

Second Price Auctions with Two-Dimensional Private Information on Values and Participation Costs*

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Abstract

This paper studies equilibria of second price auctions when values and participation costs are both privation information and are drawn from general distribution functions. We consider the existence and uniqueness of equilibrium. It is shown that there always exists an equilibrium for this general economy, and further there exists a unique symmetric equilibrium when all bidders are ex ante homogenous. Moreover, we identify a sufficient condition under which we have a unique equilibrium in a heterogenous economy with two bidders. Our general framework covers many relevant models in the literature as special cases.

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Key Words: Two-Dimensional types, Private Values, Private Participation Costs, Second Price Auctions, Existence and Uniqueness of Equilibrium.

1 Introduction

While an auction is an effective way to exploit private information by increasing the competition among buyers and thus can increase allocation efficiency, it is not freely implemented actually. One may incur some cost before submitting a bid. This paper studies (Bayesian-Nash) equilibria of sealed-bid second price, or Vickrey, auctions with bidder participation costs in a general two dimensional economic environment.

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1.1 Motivation

The fundamental structure of an auction with bidder participation costs is one in which an indivisible object is to be allocated to one of many buyers via the auction and in order to participate, bidders may have to incur some participation costs¹.

There are many sources for participation costs. For instance, sellers may require that those who submit bids have a certain minimum amount of bidding funds which may compel some bidders to borrow; bidders themselves may have transportation costs to go to an auction spot; or they need spend some money to learn the rules of the auction and how to submit bids. Bidders may even have opportunity costs to attend an auction.

There are some studies on the information acquisition in auctions. A bidder may want to learn how he/she and the others value the item, and thus he/she may incur a cost in information acquisition about their valuations². A main difference between participation costs and information acquisition costs is that information acquisition costs are avoidable while participation costs are not. If a bidder does not want to collect information about her own or others' valuations, she does not incur any cost, but she can still submit bids. Some researchers, such as McAfee and McMillan (1987), Harstad (1990) and Levin and Smith (1994), combine the idea of participation costs and the idea of information acquisition costs. Compete and Jehiel (2007) investigate the advantage of using dynamic auctions in the presence of information acquisition cost only. However, information acquisition costs and participation costs can both be regarded as sunk costs after the bidders submit bids.

With participation costs, not all bidders will be involved in playing the games. If a bidder's expected revenue from participating in the auction is less than the participation cost, he will not participate. Otherwise, the bidder participates and submits a bid accordingly. Even if a bidder decides to participate in the auction, since he may expect some other bidders will not participate in the auction, his bidding behavior may not be the same as that in the standard auction without participation costs.

Addressing the question of participation costs may have important implications. One can characterize the bidding behavior in an auction with participation costs and see how the equilibria will be different from those without participation costs, and then one can derive the implications to the bidders, to the sellers and to the society which, in turn, may be helpful for

¹Some related terminology includes participation cost, participation fee, entry cost or opportunity costs. See Laffont and Green (1984), Samuelson (1986), McAfee and McMillan (1987) etc.

²Persico (2000) studied the incentives of information acquisition in auctions. He found that bidders have more incentives for information acquisition in first price auctions than in second price auctions.

the optimal selling mechanism design.

1.2 Related Literature

The study of participation costs in auctions mainly focuses on second price auctions due to the simplicity of bidding behavior. In a standard second price auction, bidding one's true valuation is a weakly dominant strategy. There are also other equilibria in the standard second price auction as shown in Blume and Heidhues (2004), for example, the bidder with the highest value bids his true value and all others bid zero. This is referred as the asymmetric bidding equilibrium in the standard second price auction. However, in second price auctions with participation costs, so long as a bidder finds participating optimal, he cannot do better than bidding his true value. Therefore in this paper we only consider equilibria in which each bidder uses a cutoff strategy; i.e., bids his true if one finds participating optimal, does not participate otherwise. All of our results about the uniqueness or multiplicity of equilibria, then, should be interpreted accordingly.

Laffont and Green (1984) studied the second price auction with participation costs in a general framework where bidders' valuations and participation costs are both private information. However, their proof on the existence and uniqueness of equilibrium is incomplete. They wanted to show the existence and uniqueness of symmetric equilibrium via contract mapping theorem. However, the condition for that theorem to hold does not satisfied. Besides, they imposed a restrictive assumption of uniform distributions for both values and participation costs and only considered symmetric equilibrium. Recently, some work in the literature has been done on equilibria of the second price auction with participation costs in simplified versions where either only valuations or participation costs are private.

Campbell (1998) and Tan and Yilankaya (2006) studied equilibria and their properties in an economic environment when bidders' values are private information and participation costs are common knowledge and the same. They did find asymmetric equilibria when bidders are ex ante homogeneous. Uniqueness of the equilibrium cannot be guaranteed. Some other studies, including Samuelson (1985), Stegaman (1996), Levin and Smith (1994), etc, also assumed that participation costs are the same across players. While the assumption of equal participation costs is stringent and unrealistic, Cao and Tian (2008) investigated the equilibria when bidders may have differentiated participation costs. They introduced the notions of monotonic equilibrium and neg-monotonic equilibrium.

Kaplan and Sela (2006) simplified the framework of Laffont and Green in another way. They studied equilibria of second price auctions with participation costs when bidders' participation

costs are private information and are drawn from the same distribution function, while valuations are common knowledge.

Thus, up to now, the problem considered in Laffont and Green (1984) has only been answered in some special settings: either participation costs are commonly known or values are publicly known. However, in reality, it is possible that both the valuations and participation costs are private information. Some participation costs are observable to the seller such as the entry fee; some are unobservable to the seller such as the learning costs. A natural way to deal with this is to allow both valuations and participation costs of bidders are private information and their distribution functions are general and may be different. This paper aims to give an answer to the question raised in Laffont and Green (1984) in a general framework³.

1.3 Objects of The Paper

This paper studies equilibria of second price auctions with general distribution functions on valuations and participation costs. The special cases of this general specification includes that either the valuations or participation costs are common knowledge, as those have been investigated in previous literature.

Under a general two-dimensional distribution of the bidders' participation costs and valuations we prove that the equilibria always exist. When bidders have the same distributions, there exists a unique symmetric equilibrium. Moreover, we identify the conditions under which we have a unique equilibrium in a simple two bidder economy. Special cases in which multiple equilibria exist are also discussed. There may exist an equilibrium in which one bidder never participates or an equilibrium in which one bidder always participates.

As compared to the work by Laffont and Green (1984), our general framework can not only establish the existence of equilibrium and uniqueness of symmetric equilibrium in the two-dimensional uniform setting, but can also do that in many other two-dimensional settings such as truncated normal distributions, exponential distributions etc. Not restricted to the symmetric equilibrium when all bidders are homogenous, our framework can deal with the asymmetric equilibria which have been seen in literature with one-dimensional private information, like those in Tan and Yilankaya (2006).

The existence of asymmetric equilibria has important consequences for the strategic behavior

³It should be pointed out that the framework considered can be applied to many other participation costs related economic issues. For instance, in order to decide whether or not enter an undeveloped market, one needs to know the possible revenue before he enters the market and compare that with the necessary costs. To do this, one must also consider the possible entrance behavior of other opponents.

of bidders and the efficiency of the auction mechanism. When an auction has a participation cost, a bidder would expect less bidders to submit their bids. When symmetric equilibrium is unique, every bidder has to follow the symmetric cutoff and has no other choices. However, when asymmetric equilibria exist, bidders may choose an equilibrium they are more desirable. In this case, some bidders may form a collusion to cooperate at the entrance stage by choosing a smaller cutoff point that may decrease the probability that other bidders enter the auction, and consequently, may reduce the competition in the bidding stage. An asymmetric equilibrium may become more desirable when an auction can run repeatedly. Also, an asymmetric equilibrium may be ex-post inefficient. The item being auctioned is not necessarily allocated to the bidder with the highest valuation.

The remainder of the paper proceeds as follows. In Section 2, we describe a general setting of economic environments. We establish the existence of equilibrium in Section 3. The uniqueness of equilibrium is discussed in section 4. In section 5 we give a brief discussion about the existence of multiple equilibria. Concluding remarks are provided in Section 6. All the proofs are relegated to the appendix.

2 The Setup

We consider an independent value economic environment with one seller and n buyers. Let $N = \{1, 2, \dots, n\}$. The seller has an indivisible object which he values at zero to sell to one of the buyers. The auction format is the sealed-bid second price auction (see Vickrey, 1961). In order to submit a bid, bidder i must pay a participation cost c_i . Buyer i 's value for the object, v_i , and participation cost c_i are private and independently drawn from the distribution function $K_i(v_i, c_i)$ with the support $[0, 1] \times [0, 1]$. Let $k_i(v_i, c_i)$ denote the corresponding density function. In particular, when v_i and c_i are independent, we have $K_i(v_i, c_i) = F_i(v_i)G_i(c_i)$ and $k_i(v_i, c_i) = f_i(v_i)g_i(c_i)$, where $F_i(v_i)$ and $G_i(c_i)$ are the cumulative distribution functions of bidder i 's valuation and participation cost, $f_i(v_i)$ and $g_i(c_i)$ ⁴ are the corresponding density functions.

Each bidder knows his own value and participation cost before he makes his entrance decision and does not know others' decisions when one makes his own. If bidder i decides to participate in the auction, he pays a non-refundable participation cost c_i and submits a bid. The bidder

⁴When v_i or c_i takes discrete values, their density functions $f_i(v)$ and $g_i(c_i)$ are reduced to the discrete probability distribution functions, which can be represented by the Dirac delta function. The density at the discrete point is infinity.

with the highest bid wins the object and pays the second highest bid. If there is only one person in the auction, he wins the object and pays 0. If there is a tie, the allocation is determined by a fair lottery. The bidder who wins the object pays his own bid.

In this second price auction mechanism with participation costs, the individually rational action set for any type of bidder is $:\{No\} \cup (0, 1]^5$, where “ $\{No\}$ ” denotes not participating in the auction. Bidder i incurs the participation cost if and only if his action is different from “ $\{No\}$ ”.

Bidders are risk neutral and they will compare the expected revenues from participating and participation costs to decide whether or not to participate. If the expected revenue from participating is less than the costs, not participate. Otherwise, participate and submit bids. Further if a bidder finds participating in this second price auction optimal, he cannot do better than bidding his true valuation (i.e., bidding his true value is a weakly dominant strategy). Therefore, we can restrict our attention to Bayesian-Nash equilibria in which each bidder uses a cutoff strategy; i.e., one bids his true valuation if his participation cost is less than some cutoff point and does not enter otherwise. An equilibrium strategy of each bidder i is then determined by the expected revenue of participating in the auction $c_i^*(v_i)$ when his value is v_i ⁶. Let $b_i(v_i, c_i)$ denote bidder i 's strategy. Then the bidding decision function can be characterized by

$$b_i(v_i, c_i) = \begin{cases} v_i & \text{if } 0 \leq c_i \leq c_i^*(v_i) \\ \text{No} & \text{otherwise.} \end{cases}$$

Remark 1 At an equilibrium, $c_i^*(v_i) > 0$ is a cost cutoff (critical) point such that individual i is indifferent from participating in the auction or not. Bidder i will participate in the auction whenever $0 < c_i \leq c_i^*(v_i)$. Note that at equilibrium, we have $c_i^*(v_i) \leq v_i$.

The description of the equilibria can be slightly different under different informational structures on $K_i(v_i, c_i)$:

- (1) v_i is a private information and c_i is common knowledge to all bidders. In this case, $K_i(v_i, c_i) = F_i(v_i)$. Campbell (1998), Tan and Yilankaya (2006) and Cao and Tian (2007) studied this special case. The equilibrium is described by a valuation cutoff v_i^* for each bidder i . Bidder i submits a bid when $v_i \geq v_i^*$.
- (2) c_i is a private information and v_i is common knowledge to all bidders. In this case, $K_i(v_i, c_i) = G_i(c_i)$. Kalpan and Sela (2006) investigated this kind of economic

⁵For completeness, we assume a bidder with valuation 0 and participation cost 0 does not participate in the auction. The strategy of $\{No\}$ will be denoted by 0.

⁶In equilibrium, $c_i^*(v_i)$ depends on the distributions of all bidders' valuations and participation costs.

environment. The equilibrium is described by a cost cutoff point cost c_i^* for each bidder i . Bidder i submits a bid when $c_i \leq c_i^*$.

3 The Existence of Equilibrium

Suppose, provisionally, there exists an equilibrium in which each bidder i uses $c_i^*(v_i)$ as his entrance decision making. Then for bidder i with value v_i , when his participation cost $c_i \leq c_i^*(v_i)$, the bidder will participate in the auction and submit his weakly dominant bid, or else he will stay out⁷. For bidder i , to submit a bid v_i , he should participate in the auction first; i.e., $c_i \leq c_i^*(v_i)$. So the density of submitting a bid v_i is

$$f_{c_i^*(v_i)}(v_i) = \int_0^{c_i^*(v_i)} k_i(v_i, c_i) dc_i.$$

Remark 2 When v_i and c_i are independent, bidder i with value v_i will submit the bid v_i with probability $G_i(c_i^*(v_i))$ and stay out with probability $1 - G_i(c_i^*(v_i))$.

$f_{c_i^*(v_i)}(0)$ refers the probability (density) that bidder i does not submit a bid. Let $F_{c_i^*(v_i)}(v_i)$ be the corresponding cumulative probability. Note that there is a mass at $v_i = 0$ for $F_{c_i^*(v_i)}(v_i)$.

For each bidder i , let the maximal bid of the other bidders be m_i . Note that, if $m_i > 0$, at least one of other bidders participates in the auction. If $m_i = 0$, no other bidders participates in the auction.

The revenue of participating in the auction for bidder i with value v_i is given by $\int_0^{v_i} (v_i - m_i) d \prod_{j \neq i} F_{c_j^*}(m_i)$, and thus the zero expected net-payoff condition for bidder i to participate in the auction when his valuation is v_i requires that

$$c_i^*(v_i) = \int_0^{v_i} (v_i - m_i) d \prod_{j \neq i} F_{c_j^*}(m_i).$$

If $m_i = 0$, none of the other bidders will participate in the auction, the probability of which is

$$\prod_{j \neq i} F_{c_j^*}(0) = \prod_{j \neq i} \int_0^1 \int_{c_j^*(\tau)}^1 k_j(\tau, c_j) dc_j d\tau.$$

Otherwise, at least one other bidder submits a bid. Then

$$\prod_{j \neq i} F_{c_j^*}(m_i) = \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau].$$

⁷ $c_i^*(v_i)$ can be interpreted as the maximal amount that bidder i would like to pay to participate in the auction when his value is v_i .

Thus, the cutoff curve for individual i , $i \in 1, 2, \dots, n$, can be characterized by

$$c_i^*(v_i) = \int_0^{v_i} (v_i - m_i) d \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] + v_i \prod_{j \neq i} [\int_0^1 \int_{c_j^*(\tau)}^1 k_j(\tau, c_j) dc_j d\tau].$$

Now, integrating the first part by parts, we have

$$\begin{aligned} c_i^*(v_i) &= \int_0^{v_i} (v_i - m_i) d \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] + v_i \prod_{j \neq i} [\int_0^1 \int_{c_j^*(\tau)}^1 k_j(\tau, c_j) dc_j d\tau] \\ &= (v_i - m_i) \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] \Big|_0^{v_i} + v_i \prod_{j \neq i} [\int_0^1 \int_{c_j^*(\tau)}^1 k_j(\tau, c_j) dc_j d\tau] \\ &\quad + \int_0^{v_i} \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_i \\ &= -v_i \prod_{j \neq i} [1 - \int_0^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] + v_i \prod_{j \neq i} [\int_0^1 \int_{c_j^*(\tau)}^1 k_j(\tau, c_j) dc_j d\tau] \\ &\quad + \int_0^{v_i} \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_i \end{aligned}$$

Since

$$\int_0^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau + \int_0^1 \int_{c_j^*(\tau)}^1 k_j(\tau, c_j) dc_j d\tau = \int_0^1 \int_0^1 k_j(\tau, c_j) dc_j d\tau = 1,$$

we have

$$c_i^*(v_i) = \int_0^{v_i} \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_i. \quad (1)$$

Remark 3 When v_i and c_i are independent, $K_i(v_i, c_i) = F_i(v_i)G_i(c_i)$ and $k_i(v_i, c_i) = f_i(v_i)g_i(c_i)$, we have

$$c_i^*(v_i) = \int_0^{v_i} \prod_{j \neq i} [1 - \int_{m_i}^1 G_j(c_j(\tau)) f_j(\tau) d\tau] dm_i.$$

Take derivative of equation (1) with respect to v_i , we have

$$c_i^{*'}(v_i) = \prod_{j \neq i} [1 - \int_{v_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau]. \quad (2)$$

Notice that $c_i^*(0) = 0$, thus the above equation is a functional differential equation with the initial condition. Specially when v_i and c_i are independent,

$$c_i^{*'}(v_i) = \prod_{j \neq i} [1 - \int_{v_i}^1 G_j(c_j^*(\tau)) f_j(\tau) d\tau].$$

Lemma 1 $c_i^*(v_i)$ has the following properties:

$$(i) \ c_i^*(0) = 0.$$

$$(ii) \ 0 \leq c_i^*(v_i) \leq v_i.$$

$$(iii) \ c_i^{*'}(1) = 1.$$

$$(iv) \ \frac{dc_i^*(v_i)}{dn} < 0.$$

$$(v) \ \frac{dc_i^*(v_i)}{dv_i} \geq 0 \text{ and } \frac{d^2c_i^*(v_i)}{dv_i^2} \geq 0$$

(i) means that, when bidder i 's value for the object is 0, the value of participating in the auction for bidder i is zero and thus the cutoff cost point for the bidder to enter the auction is zero. Then, as long as the bidder has participation cost bigger than zero, he will not participate in the auction.

(ii) means that a bidder will not be willing to pay more than his value to participate in the auction.

(iii) means that, when a bidder's value is 1, the marginal willingness to pay to enter the auction is 1. The intuition is that when his value for the object is 1, he will win the object almost surely. Then the marginal willingness to pay is equal to the marginal increase in the valuation.

(iv) states that the participation cutoff point is a nondecreasing function in the number of bidders. As the number of bidders increases, the probability to win the object will decrease, holding other things constant. More bidders will increase the competition among the bidders and thus reduce the expected revenue.

(v) states that the marginal willingness to pay is positive and increasing. The intuition is that when a bidder's value increases, the probability of winning the auction increases. The willingness to pay increases and so is the marginal willingness to pay.

Definition 1 Given the economic environment and the properties described above, a cutoff curve equilibrium is a n -dimensional plane compromised by $(c_1^*(v_1), c_2^*(v_2), \dots, c_n^*(v_n))$ that is a solution of the following equation system:

$$(P1) \left\{ \begin{array}{l} c_1^*(v_1) = \int_0^{v_1} \prod_{j \neq 1} [1 - \int_{m_1}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_1 \\ c_2^*(v_2) = \int_0^{v_2} \prod_{j \neq 2} [1 - \int_{m_2}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_2 \\ \vdots \\ c_n^*(v_n) = \int_0^{v_n} \prod_{j \neq n} [1 - \int_{m_n}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_n, \end{array} \right.$$

or equivalently the following differential equation system problem with initial conditions:

$$(P2) \begin{cases} c_1^{*'}(v_1) = \prod_{j \neq 1} [1 - \int_{v_1}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] \text{ with } c_1^*(0) = 0 \\ c_2^{*'}(v_2) = \prod_{j \neq 2} [1 - \int_{v_2}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] \text{ with } c_2^*(0) = 0 \\ \vdots \\ c_n^{*'}(v_n) = \prod_{j \neq n} [1 - \int_{v_n}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] \text{ with } c_n^*(0) = 0. \end{cases}$$

We then have the following result on the existence of equilibrium $(c_1^*(v_1), \dots, c_i^*(v_i), \dots, c_n^*(v_n))$ for $v_i \in [0, 1]$, $i \in \{1, 2, \dots, n\}$.

Proposition 1 (The Existence Theorem) *For the economic environment under consideration, the integral equation system (P1) or the differential equation system (P2) with initial conditions $c_i(0) = 0$ for all i has at least one solution $(c_1^*(v_1), c_2^*(v_2), \dots, c_n^*(v_n))$, i.e., there is always an equilibrium in which every bidder i uses his cutoff curve $c_i^*(v_i)$.*

The differential equation system above is a partial functional differential equation system, but not a partial differential equation system. The derivatives of $c_i^*(v_i)$ at v_i depends not only on v_i itself, but also on the future path of $c_i^*(v_j)$ with $j \neq i$ and $v_j \geq v_i$. Beyond that, we have multiple variables in the functional differential equation system which increases the difficulty to show the existence of equilibrium. However, we can transfer the original differential equation system to the following differential equation system

$$(P3) \begin{cases} c_1^*(v) = \int_0^v \prod_{j \neq 1} [1 - \int_{m_1}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_1 \\ c_2^*(v) = \int_0^v \prod_{j \neq 2} [1 - \int_{m_2}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_2 \\ \vdots \\ c_n^*(v) = \int_0^v \prod_{j \neq n} [1 - \int_{m_n}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_n. \end{cases}$$

Lemma 2 Problem (P1) and problem (P3) are equivalently solvable in the sense that

(1), if $(c_1^*(v_1), c_2^*(v_2), \dots, c_n^*(v_n))$ is a solution to problem (P1), then $(c_1^*(v), c_2^*(v), \dots, c_n^*(v))$ is a solution to problem (P3).

(2), if $(c_1^*(v), c_2^*(v), \dots, c_n^*(v))$ to problem (P3), then $(c_1^*(v_1), c_2^*(v_2), \dots, c_n^*(v_n))$ is a solution to problem (P1).

Thus we have reduced the multiple variables functional differential equation system to a single variable functional equation system.

Remark 4 When v_i and c_i are independent, the equilibrium is a n -dimensional plane composed

by $(c_1^*(v), c_2^*(v), \dots, c_n^*(v))$ that is a solution of the following equation system:

$$(P4) \begin{cases} c_1^*(v) = \int_0^v \prod_{j \neq 1} [1 - \int_{m_1}^1 G_j(c_j^*(\tau)) f_j(\tau) d\tau] dm_1 \\ c_2^*(v) = \int_0^v \prod_{j \neq 2} [1 - \int_{m_2}^1 G_j(c_j^*(\tau)) f_j(\tau) d\tau] dm_2 \\ \vdots \\ c_n^*(v) = \int_0^v \prod_{j \neq n} [1 - \int_{m_n}^1 G_j(c_j^*(\tau)) f_j(\tau) d\tau] dm_n, \end{cases}$$

or equivalently the following differential equation system problem with initial conditions:

$$(P5) \begin{cases} c_1^{*'}(v) = \prod_{j \neq 1} [1 - \int_v^1 G_j(c_j^*(\tau)) f_j(\tau) d\tau] \text{ with } c_1^*(0) = 0, \\ c_2^{*'}(v) = \prod_{j \neq 2} [1 - \int_v^1 G_j(c_j^*(\tau)) f_j(\tau) d\tau] \text{ with } c_2^*(0) = 0, \\ \vdots \\ c_n^{*'}(v) = \prod_{j \neq n} [1 - \int_v^1 G_j(c_j^*(\tau)) f_j(\tau) d\tau] \text{ with } c_n^*(0) = 0. \end{cases}$$

This general model with two-dimensional private values and participation costs with general distribution functions is very general and contains many existing results as special cases. In the following, for simplicity, we assume v_i and c_i are independent to illustrate the generality of our setting.

Case 1. Suppose there is a subset, denoted by A , of bidders whose valuations are common knowledge. Then for all $i \in \bar{A} = N \setminus A$, we have

$$\begin{aligned} c_i^*(v) &= \int_0^v \prod_{j \in \bar{A} \setminus \{i\}} [1 - \int_{m_i}^1 G_j(c_j^*(\tau)) f_j(\tau) d\tau] \prod_{j \in A \setminus \{i, v_j > v\}} [1 - G_j(c_j^*(v_j))] \\ &\quad \times \prod_{j \in \bar{A} \setminus \{i, v_j < v\}} [1 - \int_{m_i}^1 G_j(c_j^*(\tau)) f_j(\tau) d\tau] dm_i. \end{aligned}$$

For all $i \in A$,

$$\begin{aligned} c_i^*(v_i) &= \int_0^{v_i} \prod_{j \in \bar{A} \setminus \{i\}} [1 - \int_{m_i}^1 G_j(c_j^*(\tau)) f_j(\tau) d\tau] \prod_{j \in A \setminus \{i, v_j > v_i\}} [1 - G_j(c_j^*(v_j))] \\ &\quad \times \prod_{j \in A \setminus \{i, v_j < v_i\}} [1 - \int_{m_i}^1 G_j(c_j^*(\tau)) f_j(\tau) d\tau] dm_i. \end{aligned}$$

In this case, one needs to distinguish the difference between $v_i > v_j$ and $v_j > v_i$, since under these two situations the expected revenue has different expressions.

Example 1 Suppose $n = 2$ and $v_1 < v_2$ is common knowledge, we have two bidders. Then for

the bidder with value v_1 ,

$$\begin{aligned}
c_2^*(v_2) &= \int_0^{v_2} [1 - \int_{m_2}^1 G_1(c_1^*(\tau))f_1(\tau)d\tau]dm_2 \\
&= \int_0^{v_1} [1 - \int_{m_2}^1 G_1(c_1^*(\tau))f_1(\tau)d\tau]dm_2 \\
&\quad + \int_{v_1}^{v_2} [1 - \int_{m_2}^1 G_1(c_1^*(\tau))f_1(\tau)d\tau]dm_2 \\
&= v_1(1 - G_1(c_1^*(v_1))) + (v_2 - v_1),
\end{aligned}$$

and

$$\begin{aligned}
c_1^*(v_1) &= \int_0^{v_1} [1 - \int_{m_1}^1 G_2(c_2^*(\tau))f_2(\tau)d\tau]dm_1 \\
&= \int_0^{v_1} [1 - G_2(c_2^*(v_2))]dm_1 = v_1(1 - G_2(c_2^*(v_2))).
\end{aligned}$$

which can be reduced to the formula obtained in Kaplan and Sela (2006) when the cost distribution functions are the same.

Case 2. On the contrary, suppose there is a subset, denoted by B , of bidders whose participation costs are common knowledge, as discussed in Tan and Yilankaya (2006) and Cao and Tian (2007). Let $\bar{A} = N \setminus A$. Then, for all $i \in N$, we have

$$\begin{aligned}
c_i^*(v_i) &= \int_0^{v_i} \prod_{j \in B \setminus \{i\}} [1 - \int_{m_i}^1 G_j(c_j^*(\tau))f_j(\tau)d\tau] \prod_{j \in \bar{B} \setminus \{i\}} [1 - \int_{m_i}^1 G_j(c_j^*(\tau))f_j(\tau)d\tau]dm_i \\
&= \int_0^{v_i} \prod_{j \in B \setminus \{i\}} [1 - \int_{m_j^*}^1 G_j(c_j^*(\tau))f_j(\tau)d\tau] \prod_{j \in \bar{B} \setminus \{i\}} [1 - \int_{m_j}^1 G_j(c_j^*(\tau))f_j(\tau)d\tau]dm_i \\
&= \int_0^{v_i} \prod_{j \in B \setminus \{i\}, m_j > v} [1 - \int_{m_j}^1 G_j(c_j^*(\tau))f_j(\tau)d\tau] \prod_{j \in B \setminus \{i\}, m_j < v} [1 - \int_{m_j}^1 G_j(c_j^*(\tau))f_j(\tau)d\tau] \\
&\quad \times \prod_{j \in \bar{B} \setminus \{i\}} [1 - \int_{m_i}^1 G_j(c_j^*(\tau))f_j(\tau)d\tau]dm_i,
\end{aligned}$$

where m_j^* is determined by $c_j^*(m_j) = c_j$ for $j \in B$. It may be remarked that $c_i^*(v_i)$ may have different functional forms when v_i is in the different regions of $v_i > m_j^*$ and $v_i \leq m_j^*$.

Example 2 Consider an economic environment with two bidders whose values are drawn from the same continuous distribution function $F(v)$. Bidders' participation costs are common knowledge and the same, $c_1 = c_2 = c$. This is an economy studied in Tan and Yilankaya (2006) for $n = 2$. Let $c_1^*(m_1^*) = c_2^*(m_2^*) = c$.

Then for bidder 1, we have

$$c_1^*(v_i) = \int_0^{v_i} [1 - \int_{m_1}^1 G(c_2^*(\tau))f(\tau)dm_1]d\tau.$$

As such, we have

$$c_1^*(v_1) = \int_0^{v_1} [1 - \int_{m_2^*}^1 G(c_2^*(\tau))f(\tau)dm_1]d\tau = F(m_2^*)v_1$$

when $v_1 < m_2^*$, and

$$\begin{aligned} c_1^*(v_1) &= \int_0^{m_2^*} [1 - \int_{m_2^*}^1 G(c_2^*(\tau))f(\tau)dm_1]d\tau + \int_{m_2^*}^{v_1} [1 - \int_{m_2^*}^1 G(c_2^*(\tau))f(\tau)dm_1]d\tau \\ &= F(m_2^*)m_2^* + \int_{m_2^*}^{v_1} F(m_1)dm_1 \end{aligned}$$

when $v_1 \geq m_2^*$.

Similarly, for bidder 2, we have

$$c_2^*(v_2) = \int_0^{v_2} [1 - \int_{m_2}^1 G(c_1^*(\tau))f(\tau)d\tau]dm_2.$$

Then, we have $c_2^*(v_2) = F(m_1^*)v_2$ when $v_2 < m_1^*$, and $c_2^*(v_2) = F(m_1^*)m_1^* + \int_{m_1^*}^{v_2} F(m_2)dm_2$ when $v_2 \geq m_1^*$.

We can use these equations to find the cutoff points. It is clear that there is a symmetric equilibrium in which both bidders use the same cutoff point $m_1^* = m_2^* = m^*$, which satisfies the equation

$$m^*F(m^*) = c.$$

Indeed, by the monotonicity of $m^*F(m^*)$, the symmetric equilibrium exists and is unique.

Now if we provisionally suppose that $m_1^* < m_2^*$, then we should have

$$c_1^*(m_1^*) = m_1^*F(m_2^*) = c,$$

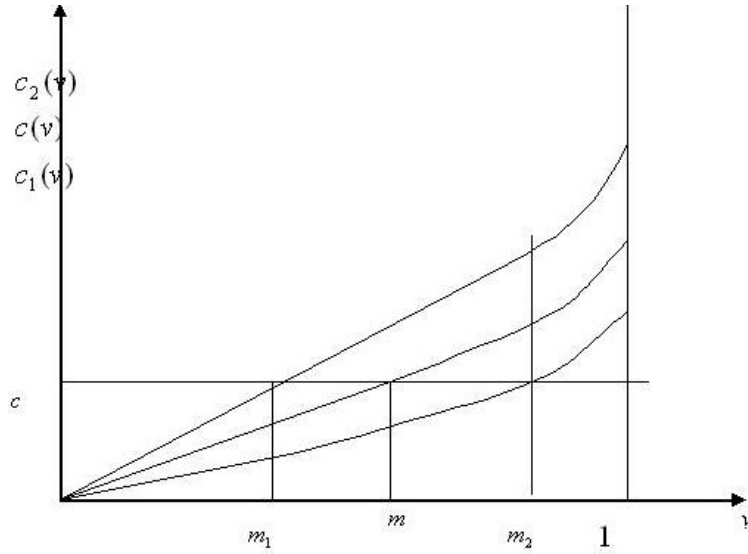
and

$$c_2^*(m_2^*) = m_1^*F(m_1^*) + \int_{m_1^*}^{m_2^*} F(m_2)dm_2 = c.$$

Tan and Yilankaya (2006) showed that when $F(v)$ is strictly convex, there exists $m_1^* < m_2^*$ satisfying the above two equations.

We can use a graph to illustrate the equilibria in Example 2. There are three curves in the graph. The middle curve indicates both bidders use the same cutoff point $c^*(v)$, and then they have the same cutoff point m^* . The highest curve is bidder 1's reaction curve $c_1^*(v_1)$. There is a kink at $v_1 = m_2^*$. Before this point, the curve is a straight line passing through the original point with slope $F(m_2^*)$. After m_2^* , it is a smooth curve with the slope changing along the curve, which is $F(v)$. We can see as $v \rightarrow 1$, the slope goes to 1, which is consistent with properties of

Figure 1: Symmetric & Asymmetric Equilibrium With 2 Bidders



the cutoff curves described in Lemma 1. The lowest curve is bidder 2's reaction curve $c_2^*(v_2)$. The equilibrium is the intersection of the horizontal line c and each bidder's cutoff curve.

Case 3. When all participation costs are zero, $G_i(c_i^*(\tau)) = 1$ for all τ and all i . Then

$$c_i^*(v) = \int_0^v \prod_{j \neq i} [1 - \int_{m_i}^1 f_j(\tau) d\tau] dm_i = \int_0^v \prod_{j \neq i} F_j(m_i) dm_i > 0,$$

and thus, a bidder with positive value for the object will always participate in the auction and submit a bid. Under this circumstance the entrance equilibrium curve is unique.

Case 4. When all participation costs are 1, $G_j(c_j^*(\tau)) = 0$ for all $c_j^*(\tau) < 1$, and thus $c_i^*(v) = 1$. Considering the initial condition, we have $c_i^*(v_i) = v_i$, i.e., a bidder with value v_i would like to pay at most v_i to enter the auction. Now since the designed participation cost is 1 for all bidders, then there will be no one participating in the auction.

4 Uniqueness of Equilibrium

To investigate the uniqueness of the equilibrium $c^*(v)$, we can focus on uniqueness of the solution of (P3) by Lemma 2. We first consider the case that all bidders are ex ante homogeneous in the sense that they have the same joint distribution functions of valuations and participation costs and focus on the symmetric equilibrium in which all bidders use the same cutoff curve, and then study the uniqueness of equilibrium for a more general case. Then (P3) can be simply written as

$$c^*(v) = \int_0^v [1 - \int_m^1 \int_0^{c^*(\tau)} k_j(\tau, c) dc d\tau]^{n-1} dm, \quad (3)$$

and correspondingly we have

$$c^{*'}(v) = [1 - \int_v^1 \int_0^{c^*(\tau)} k(\tau, c) dc d\tau]^{n-1}, c^*(0) = 0. \quad (4)$$

We first give the uniqueness of the symmetric equilibrium when all bidders are ex ante homogeneous.

Proposition 2 (Uniqueness of Symmetric Equilibrium) *For the economic environment under consideration in this section, suppose that all bidders have the same distribution function $K(v, c)$. There is a unique solution $c^*(v)$ to integral equation (3) or differential equation (4) with initial condition. Consequently, there exists a unique symmetric equilibrium at which each bidder uses the same cutoff curve for his entrance decision making.*

Remark 5 Uniqueness of the symmetric equilibrium has been established in some special cases.

- 1) In Campbell (1998) and Tan and Yilankaya (2006), when bidders have the same participation cost and continuously differentiable valuation distribution function, there is a unique symmetric equilibrium in which each bidder uses a same cutoff point v^s for their entrance decision making.
- 2) In Kaplan and Sela (2006), when all bidders have the same valuations for the object and continuously differentiable participation cost distribution functions, there is a unique symmetric cutoff point c^* .
- 3) More earlier, Laffont and Green (1984) investigated the existence of equilibria when both valuation and participation costs are uniform distributed. They got the uniqueness of the symmetric equilibrium under the simple two-dimensional economic environment. However their proof is incomplete.

Remark 6 Note that the above proposition only shows that the uniqueness of symmetric equilibrium when bidders are ex ante homogeneous. It does not exclude the possibility of the asymmetric equilibrium. As those in Tan and Yilankaya (2006), Kalpan and Sela (2006), there are some examples where ex-ante homogeneous bidders may use different cutoffs which means the equilibria are not unique.

Now under the assumption of independence of v_i and c_i , We consider the uniqueness of the functional differential equation system (P3). For simplicity we consider a simple economy with only two bidders. The corresponding functional differential equation system can be written as:

$$(P6) \begin{cases} c_1^*(v) = [1 - \int_v^1 G_2(c_2^*(\tau))f_2(\tau)d\tau], c_1^*(0) = 0, \\ c_2^*(v) = [1 - \int_v^1 G_1(c_1^*(\tau))f_1(\tau)d\tau], c_2^*(0) = 0. \end{cases}$$

Proposition 3 (Uniqueness of Equilibrium) In the two bidder economy with $G_i(c)$ is continuously differentiable on $[0, 1]$ and $\delta_i = \max_c g_i(c)$, there is a unique equilibrium when $\delta_i \int_0^1 (1 - F_i(s))ds < 1$.

When $G_i(c_i)$ is uniform on $[0, 1]$, $\delta_i = 1$ and $\int_0^1 (1 - F_i(s))ds < 1$, we have a unique equilibrium. Specially when bidders are ex ante homogenous, the unique equilibrium is symmetric. To see this, consider the following examples.

Example 3 This example follows from Example 1. Assume that $G_i(c_i)$ is uniform on $[0, 1]$. Then we have

$$\begin{cases} c_1^*(v_1) = v_1(1 - c_2^*(v_2)), \\ c_2^*(v_2) = v_2 - v_1 c_1^*(v_1). \end{cases}$$

There is a unique equilibrium given by $c_1^*(v_1) = \frac{v_1(1-v_2)}{1-v_1^2}$ and $c_2^*(v_2) = \frac{v_2(1-v_2)}{1-v_1^2}$. Further we can check that when $v_1 = v_2 = v$, the unique equilibrium is symmetric with $c_1^*(v) = c_2^*(v) = \frac{v}{1+v}$.

Example 4 Now we assume $G_i(c)$ and $F_i(v)$ are both uniform on $[0, 1]$. At equilibrium we have

$$\begin{aligned} c_1^*(v) &= 1 - \int_v^1 c_2^*(\tau)d\tau, \\ c_2^*(v) &= 1 - \int_v^1 c_1^*(\tau)d\tau. \end{aligned}$$

Then $c_1^{*''}(v) = c_2^*(v)$ and $c_2^{*''}(v) = c_1^*(v)$. Thus we have $c_1^{*(4)}(v) = c_1^*(v)$ and $c_2^{*(4)}(v) = c_2^*(v)$ with $c_1^*(0) = 0$, $c_1^{*'}(1) = 1$, $c_2^*(0) = 0$ and $c_2^{*'}(1) = 1$. One can check that the only equilibrium is $c_1^*(v) = c_2^*(v) = ae^v - ae^{-v}$, where $a = \frac{e}{e^2+1}$.

5 Discussions

There are in general multiple equilibria in the setting under consideration. Examples can be found in Campbell (1998), Tan and Yilankaya (2006), Cao and Tian (2007) and Kaplan and Sela (2006) where either participation costs or valuations are common knowledge. In this section we provide evidence for the multiplicity of equilibria even when both the participation costs and valuations are private information.

Suppose the support of v_i and c_i to be $[0, 1] \times [\epsilon, \delta]$, where $[\epsilon, \delta]$ is a subset of $[0, 1]$ and $\epsilon > 0$. To investigate the existence of equilibrium, we construct a new density function $\tilde{k}_i(v_i, c_i)$ with

support $[0, 1] \times [0, 1]$ which has the same density as $k_i(v_i, c_i)$ on the interval $[0, 1] \times [\epsilon, \delta]$ and 0 otherwise and $\widetilde{K}_i(v_i, c_i)$ is the corresponding cumulative density function. The same as in Section 3, the equilibrium cutoff curve for individual i , $i \in 1, 2, \dots, n$, is given by

$$c_i^*(v_i) = \int_0^{v_i} (v_i - m_i) d \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} \widetilde{k}_j(\tau, c_j) dc_j d\tau] + v_i \prod_{j \neq i} [\int_0^1 \int_{c_j^*(\tau)}^1 \widetilde{k}_j(\tau, c_j) dc_j d\tau].$$

After integration by parts we have

$$c_i^*(v_i) = \int_0^{v_i} \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} \widetilde{k}_j(\tau, c_j) dc_j d\tau] dm_i.$$

Then

$$c_i^{*'}(v_i) = \prod_{j \neq i} [1 - \int_{v_i}^1 \int_0^{c_j^*(\tau)} \widetilde{k}_j(\tau, c_j) dc_j d\tau]. \quad (5)$$

By the fixed point theorem, an equilibrium exists. However the uniqueness of the equilibrium cannot be guaranteed. Specially when bidders are ex ante homogenous, asymmetric equilibrium may exist.

One special type of asymmetric equilibrium is that some bidders may never participate in the auction. This can happen when the support of participation costs, c , has non-zero lower bound. Such an equilibrium can be called a corner equilibrium. One implication of such equilibrium is that in this economic environment, some of the bidders can form a collusion to enter the auction regressively so that they can prevent some others enter the auction and thus can reduce the competition among those who participate in the auction which in turn will increase the benefits from participating.

The expected revenue of participating in the auction is a non-decreasing function of one's true value. Thus the sufficient and necessary condition for a bidder to never participate is when his value is 1, participating in the auction still gives him an expected revenue that is less than the minimum participation cost, ϵ , giving the strategies of other bidders. Formally, suppose in equilibrium, a subset $A = \{1, 2, \dots, k\} \subset \{1, 2, 3, \dots, n\}$ of bidders choose to participate in the auction when their valuations are big enough and bidders in $B = \{k + 1, \dots, n\}$ choose never participating in the auction. Then for all $i \in A$ we have

$$c_i^*(v_i) = \int_0^{v_i} \prod_{j \neq i, j \in A} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} \widetilde{k}_j(\tau, c_j) dc_j d\tau] dm_i.$$

For bidders in B never participate, it is required that for all $j \in B$,

$$c_j^*(1) = \int_0^1 \prod_{i \in A} [1 - \int_{m_j}^1 \int_0^{c_i^*(\tau)} \widetilde{k}_i(\tau, c_j) dc_j d\tau] dm_j < \epsilon,$$

which raises a requirement for the lower and upper bound of the participation costs and the distributions of valuations and participation costs. To see this, we assume that there are only two bidders and v_i and c_i are independent. The distribution functions are $F(v_i)$ and $G(c_i)$ separately.

Suppose bidder 2 never participates, then bidder 1 enters if and only if $v_1 \geq c_1$ and thus we have $c_1^*(v_1) = v_1$. Given this, the expected revenue of bidder 2 when he participates in the auction is

$$F(\epsilon) + \int_{\epsilon}^{\delta} [(1 - v_2)G(v_2) + (1 - G(v_2))]dF(v_2) + \int_{\delta}^1 (1 - v_2)dF(v_2)$$

when $v_2 = 1$. We have three terms in the above equation. When bidder 1's value is less than ϵ he will not enter the auction and bidder 2 will get revenue 1, the probability is $F(\epsilon)$; the second term is the revenue when bidder 1's value is between ϵ and δ . For any $v_2 \in (\epsilon, \delta)$, bidder 2's revenue is $1 - v_2$ when bidder 1 participates, and is 1 when bidder 1 does not participate, the probabilities are $G(v_2)$ and $1 - G(v_2)$ separately. The third term is the revenue when bidder 1's value is greater than δ and in this case bidder 1 participates for sure.

In order to have a corner equilibrium, we need

$$F(\epsilon) + \int_{\epsilon}^{\delta} [(1 - v_2)G(v_2) + (1 - G(v_2))]dF(v_2) + \int_{\delta}^1 (1 - v_2)dF(v_2) < \epsilon. \quad (6)$$

It can be seen that in the two homogenous bidders economy, when $F(\cdot)$ is concave, there is no corner equilibrium. To see this, note that when $F(\cdot)$ is concave, we have $F(v_i) \geq v_i$, equation (6) can not hold; i.e., corner equilibrium does not exist.

Remark 7 if $\epsilon = \delta$; i.e., c_i is common knowledge to all bidders, (6) can be simplified to $F(\epsilon) + \int_{\epsilon}^1 (1 - v_2)dF(v_2) < \epsilon$; i.e., $\epsilon F(\epsilon) + \int_{\epsilon}^1 F(v_2)dv_2 < \epsilon$.

Example 5 Assume v_i and c_i to be joint uniform distributed (then they are independent) and there are only two bidders. Suppose bidder 2 never participates. We have $c_1^*(v_1) = v_1$.

Then we have

$$c_2^*(v_2) = 1 - \int_{v_2}^1 G(c_1(\tau))d\tau = 1 - \int_{v_2}^1 \min\{1, \max\{\frac{\tau - \epsilon}{\delta - \epsilon}, 0\}\}d\tau,$$

which results

$$c_2^*(v_2) = \begin{cases} 1 - \int_{\epsilon}^{\delta} \frac{\tau - \epsilon}{\delta - \epsilon} d\tau - \int_{\delta}^1 d\tau = \frac{\epsilon + \delta}{2} & \text{if } v_2 < \epsilon \\ \delta - \frac{\delta^2 - 2\epsilon\delta - v_2^2 + 2\epsilon v_2}{2(\delta - \epsilon)} = \frac{\delta^2 + v_2^2 - 2\epsilon v_2}{2(\delta - \epsilon)} & \text{if } \epsilon \leq v_2 < \delta \\ v_2 & \text{if } v_2 \geq \delta \end{cases} .$$

Given the above and the initial condition $c_2^*(0) = 0$, we have

$$c_2^*(v_2) = \begin{cases} \frac{\epsilon + \delta}{2} v_2 & \text{if } v_2 < \epsilon \\ \frac{v_2^3 - 3\epsilon v_2^2 - \epsilon^3 + 3\delta^2 v_2}{6(\delta - \epsilon)} & \text{if } \epsilon \leq v_2 < \delta \\ \frac{\delta^2 + \delta\epsilon + \epsilon^2 + 3v_2^2}{6} & \text{if } v_2 \geq \delta \end{cases}$$

For bidder 2 never participates, we need $c_2^*(1) = \frac{\delta^2 + \delta\epsilon + \epsilon^2 + 3}{6} \leq \epsilon$, which is equivalent to $\epsilon^2 + (\delta - 6)\epsilon + \delta^2 + 3 \leq 0$. So when

$$\frac{(6 - \delta) - \sqrt{-3(\delta^2 + 4\delta - 8)}}{2} \leq \epsilon \leq \frac{(6 - \delta) + \sqrt{-3(\delta^2 + 4\delta - 8)}}{2},$$

the required condition is satisfied. For this to be true, we need $\frac{(6 - \delta) - \sqrt{-3(\delta^2 + 4\delta - 8)}}{2} < \delta$ and thus $\delta^2 - 2\delta + 1 < 0$ which cannot be true.

However when $F(\cdot)$ is strictly convex, given proper ϵ and δ , there may be an equilibrium in which one bidder never participates while the other enters the auction whenever his valuation is greater than his participation cost. As an illustration, we assume $F(v_i) = v_i^2$ and $G(c_i)$ is uniformly distributed on $[\epsilon, \delta]$. (6) becomes

$$\frac{\delta^3 + \delta\epsilon^2 + \delta^2\epsilon + \epsilon^3 + 2}{6} < \epsilon.$$

One can check that when $\epsilon = 0.5$ and $\delta = 0.744$, there exists a corner equilibrium. It can be concluded that if there is a corner equilibrium in the homogenous two bidder economy, there exists a corner equilibrium in which $n - 1$ bidders never participate in the homogenous n bidder economy.

When the lower bound of valuation is positive, there may exist an asymmetric equilibrium in which one bidder always participates. To see this, suppose the c_i is distributed on $[c_l, c_h]$ with distribution $G_i(c_i)$ and v_i is distributed on $[v_l, v_h]$ with distribution $F_i(v_i)$. Assume $v_h > v_l > c_h > c_l$. Suppose we have an equilibrium in which bidder 1 always enters and bidder 2 never participates. Then bidder 1 always participating is a best response. For bidder 2's strategy to be a best response, we need

$$\int_{v_l}^{v_h} (v_h - v_1) dF_1(v_1) - c_l < 0,$$

the maximum expected revenue is less than the lowest participation cost. Integration by parts we have

$$\int_{v_l}^{v_h} F_1(v_1) dv_1 < c_l.$$

One sufficient condition for this to be true is $v_h - v_l < c_l$.

6 Conclusion

This paper investigates equilibria of second price auctions with general distribution functions of private values and participation costs. We show that there always exists an equilibrium cutoff curve for each bidder. Moreover, when all bidders are ex ante homogeneous, there is a unique symmetric equilibrium. In a simple two bidder economy, a sufficient condition for the uniqueness of the equilibrium is identified. This general two-dimensional framework covers many models as special cases.

We find evidence that multiple equilibria exist. Specifically, when bidders are ex ante homogeneous, besides the symmetric equilibrium, there may be an equilibrium at which one bidder always participates or never participates. Future research may be focused on identifying sufficient conditions to guarantee the uniqueness of equilibrium but not only the uniqueness of the symmetric equilibrium. Uniqueness of the equilibrium has important policy implications. The seller can modify the economic environment such that the economy has a unique equilibrium and thus have more predictable power for the final outcomes. Welfare analysis with participation costs is another interesting topic to be tackled.

Appendix

Proof of Lemma 1:

Proof: (i) Letting $v_i = 0$ in the expression of $c_i^*(v_i)$, we have the result.

(ii) Since

$$c_i^*(v_i) = \int_0^{v_i} \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_i \leq \int_0^{v_i} dv_i = v_i$$

by the nonnegativity of $\int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j$ and

$$\int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau \leq \int_{m_i}^1 \int_0^1 k_j(\tau, c_j) dc_j d\tau \leq \int_0^1 \int_0^1 k_j(\tau, c_j) dc_j d\tau = 1,$$

we have $0 \leq c_i^*(v_i) \leq v_i$.

(iii) Letting $v_i = 1$ in (5), we have the result.

(iv) Since n is the number of bidders, as n increases, say, from n to $n + 1$, the product term inside the integral will be increased by one more term. Also, note that $0 < 1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau < 1$. So given more bidders, $c_i^*(v_i)$ will decrease.

(v)

$$\frac{dc_i^*(v_i)}{dv_i} = \prod_{j \neq i} [1 - \int_{v_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] \geq 0$$

by noting that

$$\int_{v_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau \leq \int_0^1 \int_0^1 k_j(\tau, c_j) dc_j d\tau = 1.$$

We then have

$$\frac{d^2 c_i^*(v_i)}{dv_i^2} = \sum_{k \neq i} \prod_{j \neq i, j \neq k} [1 - \int_{v_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] \int_0^{c_k^*(v_i)} k_k(\tau, c_j) dc_j d\tau \geq 0.$$

Proof of Proposition 1:

Proof: For $i = 1, 2, \dots, n$, let

$$\phi_i(c^*(v_i)) = \int_0^{v_i} \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_i$$

where $c^*(v) = (c_1^*(v_1), c_2^*(v_2), \dots, c_n^*(v_n))$. Then $\phi_i(c^*)$ is a continuous function and $0 \leq \phi_i(c^*) \leq v_i$ by Lemma 1.(ii). Thus, $\phi_1(\cdot), \phi_2(\cdot), \dots, \phi_n(\cdot)$ is a continuous mapping from the non-empty compact and convex domain $[0, v_1] \times [0, v_2] \times \dots \times [0, v_n]$ to itself, and therefore, by Brouwer's Fixed Point Theorem, there exists $c^*(v) = (c_1^*(v_1), c_2^*(v_2), \dots, c_n^*(v_n))$ such that $c_i^*(v_i) = \phi_i(c^*(v))$, and consequently, it is a solution to (P2) or (P3) with initial condition.

Proof of Lemma 2:

Proof: Suppose $(c_1^*(v_1), c_2^*(v_2), \dots, c_n^*(v_n))$ is a solution to problem (P1), then we have for any $i \in \{1, 2, \dots, n\}$,

$$c_i^*(v_i) = \int_0^{v_i} \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_i,$$

then by changing the variable v_i to v we have

$$c_i^*(v) = \int_0^v \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_i$$

for all $i \in \{1, 2, \dots, n\}$. So $(c_1^*(v), c_2^*(v), \dots, c_n^*(v))$ is a solution to (P3). On the contrary, if $(c_1^*(v), c_2^*(v), \dots, c_n^*(v))$ is a solution to (P3), then we have for any $i \in \{1, 2, \dots, n\}$,

$$c_i^*(v) = \int_0^v \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_i.$$

Then by changing the variable v to v_i in the i^{th} equation we have

$$c_i^*(v_i) = \int_0^{v_i} \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_i.$$

Thus $(c_1^*(v_1), c_2^*(v_2), \dots, c_n^*(v_n))$ is a solution to (P1).

Proof of Proposition 2:

Proof: The existence of the symmetric equilibria can be established by the Brower' Fixed point Theorem. Here we only need to prove the uniqueness of the symmetric equilibrium. Suppose not, by way of contradiction, we have two different symmetric equilibria $x(v)$ and $y(v)$ to the economic environment we consider. Then we have

$$x'(v) = [1 - \int_v^1 \int_0^{x(\tau)} k(\tau, c) dc d\tau]^{n-1}$$

$$y'(v) = [1 - \int_v^1 \int_0^{y(\tau)} k(\tau, c) dc d\tau]^{n-1}.$$

Suppose $x(1) > y(1)$, then by the continuity of $x(v)$ and $y(v)$ we can find a v^* such that $x(v^*) = y(v^*) = c(v^*)$ and $x(v) > y(v)$ for all $v \in (v^*, 1]$ by noting that $x(0) = y(0)$.

Case 1: if $k(v, c) > 0$ with positive probabilitiy measure on $(v^*, 1) \times (c(v^*), 1)$, then for $\tau \in (v^*, 1]$ we have

$$\int_0^{x(\tau)} k(\tau, c) dc > \int_0^{y(\tau)} k(\tau, c) dc$$

for $\tau \in (v^*, 1)$. Then we have $x'(v^*) < y'(v^*)$ which is a contradiction to $x(v) > y(v)$ for $v > v^*$. So we have $x(1) = y(1)$. By the same logic above we can prove that $x(v) = y(v)$ for all $v \in [0, 1]$ and thus the symmetric equilibrium is unique.

Case 2: if $k(v, c) > 0$ with zero probability measure on $(v^*, 1) \times (c(v^*), 1)$, then we have $x'(v) = y'(v)$ for all $v \in (v^*, 1]$. By $x(v^*) = y(v^*)$ we have $x(v) = y(v)$ for all $v > v^*$, which is a contradiction to $x(v) > y(v)$. Thus there is a unique symmetric equilibrium.

Then in both cases we proved that there is a unique symmetric equilibrium.

Proof of Proposition 3:

Proof: Define a mapping

$$(Pc)(v) = \int_0^v ds - \int_0^v \int_s^1 \begin{pmatrix} 0 & f_1(\tau) \\ f_2(\tau) & 0 \end{pmatrix} \begin{pmatrix} G_1(c_1(\tau)) \\ G_2(c_2(\tau)) \end{pmatrix} d\tau ds,$$

where $c = (c_1, c_2)'$.

Take any $x(v) = (x_1(v), x_2(v))'$ and $y(v) = (y_1(v), y_2(v))'$ with $x(v), y(v) \in \varphi$ where φ is the space of monotonic increasing continuous functions defined on $[0, 1] \rightarrow [0, 1]$. Then we have

$$\begin{aligned} |(Px)(v) - (Py)(v)| &\leq \int_0^v \int_s^1 \begin{pmatrix} 0 & g_1(\hat{x}_1(\tau))f_1(\tau) \\ g_2(\hat{x}_2(\tau))f_2(\tau) & 0 \end{pmatrix} \begin{pmatrix} x_1(\tau) - y_1(\tau) \\ x_2(\tau) - y_2(\tau) \end{pmatrix} |d\tau ds \\ &= \int_0^v \int_s^1 \begin{pmatrix} 0 & g_1(\hat{x}_1(\tau))f_1(\tau) \\ g_2(\hat{x}_2(\tau))f_2(\tau) & 0 \end{pmatrix} d\tau ds \sup_{0 < v \leq 1} |x(v) - y(v)| \\ &\leq \int_0^1 \int_s^1 \begin{pmatrix} 0 & g_1(\hat{x}_1(\tau))f_1(\tau) \\ g_2(\hat{x}_2(\tau))f_2(\tau) & 0 \end{pmatrix} d\tau ds \sup_{0 < v \leq 1} |x(v) - y(v)| \\ &\leq \int_0^1 \begin{pmatrix} 0 & \delta_1(1 - F_1(s)) \\ \delta_2(1 - F_2(s)) & 0 \end{pmatrix} ds \sup_{0 < v \leq 1} |x(v) - y(v)|, \end{aligned} \quad (7)$$

where the first equality comes from mean value theorem, and $\hat{x}_i(\tau)$ is some number between $x_i(\tau)$ and $y_i(\tau)$, δ_i is the maximum of $g_i(c)$, $i = 1, 2$. Thus when $\delta_i \int_0^1 (1 - F_i(s)) ds < 1$, the above mapping is a contraction, there exists a unique equilibrium.

An Alternative Proof for The Existence of Equilibria:

We give the proof of the existence of equilibrium based on (P3), the transferred single variable functional differential equation system.

Proposition 4 (The Existence Theorem) *For the general economic environment under consideration in the paper, the integral equation system (P3) has at least one solution $(c_1^*(v), c_2^*(v), \dots, c_n^*(v))$; i.e., there is always an equilibrium in which every bidder i uses his own cutoff curve $c_i^*(v)$.*

To prove the above proposition we introduce the following lemma:

Lemma 3 (Schauder-Tychonov Fixed-point Theorem Cf. Smart (1980, p.15)) *Let M be a compact convex nonempty subset of a locally convex topological space and $P : M \rightarrow M$ be continuous. Then P has a fixed point.*

Proof of Proposition 4:

Proof: Let $h_i = \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c) dc d\tau]$ and $H = (h_1, h_2, \dots, h_n)'$. Define

$$M = \{c \in \varphi \mid |c| \leq n, |c(v_1) - c(v_2)| < n|v_1 - v_2|\},$$

where φ is the space of continuous of function ϕ defined on $[0, 1] \rightarrow R^n$ with the supremum norm. Then by Ascoli's theorem M is compact and M is certainly convex. Define an operator $P : M \rightarrow M$ by

$$(Pc)(v) = \int_0^v H(s, c(\cdot)) ds$$

To see $P : M \rightarrow M$, note that

$$|(P\psi)(v_1) - (P\psi)(v_2)| \leq \left| \int_{v_2}^{v_1} H(s, c(\cdot)) ds \right| \leq n|v_1 - v_2|,$$

and also it can be easily check that

$$|(P\psi)(v) - 0| \leq n.$$

To see P is continuous, let $\phi \in M$ and let $\mu > 0$ be given. We must find $\eta > 0$ such that $\|\phi - \psi\| < \eta$ implies $\|(P\phi)(t) - (P\psi)(t)\| \leq \mu$. Now

$$|(P\phi)(t) - (P\psi)(t)| = \left| \int_0^t [H(s, \phi(s)) - H(s, \psi(s))] ds \right|$$

and H is uniformly continuous so for the $\mu > 0$ there is an $\eta > 0$ such that $|\phi(s) - \psi(s)| < \eta$ implies $|H(s, \phi(s)) - H(s, \psi(s))| \leq \mu$ and thus $|(P\phi)(t) - (P\psi)(t)| \leq \mu$ by noting that $0 < t \leq 1$ as required. Then by Lemma 3, there exists a fixed point; i.e., a solution for the functional differential equation system, also the solution is continuously differentiable.

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