidder has a positive probability to win the object if he participants in the auction. First from Maskin and Reiley (2003), the upper endpoint of the support of the distributions of the valuations is the the same for all bidders and thus the upper endpoints in the supports of all buyers' equilibrium bid distributions are the same. Thus $\overline{b}_1 = \overline{b}_2 = \overline{b}$ and $v_1(\overline{b}) = v_2(\overline{b}) = 1$

Also from Maskin and Reiley (2003) we have $\underline{b}_1 < \underline{b}_2 = \underline{b}$ which indicates that the minimum bid of a bidder in type 1 is always less than that of bidders type 2 since bidders in type 2 have an advantage in valuation distribution.

Below $\underline{b}$, type 1 bidder has no chance to win the auction and bids his true value, so $v_1(\underline{b}) = \underline{b}$. For bidders in type 2, when $v_2 = v_2(\underline{b}) = v_2^*$, bidding $\underline{b}$ is his best strategy. Again, from Maskin and Reiley (2003), $\underline{b} = \max \arg \max_b (F(b) - F(v_1^*))^{k_1} (F(b) - F(v_2^*))^{k_2 - 1} (v_2^* - b)$

In the interval $[\underline{b}, \overline{b}]$, a bidder in type i bids b which is determined by the following maximization problem:

$$\max_b \left(\frac{F(v_j(b)) - F(v_j^*)}{1 - F(v_j^*)}\right)^{k_j} \left(\frac{F(v_i(b)) - F(v_i^*)}{1 - F(v_i^*)}\right)^{k_i - 1} (v_i - b), j \neq i.$$

First order conditions give us

$$\begin{cases} \frac{k_1 f(v_1(b)) v_1'(b)}{F(v_1(b)) - F(v_1^*)} + \frac{(k_2 - 1) f(v_2(b)) v_2'(b)}{F(v_2(b)) - F(v_2^*)} = \frac{1}{v_2(b) - b} \\ \frac{(k_1 - 1) f(v_1(b)) v_1'(b)}{F(v_1(b)) - F(v_1^*)} + \frac{k_2 f(v_2(b)) v_2'(b)}{F(v_2(b)) - F(v_2^*)} = \frac{1}{v_1(b) - b}. \end{cases}$$

The boundary conditions for the differential equation system are $v_2(\underline{b}) = v_2^*$, $v_1(\underline{b}) = \underline{b}$ and $v_1(\overline{b}) = v_2(\overline{b}) = 1$. ■

Proof of Proposition 1:

Proof. (1) The existence and uniqueness of symmetric equilibrium is obvious. The proof is omitted here.

(2) Suppose $F(\cdot)$ is elastic at v^s so that $F(v^s) < v^s f(v^s)$. Consider the following two equations:

$$\begin{aligned} c &= x F(x)^{m-1} F(y)^{n-m} \\ c &\geq y F(x)^m F(y)^{n-m-1} + F(y)^{n-m-1} \sum_{k=1}^{m} C_m^k F(x)^{m-k} (F(\underline{b}_k) - F(x))^k (y - \underline{b}_k). \end{aligned}$$

where $\underline{b}_k$ satisfies $\underline{b}_k + \frac{F(\underline{b}_k)-F(x)}{kf(\underline{b}_k)} = y$, x corresponds to the cutoff point used by bidders in the first group, and y corresponds to the cutoff point used by bidders in the second group. Let v^s satisfy $c = v^s F(v^s)^{m-1} F(v^s)^{n-m}$. Define $x = \phi(y)$ implicitly from $c = xF(x)^{m-1}F(y)^{n-m}$. Notice that $\phi(y)$ is continuously differentiable and $\phi(v^s) = v^s$. Since $x \le y$ we have $x = \phi(y)$ with $y \ge v^s$. Then we have

$$\phi'(y) = -\frac{(n-m)f(y)xF(x)}{(F(x)+(m-1)xf(x))F(y)},$$

and thus

$$\phi'(v^s) = -\frac{(n-m)v^s f(v^s)}{F(v^s)+(m-1)v^s f(v^s)}.$$

Define

$$h(y) = F(y)^{n-m-1}[yF(\phi(y))^m + \sum_{k=1}^{s} C_m^k F(\phi(y))^{m-k}(F(\underline{b}_k(y)) - F(\phi(y)))^k (y - \underline{b}_k(y))] - c.$$

with $y \ge v^s$. Notice that $h(y)$ is continuously differentiable and $\underline{b}_k(y) = v^s$ when $y = v^s$. So $h(v^s) = 0$. In order to have an asymmetric equilibrium, we only need to show that *either* there exists a $y^* \in (v^s, 1]$ such that $h(y^*) = 0$ (in which case we have $v_2^* = y^*$ and $v_1^* = h(v_2^*) < v^s$ as our asymmetric cutoff equilibrium.) *or* $h(1) < 0$ (in which case $v_2^* > 1$ and $v_1^* = c.$). So if $h(1) < 0$, then it is done.

Suppose $h(1) > 0$. Since $h(.)$ is continuous with $h(v^s) = 0$ and $h(1) > 0$, when $h(y)$ is decreasing at v^s, then there exists a $y^* \in (v^s, 1]$ such that $h(y^*) = 0$. This is true when $F(\cdot)$ is elastic at v^s. Indeed,

$$h'(y) = I(y) + F(y)^{n-m-1}[II(y) + \sum_{k=1}^{m} C_s^k (III(y) + IV(y))],$$

where

$$\begin{aligned}
I(y) &= (n-m-1)F(y)^{n-m-2}f(y)[yF(\phi(y))^m + \sum_{k=1}^{m} F(\phi(y))^{m-k}(F(\underline{b}_k(y)) - F(\phi(y)))^k (y - \underline{b}_k(y))],\\
II(y) &= F(\phi(y))^m + y.mF(\phi(y))^{m-1}f(\phi(y))\phi'(y),\\
III(y) &= (m-k)F(\phi(y))^{m-k-1}f(\phi(y))\phi'(y)(F(\underline{b}_k(y)) - F(\phi(y)))^k (y - \underline{b}_k(y)),\\
IV(y) &= F(\phi(y))^{m-k}[k[F(\underline{b}_k(y)) - F(\phi(y))]^{k-1}(f(\underline{b}_k(y))\underline{b}_k'(y) - f(\phi(y))\phi'(x))(y - \underline{b}_k(y))\\
&+ (F(\underline{b}_k(y)) - F(\phi(y)))^k (1 - \underline{b}_k'(y))].
\end{aligned}$$

When $x = y = v^s$, we have $\underline{b}_k(v^s) = v^s$. Then,

$$\begin{aligned}
I(v^s) &= (n-m-1)F(v^s)^{n-m-2}f(v^s)v^s F(v^s)^m = (n-m-1)F(v^s)^{n-2}v^s f(v^s),\\
II(v^s) &= F(v^s)^m + v^s.mF(v^s)^{m-1}f(v^s)\phi'(v^s),\\
III(v^s) &= IV(v^s) = 0
\end{aligned}$$

and thus

$$h'(v^s) = F(v^s)^{n-2}[(n-m-1)v^s f(v^s) + mv^s f(v^s)\phi'(v^s) + F(v^s)].$$

Thus, $h'(v^s) < 0$ if and only if

$$|\phi'(v^s)| = \frac{(n-m)v^s f(v^s)}{F(v^s) + (m-1)v^s f(v^s)} > \frac{(n-m-1)v^s f(v^s) + F(v^s)}{mv^s f(v^s)},$$

which is true when $F(\cdot)$ is elastic at v^s. Indeed, when $F(\cdot)$ is elastic at v^s we have $v^s f(v^s) > F(v^s)$. So $F(v^s) + (m-1)v^s f(v^s) < mv^s f(v^s)$ and at the same time $(n-m)v^s f(v^s) > (n-m-1)v^s f(v^s) + F(v^s)$. Then if $h(1) > 0$, we have an asymmetric equilibrium in which $v_1^* < v^s < v_2^* \leq 1$, otherwise there is an asymmetric equilibrium in which bidders in group 2 never participate in the auction.

(3) When $F(\cdot)$ is concave, we prove the nonexistence of asymmetric equilibrium by way of contradiction. Suppose there is an asymmetric equilibrium with $v_1^* < v_2^*$. Then

$$\begin{aligned} c &= v_1^* F(v_1^*)^{m-1} F(v_2^*)^{n-m}, \\ c &\geq v_2^* F(v_1^*)^{m} F(v_2^*)^{n-m-1} + F(v_2^*)^{n-m-1} \sum_{k=1}^{m} C_s^k F(v_1^*)^{m-k} (F(\underline{b}_k) - F(v_1^*))^k (v_2^* - \underline{b}_k). \end{aligned}$$

One necessary condition for the system of these equations above to hold is

$$v_1^* F(v_1^*)^{m-1} F(v_2^*)^{n-m} \geq v_2^* F(v_1^*)^{m} F(v_2^*)^{n-m-1},$$

i.e., $\frac{F(v_2^*)}{v_2^*} \geq \frac{F(v_1^*)}{v_1^*}$, which cannot be true when $F(\cdot)$ is concave and $v_2^* > v_1^*$. Following the same procedures above, we can prove there is no asymmetric equilibrium in which $v_1^* > v_2^*$. ∎

Proof of Equation (1):

Proof. Rewrite

$$R = \sum_{k=2}^{n} C_n^k F(v_s)^{n-k} \int_{v^s}^{1} (v_i - \frac{\int_{v^s}^{v_i} (F(y) - F(v^s))^{k-1} dy}{(F(v_i) - F(v^s))^{k-1}}) k (F(v_i) - F(v^s))^{k-1} f(v_i) dv_i$$

as

$$R = \int_{v^s}^{1} \{v_i \sum_{k=2}^{n} C_n^k F(v_s)^{n-k} k (F(v_i) - F(v^s))^{k-1} - \int_{v^s}^{v_i} \sum_{k=2}^{n} C_n^k F(v_s)^{n-k} k (F(y) - F(v^s))^{k-1} dy\} dF(v_i).$$

Integrating by parts for $\int_{v^s}^{v_i}\sum_{k=2}^{n} C_n^k F(v_s)^{n-k}k(F(y)-F(v^s))^{k-1}dy$ and making simplifications, we have

$$\begin{aligned} R &= \int_{v^s}^{1}\int_{v^s}^{v_i}\sum_{k=2}^{n} C_n^k F(v_s)^{n-k}k(k-1)(F(y)-F(v^s))^{k-2}ydyf(v_i)dv_i \\ &= \int_{v^s}^{1}\int_{v^s}^{v_i}\sum_{k=2}^{n} C_{n-2}^{k-2}n(n-1)F(v_s)^{n-k}(F(y)-F(v^s))^{k-2}ydyf(v_i)dv_i \\ &= n(n-1)\int_{v^s}^{1}\int_{v^s}^{v_i} F(y)^{n-2}ydyf(v_i)dv_i \\ &= n(n-1)\int_{v^s}^{1}(1-F(x))xf(x)F(x)^{n-2}dx, \end{aligned}$$

where the second line comes from the fact that $C_n^k k(k-1) = n(n-1)C_{n-2}^{k-2}$ and the last line comes from changing the order of integration in the double integral. ■

Proof of Proposition 4:

Now consider the following two equations:

$$\begin{aligned} c &= xF_1(x)^{n_1-1}F_2(y)^{n_2} \\ c &\geq yF_1(x)^{n_1}F_2(y)^{n_2-1} + \sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}F_2(y)^{n_2-1}(F_1(\underline{b}_k)-F_1(x))^k(y-\underline{b}_k), \end{aligned}$$

with $c \leq x \leq y < 1$, where x corresponds to the cutoff point used by the strong bidders and y corresponds to the cutoff point used by the weak bidders. Let v_1^s satisfy $v_1^s F_1(v_1^s)^{n_1-1}F_2(v_1^s)^{n_2} = c$. Note that $\theta(v_1^s) = v_1^s F_1(v_1^s)^{n_1-1}F_2(v_1^s)^{n_2}$ is an increasing function of v_1^s with $\theta(1) = 1 > c$, so we have $v_1^s < 1$. For $y \geq v_1^s$, define $x = \phi(y)$ from $c = xF_1(x)^{n_1-1}F_2(y)^{n_2}$. Then x is a decreasing function of y and $\phi(v_1^s) = v_1^s$. Now let

$$h(y) = yF_1(\phi(y))^{n_1}F_2(y)^{n_2-1} + \sum_{k=1}^{n_1} C_{n_1}^k F_1(\phi(y))^{n_1-k}F_2(y)^{n_2-1}(F_1(\underline{b}_k(y))-F_1(\phi(y)))^k(y-\underline{b}_k(y))-c.$$

Then $h(y)$ is a continuous function of $y \geq v_1^s$. The remainder of the proof is based on the following two lemmas:

Lemma 2 There always exists a type-symmetric equilibrium with $v_1^* < v_2^*$.

Proof. Note that we have $x \leq \underline{b}_k \leq y$. When $y = v_1^s$, we have $\underline{b}_k = v_1^s$. Then

$$h(v_1^s) = v_1^s F_1(v_1^s)^{n_1}F_2(v_2^s)^{n_2-1} - c < v_1^s F_1(v_1^s)^{n_1-1}F_2(v_1^s)^{n_2} - c = 0$$

since $F_1(v_1^s) < F_2(v_2^s)$ by assumption. We also have

$$h(1) = F_1(\phi(1))^{n_1} + \sum_{k=1}^{n_1} C_{n_1}^k F_1(\phi(1))^{n_1-k}(F_1(\underline{b}_k(1))-F_1(\phi(1)))^k(1-\underline{b}_k(1))-c.$$

Now if $h(1) \geq 0$, then by the mean value theorem, there exists a $y = v_2^* \in (v_1^s, 1]$ such that $h(v_2^*) = 0$ so that there is an equilibrium in which $v_1^* = \phi(v_2^*) < v_1^s < v_2^* \leq 1$. Otherwise if $h(1) < 0$, then there is an equilibrium in which $v_1^* = \phi(1) < 1$ and $v_2^* > 1$; i.e., weak bidders never participate in the auction. ■

Lemma 3 When $F_1(\cdot)$ and $F_2(\cdot)$ are both concave and $F_1(v) < F_2(v)$ for all $v \in (0,1)$, there exists a unique type-symmetric equilibrium with $v_1^* < v_2^*$.

Proof. Suppose $y \leq 1$. Substituting $c = xF_1(x)^{n_1-1}F_2(y)^{n_2}$ into

$$c = yF_1(x)^{n_1}F_2(y)^{n_2-1} + \sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}F_2(y)^{n_2-1}(F_1(\underline{b}_k) - F_1(x))^k(y - \underline{b}_k)$$

and making simplifications, we have

$$yF_1(x)^{n_1} + \sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}(F_1(\underline{b}_k) - F_1(x))^k(y - \underline{b}_k) - xF_1(x)^{n_1-1}F_2(y) = 0. \qquad (2)$$

We claim that the above equation implicitly defines x as a strictly increasing function of y. Consequently, it *either* has a unique intersection with $x = \phi(y)$ (which is strictly decreasing), *or* it does not intersect with $x = \phi(y)$, in which case the unique equilibrium is given by $x = \phi(1)$ and $y > 1$ (weak bidders never participate).

To see this, taking derivatives with respect to y (notice that $\underline{b}_k$ is also a function of y) on both sides of the above equation, we have

$$\begin{aligned}
0 &= F_1(x)^{n_1} + n_1 yF_1(x)^{n_1-1}f_1(x)\frac{dx}{dy} - xF_1(x)^{n_1-1}f_2(y) - F_2(y)(F_1(x)^{n_1-1} \\
&+ (n_1-1)xf_1(x)F_1(x)^{n_1-2})\frac{dx}{dy} \\
&+ \sum_{k=1}^{n_1} C_{n_1}^k\{(n_1-k)F_1(x)^{n_1-k-1}f_1(x)(F_1(\underline{b}_k) - F_1(x))^k(y-\underline{b}_k)\frac{dx}{dy} \\
&+ F_1(x)^{n_1-k}(F_1(\underline{b}_k) - F_1(x))^{k-1}[(F_1(\underline{b}_k) - F_1(x))(1-\underline{b}_k') \\
&+ k(y-\underline{b}_k)(f(\underline{b}_k)\underline{b}_k' - f_1(x)\frac{dx}{dy})]\},
\end{aligned}$$

where

$$\begin{aligned}
(F_1(\underline{b}_k) - F_1(x))(1-\underline{b}_k') &+ k(y-\underline{b}_k)(f(\underline{b}_k)\underline{b}_k' - f_1(x)\frac{dx}{dy}) \\
&= F_1(\underline{b}_k) - F_1(x)) - k(y-\underline{b}_k)f_1(x)\frac{dx}{dy}
\end{aligned}$$

by noting that $F_1(\underline{b}_k(y)) - F_1(x) = kf_1(\underline{b}_k(y))(y - \underline{b}_k(y))$. Thus we have

$$\begin{aligned}
0 &= F_1(x)^{n_1} + n_1 y F_1(x)^{n_1-1} f_1(x)\frac{dx}{dy} - xF_1(x)^{n_1-1}f_2(y) - F_2(y)(F_1(x)^{n_1-1} \\
&+ (n_1-1)xf_1(x)F_1(x)^{n_1-2})\frac{dx}{dy} \\
&+ \sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k-1} f_1(x)(F_1(\underline{b}_k) - F_1(x))^{k-1}(y-\underline{b}_k)\frac{dx}{dy}\{n_1(F_1(\underline{b}_k) - F_1(x)) - kF_1(\underline{b}_k)\} \\
&+ \sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}(F_1(\underline{b}_k) - F_1(x))^k.
\end{aligned}$$

Then

$$\frac{dx}{dy} = \frac{F_1(x)^{n_1} + \sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}(F_1(\underline{b}_k) - F_1(x))^k - xF_1(x)^{n_1-1}f_2(y)}{-n_1 y F_1(x)^{n_1-1} f_1(x) - II + F_2(y)(F_1(x)^{n_1-1} + (n_1-1)xf_1(x)F_1(x)^{n_1-2})}$$

with

$$\begin{aligned}
II &= \sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k-1} f_1(x)(F_1(\underline{b}_k) - F_1(x))^{k-1}(y-\underline{b}_k)\{n_1(F_1(\underline{b}_k) - F_1(x)) - kF_1(\underline{b}_k)\} \\
&= I - \alpha,
\end{aligned}$$

where

$$\begin{aligned}
I &= n_1 \sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k-1} f_1(x)(F_1(\underline{b}_k) - F_1(x))^k (y-\underline{b}_k), \\
\alpha &= \sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k-1} f_1(x)(F_1(\underline{b}_k) - F_1(x))^{k-1}(y-\underline{b}_k)kF_1(\underline{b}_k) \geq 0.
\end{aligned}$$

Now we prove the denominator and numerator are strictly positive separately. First we prove the numerator is positive. From equation (2), we have

$$yF_1(x)^{n_1} - xF_1(x)^{n_1-1}F_2(y) = -\sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}(F_1(\underline{b}_k) - F_1(x))^k (y-\underline{b}_k).$$

When $F_2(\cdot)$ is concave, we have

$$\begin{aligned}
&F_1(x)^{n_1} + \sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}(F_1(\underline{b}_k) - F_1(x))^k - xF_1(x)^{n_1-1}f_2(y) \\
&\geq F_1(x)^{n_1} + \sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}(F_1(\underline{b}_k) - F_1(x))^k - xF_1(x)^{n_1-1}\frac{F_2(y)}{y}.
\end{aligned}$$

Then,

$$\begin{aligned}
yF_1(x)^{n_1} &+ y\sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}(F_1(\underline{b}_k) - F_1(x))^k - xF_1(x)^{n_1-1}F_2(y) \\
&= -\sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}(F_1(\underline{b}_k) - F_1(x))^k (y-\underline{b}_k) + y\sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}(F_1(\underline{b}_k) - F_1(x))^k \\
&= \sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}(F_1(\underline{b}_k) - F_1(x))^k \underline{b}_k > 0.
\end{aligned}$$

So the numerator is positive.

We now prove the denominator is also positive. Again from (2) we have

$$\begin{aligned}
-I &- n_1 y F_1(x)^{n_1-1} f_1(x) = -n_1 f_1(x)/F_1(x) \sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k} (F_1(\underline{b}_k) - F_1(x))^k (y - \underline{b}_k) \\
&= -n_1 f_1(x)/F_1(x)(x F_1(x)^{n_1-1} F_2(y) - y F_1(x)^{n_1}) - n_1 y F_1(x)^{n_1-1} f_1(x) \\
&= n_1 f_1(x)(-x F_1(x)^{n_1-2} F_2(y) + y F_1(x)^{n_1-1} - y F_1(x)^{n_1-1}) \\
&= -n_1 f_1(x) x F_1(x)^{n_1-2} F_2(y).
\end{aligned}$$

Then we have

$$\begin{aligned}
-n_1 f_1(x) x F_1(x)^{n_1-2} F_2(y) + F_2(y)(F_1(x)^{n_1-1} + (n_1-1) x f_1(x) F_1(x)^{n_1-2}) \\
= F_2(y)(F_1(x)^{n_1-1} - x f_1(x) F_1(x)^{n_1-2}) > 0
\end{aligned}$$

since $F_1(x) > x f_1(x)$ by the concavity of $F_1(.)$. Thus we have $\frac{dx}{dy} > 0$. The uniqueness of the equilibrium is established. ■

Proof of Proposition 5:

Proof. We first prove that when $F_2(\cdot)$ is concave, there is no type symmetric equilibrium with $v_1^* \geq v_2^*$. Suppose not. Then a necessary condition is

$$v_2^* F_2(v_2^*)^{n_2-1} F_1(v_1^*)^{n_1} \geq v_1^* F_2(v_2^*)^{n_2} F_1(v_1^*)^{n_1-1},$$

or

$$\frac{F_1(v_1^*)}{v_1^*} \geq \frac{F_2(v_2^*)}{v_2^*}.$$

Note that when $F_2(\cdot)$ is concave and $v_1^* \geq v_2^*$, we have $\frac{F_2(v_2^*)}{v_2^*} \geq \frac{F_2(v_1^*)}{v_1^*}$, and thus $\frac{F_1(v_1^*)}{v_1^*} \geq \frac{F_2(v_1^*)}{v_1^*}$ which cannot be true since $F_2(v_1^*) > F_1(v_1^*)$ by assumption.

We now show that when $F_2(\cdot)$ is strictly convex, there exists an equilibrium in which $v_1^* \geq v_2^*$ when c is sufficiently large.

Let v_2^s satisfy

$$c = v_2^s F_2(v_2^s)^{n_2-1} F_1(v_2^s)^{n_1}$$

and let v_1^s satisfy

$$c = v_1^s F_2(v_1^s)^{n_2-1}.$$

For $y \in [v_1^s, v_2^s]$, define $x = \phi(y)$ from $c = y F_2(y)^{n_2-1} F_1(x)^{n_1}$. Then x is a decreasing function of y satisfying $\phi(v_2^s) = v_2^s$ and $\phi(v_1^s) = 1$. Now define

$$h(y) = \phi(y) F_2(y)^{n_2} F_1(\phi(y))^{n_1-1} + \sum_{k=1}^{n_2} C_{n_2}^k F_2(y)^{n_2-k} F_1(\phi(y))^{n_1-1} (F_2(\underline{b}_k(y)) - F_2(y))^k (\phi(y) - \underline{b}_k(y)) - c.$$

There is the required equilibrium if $\exists y \in [v_1^s, v_2^s]$ with $h(y) = 0$. Note that

$$h(v_2^s) = v_2^s F_2(v_2^s)^{n_2} F_1(v_2^s)^{n_1-1} - c > v_2^s F_2(v_2^s)^{n_2-1} F_1(v_2^s)^{n_1} - c = 0$$

since $F_2(v_2^s) > F_1(v_1^s)$ by assumption. Since $h(y)$ is continuous, we only need

$$h(v_1^s) = F_2(v_1^s)^{n_2} + \sum_{k=1}^{n_2} C_{n_2}^k F_2(v_1^s)^{n_2-k} (F_2(\underline{b}_k(v_1^s)) - F_2(v_1^s))^k (1 - \underline{b}_k(v_1^s)) - c < 0.$$

From the definition we know v_1^s is an monotonically increasing function of c, denoted by $v_1^s(c)$. It is obvious that $v_1^s(1) = 1$ and $v_1^{s\prime}(c) = \frac{F_2(v_1^s)v_1^s}{c(F_2(v_1^s)+(n_2-1)f_2(v_1^s))}$. So we have $v_1^{s\prime}(1) = \frac{1}{1+(n_2-1)f_2(1)}$. It suffices to show

$$\widehat{h}(c) = F_2(v_1^s(c))^{n_2} + \sum_{k=1}^{n_2} C_{n_2}^k F_2(v_1^s(c))^{n_2-k} (F_2(\underline{b}_k(v_1^s(c))) - F_2(v_1^s(c)))^k (1 - \underline{b}_k(v_1^s(c))) - c < 0$$

for some c. Note that we have $\widehat{h}(1) = 0$ and

$$\begin{aligned}
\widehat{h}'(c) &= n_2 F_2(v_1^s(c))^{n_1-1} f_2(v_1^s(c)) v_1^{s\prime}(c) \\
&+ \sum_{k=1}^{n_2} C_{n_2}^k [(n_2-k) F_2(v_1^s(c))^{n_2-k-1} f_2(v_1^s(c)) v_1^{s\prime}(c) (F_2(\underline{b}_k(v_1^s(c))) - F_2(v_1^s(c)))^k (1 - \underline{b}_k(v_1^s(c))) \\
&+ F_2(v_1^s(c))^{n_2-k} (k(F_2(\underline{b}_k(v_1^s(c))) - F_2(v_1^s(c)))^{k-1} (1 - \underline{b}_k(v_1^s(c)))(f_2(\underline{b}_k(v_1^s(c)))\underline{b}_k{}'(v_1^s(c)) \\
&- f_2(v_1^s(c))))v_1^{s\prime}(c) - (F_2(\underline{b}_k(v_1^s)) - F_2(v_1^s))^k \underline{b}_k{}'(v_1^s(c))] - 1
\end{aligned}$$

As $c \to 1$, we have $\underline{b}_k(v_1^s(1)) \to 1$, and thus

$$\widehat{h}'(1) = n_2 f_2(1) v_1^{s\prime}(1) - 1 = \frac{f_2(1) - 1}{1 + (n_2 - 1) f_2(1)} > 0$$

when $F_2(\cdot)$ is strictly convex. Hence, $\exists c^* < 1$ s.t. $\widehat{h}(c) < 0$ whenever $c > c^*$. ■

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