Second Price Auctions with Differentiated Participation Costs *

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Abstract

This paper studies equilibria of second price auctions with differentiated participation costs. We consider equilibria in independent private values environments where bidders' entry costs are common knowledge while valuations are private information. We identify two types of equilibria: monotonic equilibria in which a higher participation cost results in a higher cutoff point for submitting a bid, and neg-monotonic equilibria in which a higher participation cost results in a lower cutoff point. We show that there always exists a monotonic equilibrium, and further, that the equilibrium is unique for concave distribution functions and strictly convex distribution functions with some additional conditions. There exists a neg-monotonic equilibrium when the distribution function is strictly convex and the difference of the participation costs is sufficiently small. We also provide comparative static analysis and study the limit status of equilibria when the difference in bidders' participation costs approaches zero.

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Key Words: Private Values, Differentiated Participation Costs, Second Price Auctions, Counter-monotonic Equilibrium, Existence and Uniqueness of Equilibrium.

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1 Introduction

An auction is an effective way to extract private information by increasing the competition of potential buyers and thus can increase allocation efficiency from the perspectives of both sellers and the social optimum when we do not have complete information about bidders' types. However, not every auction can be implemented freely. This paper studies (Bayesian-Nash) equilibria of sealed-bid second price, or Vickrey, auctions for economies with private values and differentiated participation costs.

1.1 Motivation

The fundamental structure of a second price auction with participation costs is one in which, under the independent values economic environment, an indivisible object is allocated to one of many potential buyers via auction, and in order to participate in the auction, buyers must pay an entry fee.[1] The fee may be used to deter non-serious buyers, to advertise, to hire an auctioneer for the auction itself, to account for the costs of traveling to the auction site, to proxy for the process of learning the rules of the auction, or to cover the costs of the acquisition of information, etc.[2] Sometimes participating in auctions and procurement is highly costly. Hence the question of whether to participate in auctions may be more crucial than the standard question of how to bid, suggesting that such decisions should be modeled and included as part of an equilibrium.

With participation costs, bidders' behavior may change. If a bidder's expected revenue from the auction is less than the participation cost before the auction, he will choose not to participate in the auction. If the expected revenue from the auction is bigger than the costs, the bidder will participate and submit a bid accordingly. Even if a bidder decides to participate in the auction, since he may expect that some other bidders will not participate, his bidding behavior may not be the same as in the standard auction without participation costs. The number of bidders submitting a bid in the auction is less than the number of bidders submitting a bid in the standard auction without participation costs, which may in turn, alter the equilibrium bidding strategy. For example, more bidders can raise coordination costs and will not necessarily improve the revenue of the sellers (cf. Samuelson (1985), Harstad, Kagel, and Levin (1990), Levin and Smith (1994)).

[1] In the literature such as those in Green and Laffont (1984), Samuelson (1986), McAfee and McMillan (1987) etc., participation cost, participation fee, entry fee, entry cost or opportunity costs of participating in the auction are used interchangeably.

[2] Persico (2000) studied the incentive of information acquisition in auctions. They found that bidders have more incentives for information acquisition in first price auctions than in second price auctions.

1.2 Related Literature

The study of participation costs in auctions mainly focuses on the second price auction due to its simplicity of bidding behavior. In standard second price auctions, bidding one's own valuation is a weakly dominant strategy. There is also another equilibrium in standard second price auctions as shown in Blume and Heidhues (2004): the bidder with the highest value bids his true value and all others bid zero. This is referred to as the asymmetric bidding equilibrium in the standard second price auction. However, in second price auctions with participation costs, it is still true that if a bidder finds participating optimal, he cannot do better than bidding his true value. Therefore, in this paper we only consider equilibria in which potential bidders use cutoff strategies; i.e., bid their true values if they are greater than the corresponding cutoff points, do not participate otherwise.[3] All of our results about the uniqueness or multiplicity of the equilibria should be interpreted accordingly.

Green and Laffont (1984) were the first to study the second price auction with participation costs in a general framework where bidders' valuations and participation costs are both private information. However, their proof of the existence and uniqueness of an equilibrium is incomplete additionally having imposed a restrictive assumption of uniform distributions for both values and participation costs. There has been some recent work in the literature on equilibria of the second price auction with participation costs in simplified versions where either only valuations or participation costs are private, while the other is assumed to be common knowledge.

Campbell (1998) considered the equilibria in second price auctions in an economic environment with equal participation costs when bidders' values are private information and participation costs are common knowledge. He focused on the coordination of equilibrium choice when multiple equilibria exist. Tan and Yilankaya (2006) also studied equilibria of second price auctions in an economic environment with equal participation costs. They proved that the equilibrium is unique and symmetric when bidders' distribution functions for values are concave. They also considered the case in which bidders are asymmetric in the sense that they have different valuation distribution functions while maintaining identical participation costs. Some others, such as Samuelson (1985), McAfee and McMillan (1987), Levin and Smith (1994), Stageman (1996), and Menezes and Monterio (2000) also studies auctions with participation costs. All of these studies assume bidders' participation cost are the same.

The assumption of equal participation costs, however, is stringent and unrealistic in many

[3] There may exist equilibrium in which bidders do not bid their true value when they participate. See the example given Remark 4 bolow.

situations. For instance, bidders may have different transportation costs for traveling to auction spots. Bidders may also have different ability to learn more information about the auctions. Some bidders can easily know valuations of other potential bidders while others do not. Thus, it may be more meaningful for one to study individuals' behavior in second price auctions with different participation costs.

Differentiated participation costs may also have additional implications. First, they can be used to distinguish bidders. Bidders can use this information to decide whether or not to participate in the auction. One can analyze how a bidder's cutoff point will be affected by others' participation costs. Secondly, while Tan and Yilankaya (2006) mainly considered bidding behavior inside the same group, they did not consider how the interaction among the different groups would determine and affect the equilibrium behavior.

Kaplan and Sela (2006) studied equilibria of the second price auction with participation costs when bidders' participation costs are private information, while valuations are common knowledge. They considered the existence of type-symmetric equilibria.

This paper aims to investigate the equilibria when bidders have different participation costs, an analysis which will be more applicable in reality. By considering bidders with different participation costs, we can investigate how equilibria vary as participation costs change. We can also study the limit behavior of the equilibria as the difference in participation costs approaches zero.

1.3 Results of The Paper

This paper considers economic environments where bidders have private valuations for the object and different participation costs that are common knowledge. We identify two types of equilibria: monotonic equilibria in which a higher participation cost results in a higher cutoff point for entering the auction and submitting a bid, and neg-monotonic equilibria in which a higher participation cost results in a lower cutoff point for entering the auction and submitting a bid. We show that there always exists a monotonic equilibrium, and further that, it is unique for concave distribution functions and strictly convex distribution functions under some additional conditions. Uniqueness of the equilibria can greatly simplify the world. When bidders' distribution functions are strictly convex and the differences among the bidders' participation costs are sufficiently small, there is a neg-monotonic equilibrium. There is no neg-monotonic equilibrium when the difference is sufficiently large.

Our neg-monotonic equilibrium result has a policy implication. When both monotonic and neg-monotonic equilibria coexist, one may want to refine the equilibria. Since a monotonic equilibrium always exists, one can eliminate neg-monotonic equilibria by charging sufficiently different entry fees for different types of bidders. In other words, if we know the difference in participation costs is sufficiently large, we do not need to consider the neg-monotonic equilibria.

Our study on auction with differentiated participation costs is not only more realistic, but also provides us deeper insight that would help us understand individuals' equilibrium behavior better in auction with participation cost. This can be seen by studying the limit behavior of the monotonic and neg-monotonic equilibria when bidders' participation costs converge to the same value. We show that, when the distribution function of valuation is concave, the monotonic equilibrium converges to the symmetric equilibrium when bidders have the same participation costs. However, when the distribution is strictly convex, the monotonic equilibrium converges to the asymmetric equilibrium. In this case one neg-monotonic equilibrium converges to a symmetric equilibrium, and another neg-monotonic equilibrium converges to an asymmetric equilibrium.

We also provide some comparative static analysis. It is shown that the cutoff point is increasing in one's own participation costs, but decreasing in opponents' participation costs, and further, as the number of bidders increases, the cutoff points of all bidders will increase.

The organization of the paper is as follows: In Section 2, we describe a general setting of economic environments. In Section 3, we focus on two bidders with the same distribution functions and different participation costs to investigate the existence, uniqueness, and limit properties of the equilibria and to make a comparative analysis. In Section 4, we extend our basic results to more general economic environments by relaxing the assumptions made in Section 3. Concluding remarks are provided in Section 5. All the proofs are presented in the appendix.

2 The Setup

We consider the independent values economic environment with one seller and $n \geq 2$ potential buyers (bidders). The seller is risk-neutral and has an indivisible object to sell to one of the buyers. The seller values the object as 0. The auction format is the sealed-bid second price auction format (see Vickrey, 1961). However, in order to submit a bid, bidder i must pay a participation cost c_i. Buyer i's valuation for the object is v_i, which is private information to the other bidders. It is assumed that v_i is independently distributed with a cumulative distribution function $F_i(v)$ that has continuously differentiable density $f_i(v) > 0$ everywhere with support $[0, 1]$. We will study the equilibrium behavior mainly for the case where bidders have the same distribution functions and then consider the general case of the different distribution functions. The participation costs $c_i \in (0, 1]$ for all i are common knowledge.

Each bidder knows his value, participation cost, and the distributions of the others' valuations. If participating in auction, he is required to pay a non-refundable participation fee. The bidder with the highest bid wins the object and pays the second-highest bid. If there is only one person in the auction, he wins the object and pays 0. If the highest bids are equal for more than one bidder, then he pays his own bid and gains nothing.

In this second price auction mechanism with participation costs, the individually rational action set for any type of bidder is $\{No\} \cup [0, 1]$, where "$\{No\}$" denotes not participating in the auction. Bidder i incurs the participation cost if and only if his action is different from "$\{No\}$". Let $b_i(v_i, c)$ denote bidder i's strategy where $c = (c_1, ..., c_n)$.

If a bidder finds participating in this second price auction optimal, he cannot do better than bidding his true valuation (i.e., bidding his true valuation is a weakly dominant strategy). Therefore, we can restrict our attention to Bayesian-Nash equilibria in which each bidder uses a cutoff strategy denoted by $v_i^*(c)$, i.e., he bids his valuation if it is greater than or equal to the cutoff point[4] and does not enter otherwise. An equilibrium strategy of each bidder i is then determined by the cutoff point for his valuation, which is the minimum valuation bidder i needs to cover the cost. Thus the bidding decision function of each bidder is characterized by

$$b_i(v_i, c) = \begin{cases} v_i & \text{if } v_i^*(c) \leq v_i \leq 1 \\ \text{No} & \text{otherwise.} \end{cases}$$

For notational convenience, we simply denote $v_i^*(c) = v_i^*$.

Remark 1 When $v_i^* \leq 1$, bidder i will participate in the auction whenever his true value satisfies $v_i^* \leq v_i \leq 1$. However, when bidder i's expected revenue is always less than his participation cost c_i for any $v_i \in [0, 1]$, he will never participate in the auction. In this case, his equilibrium strategy (action) is "$\{No\}$". For notional convenience, and also for simplicity of discussion, we use $v_i^* > 1$ to denote the equilibrium strategy of "$\{No\}$". Thus allows us to use a unified notation v_i^* to denote an equilibrium strategy of bidder i, including the equilibrium of "$\{No\}$". The rationale behind using $v_i^* > 1$ to denote the equilibrium strategy of "$\{No\}$" is the following: If we find a value v_i^* such that bidder i's expected revenue is equal to his participation cost c_i by allowing the upper bound of the support to be greater than one, we will end up with a value v_i^* that is greater than one. But the true value is actually less than or equal to one, and thus $v_i^* > 1$ is equivalent to the equilibrium strategy of "$\{No\}$."

[4] In Milgrom and Weber (1982), the term of "screening level" is used instead of using "cutoff point."

From now on we focus exclusively on cutoff points, since they are sufficient to describe equilibria. We define them with following formal definition:

Definition 1 For the economic environment under consideration, an equilibrium is a cutoff point vector $(v_1^*, v_2^*, ..., v_n^*) \in \mathbb{R}_+^n$ such that each bidder i action's is optimal, given others' cutoff strategies.

We then immediately have the following result:

Lemma 1 *$v_i^* \leq 1$ for at least some i.*

Since bidders with higher participation costs are less likely to participate in the auction, one may come to the intuition conclusion that bidders with higher participation costs may have higher cutoff points to participate in the auction. One may also perceive that bidders with the same participation costs will use the same cutoff point when their distribution functions are the same. However, as we will show in the paper, it is possible that a bidder with a higher participation cost may actually have a lower cutoff point to enter the auction. To study these possibilities, we may distinguish two types of equilibria: monotonic equilibria and neg-monotonic equilibria.

Definition 2 An equilibrium $(v_1^*, v_2^*, \ldots, v_n^*) \in \mathbb{R}_+^n$ for the economic environment under consideration is called a *monotonic equilibrium* (*resp. neg-monotonic equilibrium*) if, for any two bidders i and j, $c_i < c_j$ implies $v_i^* < v_j^*$ (resp. $v_i^* \geq v_j^*$).

As usual, when bidders' distribution functions are the same; i.e., $F_1(\cdot) = F_2(\cdot) = \ldots = F_n(\cdot) = F(\cdot)$, we can define the usual symmetric and asymmetric equilibria.

Definition 3 An equilibrium $(v_1^*, v_2^*, \ldots, v_n^*) \in \mathbb{R}_+^n$ is called a *symmetric equilibrium* (*resp. asymmetric equilibrium*) if, for any two bidders i and j, $c_i = c_j$ implies $v_i^* = v_j^*$ (resp. $v_i^* \neq v_j^*$).

Remark 2 Campbell (1998) and Tan and Yilankaya (2006) studied the existence of symmetric and asymmetric equilibria for the second price auctions with the same participation costs. The terminology of "monotonic" used here means that two variables c and v^* vary in the same direction: a higher participation cost results in a higher cutoff point. When bidders' distribution functions are the same, as one will see in Section 3, $v_1^* = v_2^*$ cannot be an equilibrium, provided bidders' participation costs are different. Thus, $c_i < c_j$ implies $v_i^* > v_j^*$ for every neg-monotonic equilibrium, and $c_i < c_j$ implies $v_i^* < v_j^*$ for every monotonic equilibrium. However, when bidders' distribution functions are different, as we will show below, $v_1^* = v_2^*$ may be an equilibrium although bidders' participation costs are different. That is, we have a special neg-monotonic equilibrium with $v_i^* = v_j^*$ even when $c_i < c_j$.

Remark 3 We can give an simple example to understand the notion of monotonic and neg-monotonic equilibria. Suppose there is one object for sale to two bidders. Both bidders value it at 1. The participation costs are $c_1 < c_2 < 1$. If both bidders enter, they both have negative payoffs. There are two pure strategy equilibria: (bidder 1 enters, bidder 2 stays out) and (bidder 1 stays out, bidder 2 enters), which correspond to two equilibrium cutoff points $(v_1^* = c_1, v_2^* > 1)$ and $(v_1^* > 1, v_2^* = c_2)$. That is, the former one is monotonic and the latter one is neg-monotonic. Bidders may also use mixed strategies when there are multiple cutoff points. For simplicity, in this paper we focus only on the pure strategy, not the mixed strategy of using different cutoff points.

The remainder of the paper investigates whether an equilibrium exists or not. If it exists, we ask if it unique, and whether it is monotonic or neg-monotonic. We first consider the simple case of two bidders with the same distribution functions. We then consider more general cases.

3 Two Bidders with Different Participation Costs

In this section we consider an economy with two bidders who have different participation costs c_1 and c_2 with $c_1 < c_2$, and have the same distribution function $F(v)$ on $[0, 1]$, where the costs are common knowledge and valuations are private information.

We first assume, provisionally, that a monotonic equilibrium (v_1^*, v_2^*) exists, i.e., $v_1^* < v_2^*$. By Lemma 1, we must have $v_1^* \leq 1$. When bidder 1's valuation is $v_1 = v_1^*$, his expected revenue is given by $v_1^* F(v_2^*) + 0(1 - F(v_2^*))$, where $F(v_2^*)$ is the probability bidder 2 will not participate in the auction. Indeed, when he participates in the auction and bidder 2 does not participate in the auction, his value is v_1^*. When bidder 2 participates in the auction, it must be the case that $v_2 \geq v_2^*$. Then bidder 1 cannot get the object since $v_2 \geq v_2^* > v_1^* = v_1$, and thus his revenue is zero. Therefore, his expected revenue from the auction is $v_1^* F(v_2^*)$. The zero net-payoff (equilibrium) condition then requires that

$$c_1 = v_1^* F(v_2^*). \tag{1}$$

When bidder 2's participation cost is too large, he may never participate in the auction, no matter what his valuation is. In this case, bidder 1 uses $v_1^* = c_1$ as his cutoff point, and bidder

2's expected payoff must satisfy

$$F(c_1) + \int_{c_1}^{1} (1 - v)dF(v) = c_1F(c_1) + \int_{c_1}^{1} F(v)dv < c_2;$$

i.e., the expected revenue he obtains from participating even when his value is 1 is less than his participation cost, given bidder 1 uses c_1 as the cutoff point. Thus we have $v_2^* > 1$. Then, we may have a monotonic equilibrium with $v_1^* = c_1$ and $v_2^* > 1$.

Now suppose $v_2^* \leq 1$. Then, when bidder 2's valuation is $v_2 = v_2^*$, his expected revenue is

$$v_2^*F(v_1^*) + \int_{v_1^*}^{v_2^*} (v_2^* - v)dF(v),$$

where the first part is the expected revenue when bidder 1 does not enter the auction, and the second part is the expected revenue when both bidders participate in the auction. Note that bidder 2 will lose the object if $v_1 > v_2^*$. The zero expected net-payoff (equilibrium) condition then requires that

$$v_2^*F(v_1^*) + \int_{v_1^*}^{v_2^*} (v_2^* - v)dF(v) = c_2. \tag{2}$$

Integrating by parts in the left side of (2), we have

$$v_1^*F(v_1^*) + \int_{v_1^*}^{v_2^*} F(v)dv = c_2. \tag{3}$$

Note that, from (1) and (3), one can see the claim in Remark 2 is true: It is impossible for both bidders to use the same cutoff point $v_1^* = v_2^* = v^*$ when their participation costs are different. Indeed, suppose not. Then we must have $c_1 = v^*F(v^*)$ by (1) and $c_2 = v^*F(v^*)$ by (2). Thus $c_1 = c_2$, which contradicts the fact that $c_2 > c_1$.

Before we proceed to investigate the existence and uniqueness of the monotonic equilibrium, it is necessary to introduce more notation. Campbell (1998) and Tan and Yilankaya (2006) showed the existence and uniqueness of the symmetric cutoff point $v_i^* = v^s$ when bidders have the same participation cost. In our model, if both bidders have the same participation cost c_1, we have $v_1^sF(v_1^s) = c_1$, and then we can find the symmetric equilibrium cutoff point v_1^s. Now, if both bidders have the participation cost c_2, we can find the symmetric equilibrium cutoff point v_2^s by solving $v_2^sF(v_2^s) = c_2$. Such $v_1^s \leq 1$ and $v_2^s \leq 1$ do exist and are unique since the defined function $m(v) = vF(v)$ is monotonically increasing, $m(0) = 0$, and $m(1) = 1$.

The following lemma shows the relationship between a monotonic equilibrium and symmetric equilibria.

Lemma 2 *Suppose (v_1^*, v_2^*) is a monotonic equilibrium, (v_1^s, v_1^s) and (v_2^s, v_2^s) are symmetric equilibria associated with participation costs $c_1 < c_2$, respectively. Then, we have $v_1^* < v_1^s < v_2^s < v_2^*$.*

This lemma shows that, when bidders have different participation costs, the cutoff point for the bidder with the lower participation cost at the monotonic equilibrium is lower than the cutoff point at the symmetric equilibrium when bidders have the same lower participation cost c_1.

To find a monotonic equilibrium, we define the following two cutoff reaction function equations.

$$xF(y) = c_1 \tag{4}$$

$$xF(x) + \int_x^y F(v)dv = c_2 \tag{5}$$

with $x < y$, where x corresponds to v_1^*, and y corresponds v_2^*. It can be easily seen that we have $x \geq c_1$ and $y \geq c_2$. They can be regarded as cutoff reaction functions because (4) shows how bidder 1 will choose a cutoff point x, given bidder 2's action y. Equation (5) shows how bidder 2 will choose a cutoff point y, given bidder 1's a action x. A monotonic equilibrium $(v_1^*, v_2^*) \in [0,1] \times [0,1]$ is obtained when x and y satisfy these two equations simultaneously.

From (4), we have $x = x(y) = \frac{c_1}{F(y)}$. Then $\frac{dx}{dy} = -\frac{c_1 f(y)}{F^2(y)} < 0$. This implicitly defines y as a decreasing function of x, denoted by $y = y(x)$. We now substitute $y = y(x)$ into the left side of (5) and let

$$h(x) = xF(x) + \int_x^{y(x)} F(v)dv - c_2.$$

Substitute $x = x(y)$ into the left side of (5) and let

$$\lambda(y) = \frac{c_1}{F(y)} F(\frac{c_1}{F(y)}) + \int_{\frac{c_1}{F(y)}}^y F(v)dv.$$

Then $\lambda'(y) = F(y) - \frac{c_1}{F(y)} \frac{c_1}{F^2(y)} f(y) f(\frac{c_1}{F(y)})$. Since $x = \frac{c_1}{F(y)}$, by substitution, we have

$$\lambda'(y) = F(y) - \frac{x^2}{F(y)} f(y) f(x).$$

To consider the existence of neg-monotonic equilibria in which the cutoff points satisfy $v_2^* < v_1^*$ whenever $c_1 < c_2$, we can follow the above process similarly. Also by Lemma 1, we have $v_2^* \leq 1$.

For bidder 2, when $v_2 = v_2^*$, his expected revenue is given by $v_2^* F(v_1^*)$ and the zero profit condition requires that

$$c_2 = v_2^* F(v_1^*). \tag{6}$$

For bidder 1, it is possible that $v_1^* > 1$, i.e., bidder 1 will never participate. Again, this requires that

$$F(c_2) + \int_{c_2}^1 (1-v)dF(v) = c_2 F(c_2) + \int_{c_2}^1 F(v)dv < c_1.$$

In this case, we have a neg-monotonic equilibrium with $v_1^* > 1$ and $v_2^* = c_2$.

Now suppose the above inequality cannot be true. Then bidder 1 chooses a cutoff point $v_1^* \in [0, 1]$. When his valuation is $v_1 = v_1^* \leq 1$, he participates in the auction and receives a zero net-payoff so that

$$v_1^* F(v_2^*) + \int_{v_2^*}^{v_1^*} (v_1^* - v) dF(v) - c_1 = 0.$$

Integrating by parts, we get

$$c_1 = v_2^* F(v_2^*) + \int_{v_2^*}^{v_1^*} F(v) dv. \tag{7}$$

Since the distribution function $F(v)$ is non-decreasing, we have

$$c_1 > v_1^* F(v_2^*). \tag{8}$$

In order for (6), (8), and $c_2 > c_1$ to be consistent, it requires that

$$v_2^* F(v_1^*) > v_1^* F(v_2^*)$$

or

$$\frac{F(v_1^*)}{v_1^*} > \frac{F(v_2^*)}{v_2^*}. \tag{9}$$

To find neg-monotonic equilibrium, through (6) and (7), we define the two cutoff reaction functions

$$\begin{aligned} y(x) &= c_2 / F(x) \\ \phi(x) &= \frac{c_2}{F(x)} F\left(\frac{c_2}{F(x)}\right) + \int_{\frac{c_2}{F(x)}}^{x} F(v) dv. \end{aligned}$$

Again, we use x to correspond to v_1^* and y to correspond to v_2^*. Note that we have $x \geq y \geq c_2$.

From Campbell (1998) and Tan and Yilankaya (2006), we know that when two bidders have the same participation cost c_2 and $F(v)$ is strictly convex, there exists a unique symmetric equilibrium $x = y = v_2^s$ that satisfies $y = x = c_2 / F(x)$ and an asymmetric equilibrium (x_0, y_0) with $x_0 > v_2^s$ and $y_0 < v_2^s$, indicating that $\phi(x)$ intersects with c_2 when $x = v_2^s$ and $x = x_0$. Also, by the uniqueness of symmetric equilibrium, $v_1^* \geq v_2^s$ if it exists. Let c_m be the minimum of $\phi(x) = \frac{c_2}{F(x)} F(\frac{c_2}{F(x)}) + \int_{\frac{c_2}{F(x)}}^{x} F(v) dv$ in the interval $[v_2^s, 1]$.

We then have the following proposition on the existence and uniqueness of equilibria:

Proposition 1 (Existence and Uniqueness Theorem) *For the independent private values economic environment with two bidders who have different participation costs $c_2 > c_1$, we have the following conclusions:*

(1) There always exists a monotonic equilibrium.

(2) Suppose $F(v)$ is concave. Then the equilibrium is unique and monotonic.

(3) Suppose $F(v)$ is strictly convex. Then

(3.i) the monotonic equilibrium is unique when $\frac{f(v)}{F(v)^2}$ is non-increasing,

(3.ii) the neg-monotonic equilibrium is unique when $c_1 = c_m$,

(3.iii) there is no neg-monotonic equilibrium when $c_1 < c_m$, and

(3.iv) there are at least two neg-monotonic equilibria when $c_m < c_1 < c_2$.

The formal proof can be found in the appendix. Here we provide some intuition as to why the results are true. To investigate the existence and uniqueness of the equilibria, we first note the extreme case where there may be an equilibrium in which one bidder will never participate in the auction. We then exam how functions $\lambda(y)$ and $\phi(x)$ intersect with c_2 and c_1, respectively. The existence of a monotonic equilibrium can be established by the intermediate value theorem. The uniqueness of the monotonic (neg-monotonic) equilibrium comes from the fact that $\lambda(y)$ and $\phi(x)$ intersect with c_2 and c_1, respectively, at most once on the interval $y \in [v_1^s, 1]$ and $[v_2^s, 1]$. When $F(v)$ is concave, $\lambda(y)$ is a monotonic increasing function, and thus the monotonic equilibrium is unique. When $F(v)$ is strictly convex, we can also show the unique monotonic equilibrium and the existence and uniqueness of neg-monotonic equilibria for some types of convex distribution functions.

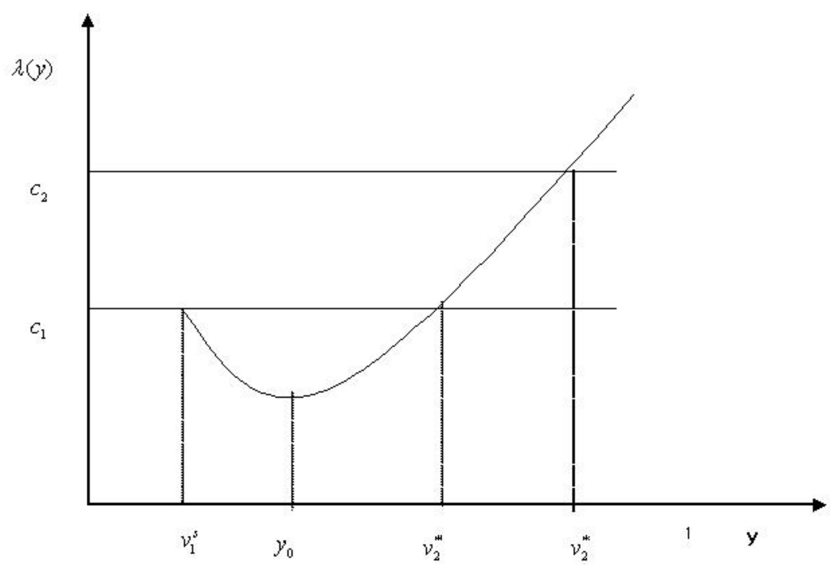

Figure 1: Uniqueness for Convex Case

Remark 4 There are some facts that may be mentioned for understanding the proof of Proposition 1:

1. For any strictly convex power functions and exponential functions, $\frac{f(v)}{F(v)^2}$ is a non-increasing function of v. Thus, the set of such strictly convex functions is not empty. A figure can be used for understanding why there is a unique monotonic equilibrium for this type of strictly convex distribution. In Figure 1, $\lambda(y)$ starts from v_1^s with negative slope. When $\lambda'(y) = 0$ has at most one solution, $\lambda(y)$ intersects with c_2 at most one time, indicating that the monotonic equilibrium is unique.

2. From the proof in the appendix, one can see that it is always true that $c_2 > c_m$. Then, as long as $c_2 - c_1$ is sufficiently small, we have $c_2 > c_1 > c_m$. Thus, we can conclude that when $c_2 - c_1$ is sufficiently small, there are two neg-monotonic equilibria that are given by (x_1, y_1) and (x_2, y_2) with $y_1 = y(x_1)$, $y_2 = y(x_2)$, and $y_1 < y_2 < v_2^s < x_1 < x_m < x_2 < x_0$. Thus, when $F(v)$ is not concave, the existence of a neg-monotonic equilibrium depends on the difference of participation costs, $c_2 - c_1$. For instance, when $c_1 = 0.3$, $c_2 = 0.32$, and $F(v) = \frac{v+v^3}{2}$ (which is strictly convex), we have one monotonic equilibrium $(0.3753, 0.8911)$ and two neg-monotonic equilibria $(0.6995, 0.6142)$ and $(0.8301, 0.4564)$. However, when $c_1 = 0.3$, $c_2 = 0.4$, and $F(v) = \frac{v+v^3}{2}$, we only have one monotonic equilibrium $(0.3003, 0.9994)$. Thus, this example demonstrates that there are multiple neg-monotonic equilibria when $c_2 - c_1$ is sufficiently small, and there is no neg-monotonic equilibrium when $c_2 - c_1$ is large enough.

3. Figure 2 can help us to understand the proof and the points mentioned above. $\phi(x)$ starts from $y = v_2^s$ with negative slope. When $c_2 - c_1$ is small enough, it intersects with c_1; i.e., a neg-monotonic equilibrium exists. When $c_2 - c_1$ is big enough so that $c_1 < c_m$, $\phi(x)$ and c_1 can not intersect; i.e., no neg-monotonic equilibrium exists. From the figure, when c_1 is close to c_2, there are at least two intersection points for $y = \phi(x)$ and $y = c_1$, which means there are at least two neg-monotonic equilibria, say, (x_1, y_1) and (x_2, y_2).

4. Campbell (1998) and Tan and Yilankaya (2006) showed that there exists an asymmetric equilibrium when distribution functions are strictly convex. However, our result shows that the strict convexity of the distribution function alone is not a sufficient condition for the existence of a neg-monotonic equilibrium, unless the

difference $c_2 - c_1$ is small enough. In fact, this result implies that one can refine equilibria and always eliminate non-equilibria by making participation costs for bidders sufficiently different.

5. In the proof of Proposition 1, the condition that $F(v)$ is concave can be weakened to $F(v) \geq vf(v)$ for all $v \in [c_1, 1]$, and the condition that $F(v)$ is strictly convex can be weakened to $F(v) < vf(v)$ for all $v \in [c_2, 1]$.

6. A non-truth-telling equilibrium may exist when bidders do not use weakly dominant bidding strategies even if they participate. For example, suppose bidder 1 bids zero when he enters and bidder 2 bids 1 when he enters. For bidder 1, he only wins when bidder 2 does not enter, hence in equilibrium $v_1^* F(v_2^*) = c_1$. Now for bidder 2, he always wins once he enters and pays nothing. At equilibrium we have $v_2^* = c_2$. Thus $v_1^* = \frac{c_1}{F(c_2)}$. So if bidders do not use dominant bidding strategy when they enter, we may have other cutoff equilibria.

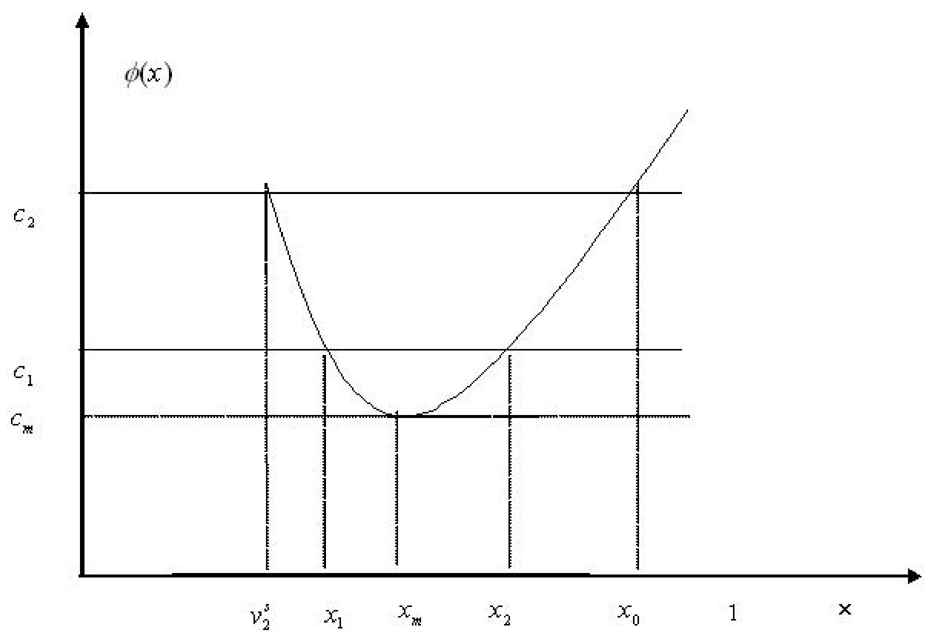

Figure 2: Existence of Counter-Monotonic Equilibria For Convex Case

One may wonder what would happen at the limits of monotonic and neg-monotonic equilibria as $c_2 - c_1 \to 0$. Should a monotonic equilibrium converge to a symmetric equilibrium or a neg-monotonic equilibrium converge to an asymmetric equilibrium when $c_2 \to c_1$?

Table 1 Sequences of Monotonic and Counter-Monotonic Equilibria)

c_1	c_2	$F(v) = \sqrt{v}$	$F(v) = v^2$		
0.3	0.3	(0.4481, 0.4481)	(0.6694, 0.6694)	(0.3425, 0.9358)	(0.9358, 0.3425)
0.3	0.31	(0.4387, 0.4675)	(0.6845, 0.6616)	(0.3327, 0.9426)	(0.9318, 0.3570)
0.3	0.32	(0.4303, 0.4861)	(0.7000, 0.6530)	(0.3237, 0.9627)	(0.9271, 0.3723)
0.3	0.33	(0.4226, 0.5038)	(0.7162, 0.6434)	(0.3155, 0.9751)	(0.9216, 0.3886)
0.3	0.34	(0.4156, 0.5210)	(0.7332, 0.6323)	(0.3079, 0.9870)	(0.9150, 0.4061)
0.3	0.35	(0.4091, 0.5376)	(0.7517, 0.6193)	(0.3009, 0.9985)	(0.9068, 0.4256)
0.3	0.36	(0.4032, 0.5537)	(0.7725, 0.6038)	(0.3000, 1.0000)	(0.8963, 0.4418)
0.3	0.37	(0.3976, 0.5694)	(0.7984, 0.5804)	(0.3000, 1.0000)	(0.8805, 0.4773)
0.3	0.38	(0.3923, 0.5847)	NA	(0.3000, 1.0000)	NA

For instance, suppose c_1 is constant at, 0.30, and let c_2 decrease from some point until $c_2 = c_1 = 0.3$. Will there be any convergence behavior for monotonic and neg-monotonic equilibria in this case? Do they converge to a symmetric equilibrium or an asymmetric equilibrium (if it exists) for a given distribution function? Some numerical experiments are given in Table 1.

From the table, when $F(v) = \sqrt{v}$, which is concave , we only have the monotonic equilibrium and is unique. Tan and Yilanyaka (2006) proved that when $F(v)$ is concave there is only one unique symmetric equilibrium and no asymmetric equilibrium. Then is natural that, when c_2 converges to c_1, the unique monotonic equilibrium will converge to the unique symmetric equilibrium, as can be seen from Table 1.

However, when $F(v) = v^2$, which is a strictly convex distribution function, we can see from the table that when $c_2 - c_1$ is small enough, there exist one monotonic and two neg-monotonic equilibria, but when $c_2 - c_1$ is big enough, there is only one monotonic equilibrium. We can also see from the table, somewhat surprisingly, that unlike the monotonic equilibrium, one sequence of neg-monotonic equilibria converges to the symmetric equilibrium, while the other sequence of monotonic equilibria converges to the asymmetric equilibrium. Thus, the notion of monotonic/neg-monotonic equilibrium is not a trivial generalization of symmetric/asymmetric equilibria.

Actually, these limit relationships among monotonic/neg-monotonic equilibria and symmetric/asymmetric equilibria are true for general concave and strictly convex functions.

Proposition 2 (Limit Theorem) *For the independent private values economic environment with two bidders with participation costs $c_2 > c_1$, we have the following conclusions:*

(1) Suppose $F(v)$ is concave. The unique monotonic equilibrium (no neg-monotonic equilibrium) converges to the unique symmetric equilibrium as $c_2 - c_1 \to 0$.

(2) Suppose $F(v)$ is strictly convex and $\frac{f(v)}{F(v)^2}$ is a non-increasing function of v. The unique monotonic equilibrium converges to an asymmetric equilibrium as $c_2 - c_1 \to 0$.

(3) Suppose $F(v)$ is strictly convex. When $c_2 - c_1 \to 0$, there are two neg-monotonic of which one converges to the unique symmetric equilibrium, and the other converges to an asymmetric equilibrium.

Some intuition can be given here for the convergence results of the equilibria. By the continuity of the reaction function, as the participation costs c_1 and c_2 converge, the set of equilibria will converge to the set of equilibria when $c_1 = c_2$. In particular, if we focus on the equilibrium in which bidder 1 uses the smallest cutoff point among all bidder 1's equilibrium cutoffs (which is necessarily a monotonic equilibrium), this will converge to the equilibrium for $c_1 = c_2$ in which bidder 1 uses the smallest cutoff among all of bidder 1's equilibrium cutoffs. Thus, if the equilibrium is unique when $c_1 = c_2$, and there is a unique monotonic equilibrium for all c_1 and c_2 in the sequence, that equilibrium sequence must converge to the symmetric equilibrium. However, if there are asymmetric equilibria when $c_1 = c_2$, then the equilibrium in which bidder 1 uses the smallest cutoff must converge to the asymmetric equilibrium in which bidder 1 uses the smaller cutoff. Hence, if the monotonic equilibrium is unique, then it will converge, and the equilibrium that converges to the symmetric equilibrium must be neg-monotonic.

From Figures 1 and 3, one can see that, as $c_2 - c_1 \to 0$, any monotonic/non monotonic equilibrium converges along the bidders' reaction curves determined by $\lambda(y)$ and ϕ to the nearest equilibrium, whether it is symmetric or asymmetric.

Before finishing this section, we examine the effects of changes in participation costs on equilibrium behavior.

Proposition 3 (Comparative Static Theorem) *For the independent private values economic environment with two bidders, suppose the values of bidders are drawn from a concave distribution function $F(v)$ and the participation costs c_1 and c_2 are common knowledge. Then an increase in participation cost c_i increases i's cutoff point v_i^* but decreases the opponent's cutoff point v_j^* for $j \neq i$.*

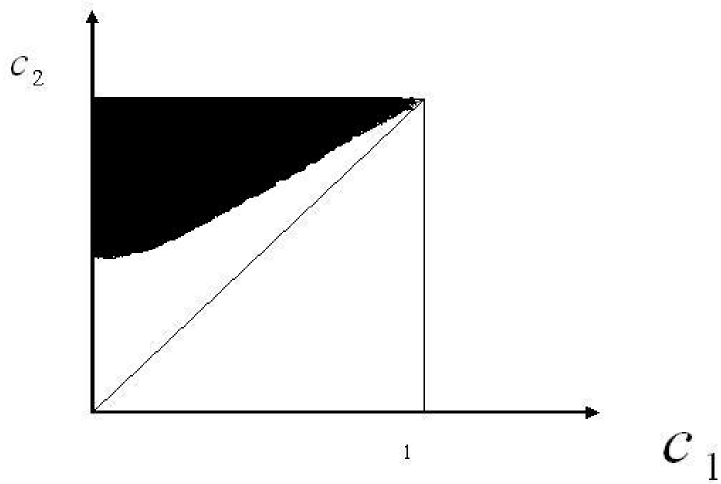

Figure 3: Uniform Case

In fact, when $F(v)$ is uniform, we can derive the unique equilibrium explicitly, and analyze equilibrium behavior directly. The condition for $v_2^* > 1$ implies $c_2 > \frac{1}{2} + \frac{1}{2}c_1^2$. In Figure 3, above the parabola $c_2 = \frac{1}{2} + \frac{1}{2}c_1^2$ and inside the square (the shaded area) is the area where bidder 2 will never participate ($v_2^* > 1$) and bidder 1 uses $v_1^* = c_1$ as his cutoff point. In the area between $c_1 = c_2$ and the parabola, we have $c_1 < c_2 \leq \frac{1}{2} + \frac{1}{2}c_1^2$. In this case, there is a unique monotonic equilibrium with $v_1^* \leq 1$ and $v_2^* \leq 1$ that can be solved explicitly.

Using (1) and (3) under the uniform distribution, we have

$$(P1)\begin{cases} v_2^* > v_1^* \\ c_1 = v_1^* v_2^* \\ \frac{1}{2}(v_1^{*2} + v_2^{*2}) = c_2. \end{cases}$$

Solving these equations, we have

$$\begin{cases} v_1^* = \frac{1}{2}(\sqrt{2(c_1+c_2)} - \sqrt{2(c_2-c_1)}) \\ v_2^* = \frac{1}{2}(\sqrt{2(c_1+c_2)} + \sqrt{2(c_2-c_1)}). \end{cases}$$

We can also require here that $v_2^* \leq 1$ to see what conditions should be satisfied. From $\frac{1}{2}(\sqrt{2(c_1+c_2)} + \sqrt{2(c_2-c_1)}) \leq 1$, we immediately have $c_2 \leq \frac{1}{2} + \frac{1}{2}c_1^2$, which is exactly the same condition required for $v_2^* \leq 1$.

Since

$$\frac{\partial v_1^*}{\partial c_1} = (2\sqrt{c_1+c_2})^{-1} + (2\sqrt{c_2-c_1})^{-1} > 0$$
$$\frac{\partial v_2^*}{\partial c_2} = (2\sqrt{c_1+c_2})^{-1} + (2\sqrt{c_2+c_1})^{-1} > 0,$$

the equilibrium cutoff points are increasing functions in their own participation costs; the higher a bidder's own participation cost is, the less likely he will participate in the auction and submit the bid.

Also, since $c_1 + c_2 > c_2 - c_1$, we have

$$\frac{\partial v_1^*}{\partial c_2} = (2\sqrt{c_1 + c_2})^{-1} - (2\sqrt{c_2 - c_1})^{-1} < 0$$

$$\frac{\partial v_2^*}{\partial c_1} = (2\sqrt{c_1 + c_2})^{-1} - (2\sqrt{c_2 - c_1})^{-1} < 0.$$

The cutoff point of each bidder is a decreasing function of the other's participation cost. The intuition behind this is clear. The higher your opponent's participation cost is, the less likely he will participate in the auction. Thus, it is more likely you will win the object, so your expected net-payoff tends to be higher. Consequently, you will be more willing to participate in the auction, so your cutoff point will be lower.

Remark 5 Here we express the comparative statics in terms of c_1 and c_2. We can also express these in terms of v_1^* and v_2^*: $\frac{\partial v_1^*}{\partial c_1} = \frac{v_2^*}{{v_2^*}^2 - {v_1^*}^2} > 0$, $\frac{\partial v_2^*}{\partial c_1} = -\frac{v_1^*}{{v_2^*}^2 - {v_1^*}^2} < 0$, $\frac{\partial v_1^*}{\partial c_2} = -\frac{v_1^*}{{v_2^*}^2 - {v_1^*}^2} < 0$, and $\frac{\partial v_2^*}{\partial c_2} = \frac{v_2^*}{{v_2^*}^2 - {v_1^*}^2} > 0$.

More generally, suppose we have a monotonic equilibrium (v_1^*, v_2^*) for the costs (c_1, c_2). Now choose (c_1', c_2') satisfying $c_1' \geq c_1$ and $c_2' \leq c_2$. Note that bidder 1's best response when bidder 2's cutoff is in $[c_2, v_2^*]$ must lie in $[v_1^*, 1]$ since c_1 has weakly increased and bidder 2's best response when bidder 1's cutoff is in $[v_1^*, 1]$ must lie in $[c_2, v_2^*]$ since c_2 has weakly decreased. Thus for costs (c_1', c_2'), the sets $[v_1^*, 1]$ for bidder 1 and $[c_2, v_2^*]$ for bidder 2 are closed under best response. So there must be an equilibrium in which each bidder uses a cutoff point from his specified set. That is, when one bidder's cost increases and the other's decreases, there is necessarily a new equilibrium in which the former uses a greater cutoff point and the latter uses a smaller cutoff. Of course when $F(v)$ is concave, which gives us a unique equilibrium, we obtain the above result.

4 Extensions

In the previous section, we studied the equilibrium behavior for an economic environment with two bidders and the same continuously differentiable distribution functions defined on the support $[0, 1]$. In this section, we briefly discuss some extensions by relaxing these assumptions.

4.1 Two Types of Bidders with Different Participation Costs

In this subsection we extend the model in Section 3 to a more general economic environment where there are two types of bidders. Type 1 possesses lower participation costs c_1. The number of bidders in this type is n_1. Type 2 possesses higher participation costs $c_2 > c_1$. There are n_2 bidders in this type. The total number of bidders is $n = n_1 + n_2$. For simplicity, we only consider the type-symmetric equilibrium by assuming that that bidders with the same participation cost use the same cutoff point. We will consider the type-asymmetric equilibrium next subsection.

Again, we first assume, provisionally, that a monotonic equilibrium exists. By Lemma 1, we have $v_1^* \leq 1$. As discussed in Section 3, for each bidder in type 1, when his valuation is $v_1 = v_1^*$, he is indifferent between participating in the auction and not participating in the auction. Thus we have the zero net-payoff equation

$$c_1 = v_1^* F(v_1^*)^{n_1 - 1} F(v_2^*)^{n_2}. \tag{10}$$

For each bidder in type 2 who has participation cost c_2 and uses v_2^* as his decision point to participate, if $v_2^* > 1$ (i.e., type 2 bidders never participate) then $v_1^* = v_1'$, where v_1' is determined by $c_1 = v_1' F(v_1')^{n_1 - 1}$. For this to be an equilibrium strategy of "$\{No\}$", by the same reason as in last section, we need

$$v_1' F(v_1')^{n_1} + \int_{v_1'}^{1} F(v)^{n_1} dv < c_2;$$

i.e., given the strategy of type 1 bidders, the expected revenue of any type 2 bidder participating in the auction is less than his participation cost even when his value is 1. Then, $v_1^* = v_1'$ for each bidder in type 1 and $v_2^* > 1$ for each bidder in type 2 comprise a monotonic equilibrium.

When a type 2 bidder chooses a cutoff point $v_2^* \leq 1$, and $v_2 = v_2^*$, we have

$$c_2 = v_2^* F(v_1^*)^{n_1} F(v_2^*)^{n_2 - 1} + F(v_2^*)^{n_2 - 1} \int_{v_1^*}^{v_2^*} (v_2^* - v) d(1 - \int_{v}^{1} dF(v))^{n_1},$$

where the first part on the right side is the expected revenue when he is the only bidder in the auction submitting the bid. The second part is the expected revenue when he is the only type 2 bidder submitting a bid, and there is at least one bidder in type 1 submitting a bid. $(1 - \int_v^1 dF(v))^{n_1}$ is the probability that at least one type 1 bidder participates in the auction and bids at most v. Simplifying the equation, we have

$$c_2 = v_2^* F(v_1^*)^{n_1} F(v_2^*)^{n_2 - 1} + F(v_2^*)^{n_2 - 1} \int_{v_1^*}^{v_2^*} (v_2^* - v) dF(v)^{n_1}. \tag{11}$$

Note that, when $n_1 = n_2 = 1$, this reduces to equation (2). Integrating by parts to equation (11), we have

$$c_2 = v_1^* F(v_1^*)^{n_1} F(v_2^*)^{n_2 - 1} + F(v_2^*)^{n_2 - 1} \int_{v_1^*}^{v_2^*} F(v)^{n_1} dv. \tag{12}$$

When a type-symmetric neg-monotonic equilibrium exists, by Lemma 1, we have $v_2^* \leq 1$. For bidders in type 1, if $v_2' F(v_2')^{n_2} + \int_{v_2'}^1 F(v)^{n_2} dv < c_1$, where v_2' is determined by $c_2 = v_2' F(v_2')^{n_2 - 1}$, then we have $v_1^* > 1$. Bidders in type 1 never participate in the auction. Then, in this case, $v_1^* > 1$ for bidders in type 1, and $v_2^* = v_2'$ for bidders in type 2 compose a type-symmetric neg-monotonic equilibrium.

Suppose that $v_2' F(v_2')^{n_2} + \int_{v_2'}^1 F(v)^{n_2} dv < c_1$ is not true. Then $v_1^* \leq 1$. Similar to the previous section, the zero net-payoff condition requires that

$$\begin{aligned} c_2 &= v_2^* F(v_2^*)^{n_2-1} F(v_1^*)^{n_1} \\ c_1 &= v_1^* F(v_1^*)^{n_1-1} F(v_2^*)^{n_2} + F(v_1^*)^{n_1-1} \int_{v_2^*}^{v_1^*} (v_1^* - v) d(F(v))^{n_2}. \end{aligned} \tag{13}$$

Integrating (13) by parts, we have

$$c_1 = v_2^* F(v_2^*)^{n_2} F(v_1^*)^{n_1-1} + F(v_1^*)^{n_1-1} \int_{v_2^*}^{v_1^*} F(v)^{n_2} dv \tag{14}$$

and thus

$$c_1 > v_2^* F(v_2^*)^{n_2} F(v_1^*)^{n_1-1} + F(v_1^*)^{n_1-1} (v_1^* - v_2^*) F(v_2^*)^{n_2} = v_1^* F(v_2^*)^{n_2} F(v_1^*)^{n_1-1}.$$

In order for this to be consistent with $c_1 < c_2$, one necessary condition required is

$$c_2 = v_2^* F(v_2^*)^{n_2-1} F(v_1^*)^{n_1} > v_1^* F(v_2^*)^{n_2} F(v_1^*)^{n_1-1},$$

or $\frac{F(v_2^*)}{v_2^*} < \frac{F(v_1^*)}{v_1^*}$, which is the same as in the case of two bidders.

Consider the following 2 equations:

$$\begin{aligned} c_2 &= y F(y)^{n_2-1} F(x)^{n_1} \\ c_1 &= y F(y)^{n_2} F(x)^{n_1-1} + F(x)^{n_1-1} \int_y^x F(v)^{n_2} dv. \end{aligned}$$

The first equation implicitly defines y as a function of x, denoted by $y(x)$, which has a fixed point v_2^s determined by $c_2 = v_2^s F(v_2^s)^{n_1+n_2-1}$. Then when $x > v_2^s$, we have $y < v_2^s$.

Insert $y(x)$ into the right side of the second equation and let

$$\phi(x) = y(x) F(y(x))^{n_2} F(x)^{n_1-1} + F(x)^{n_1-1} \int_{y(x)}^x F(v)^{n_2} dv.$$

Let c_m be the minimum of $\phi(x) = y(x) F(y(x))^{n_2} F(x)^{n_1-1} + F(x)^{n_1-1} \int_{y(x)}^x F(v)^{n_2} dv$ in the interval $[v_2^s, 1]$.

We then have the following proposition:

Proposition 4 (Existence and Uniqueness Theorem) *For the independent private values economic environment with two types of bidders who have different participation costs $c_2 > c_1$, we have the following conclusions for type-symmetric equilibria:*

(1) There is always a type-symmetric monotonic equilibrium.

(2) Suppose $F(v)$ is concave. Then, the type-symmetric equilibrium is unique.

(3) Suppose $F(v)$ is strictly convex. Then,

(3.i) the type-symmetric monotonic equilibrium is unique if $\frac{f(v)}{F^2(v)}$ and $\frac{F(v)}{vf(v)}$ are non-increasing,

(3.ii) the type-symmetric neg-monotonic equilibrium is unique when $c_1 = c_m$,

(3.iii) there is no type-symmetric neg-monotonic equilibrium when $c_1 < c_m$ and

(3.iv) there are at least two type-symmetric neg-monotonic equilibria when $c_m < c_1 < c_2$.

As in the case of two bidders, when $c_2 - c_1 \to 0$, we have similar convergence results: When $F(v)$ is concave, the unique type-symmetric monotonic equilibrium (there is no type-symmetric neg-monotonic equilibrium) converges to the unique type-symmetric equilibrium as $c_2 - c_1 \to 0$. When $F(v)$ is strictly convex, $\frac{f(v)}{F(v)^2}$ and $\frac{F(v)}{vf(v)}$ are non-increasing, the unique type-symmetric monotonic equilibrium converges to a type asymmetric equilibrium as $c_2 - c_1 \to 0$. When $F(v)$ is strictly convex, there are two type-symmetric neg-monotonic equilibria such that one converges to the unique type-symmetric equilibrium, and the other converges to a type-asymmetric equilibrium as $c_2 - c_1 \to 0$.

We can similarly examine the effects of changes in costs and in numbers of bidders on equilibrium behavior. For simplicity, we only consider the case of the uniform distribution function. As in the last section, the comparative analysis can be obtained for general distribution functions.

From (10) and (11), we have

$$\begin{aligned} c_1 &= v_1^{*n_1} v_2^{*n_2} &(15)\\ c_2 &= v_1^{*n_1+1} v_2^{*n_2-1} + v_2^{*n_2-1} \int_{v_1^*}^{v_2^*} v^{n_1} dv &(16) \end{aligned}$$

which gives us

$$c_2 = \frac{n_1}{n_1+1} v_1^{*n_1+1} v_2^{*n_2-1} + \frac{1}{n_1+1} v_2^{*n_1+n_2}. \quad (17)$$

(15) and (17) provide the conditions that should be satisfied simultaneously for the equilibrium cutoff points.

We then have the following proposition:

Proposition 5 *Suppose the values of bidders are drawn from a uniform distribution function $F(v)$ and the participation costs c_1 and c_2 are publicly known information. Then we have*

1) an increase in participation cost c_i increases i's cutoff point v_i^*, but decreases his opponents' cutoff point v_j^* for $i, j = 1, 2, j \neq i$.

2) the cutoff points for both types of bidders increase when the number of any type of bidder increases.

Proposition 5.(1) gives us results similar to those in previous section. A bidder's cutoff point is an increasing function of his own participation cost and a decreasing function of others' participation costs. Proposition 5.(2) shows that more bidders in the auction will increase the competitiveness among the potential bidders, and this will reduce the possible payoff to each bidder. Thus, bidders will be less likely to participate in the auction, and their value cutoff points will increase.

4.2 Type Asymmetric Equilibria

In this subsection we give a brief discussion on allowing asymmetric cutoff points within a group. To allow such a possibility, we consider the simplest economy with three bidders in the two groups. The first two bidders' participation costs are the same so that $c_1 = c_2$, and the third bidder's participation cost is c_3. For simplicity, we assume that distribution functions are the same for all bidders. Let v_1^* and v_2^* be the corresponding cutoff points for the two bidders in type 1 and v_3^* be the cutoff point for type 2 bidder. We assume $c_1 = c_2 < c_3$ and $v_1^* < v_2^*$. There are three cases to be considered.

Case 1: $v_1^* < v_2^* < v_3^*$. Then we have

$$\begin{aligned} c_1 &= v_1^* F(v_2^*) F(v_3^*), \\ c_2 &\geq v_2^* F(v_1^*) F(v_3^*) + F(v_3^*) \int_{v_1^*}^{v_2^*} (v_2^* - v) dF(v), \\ c_3 &\geq v_3^* F(v_1^*) F(v_2^*) + F(v_2^*) \int_{v_1^*}^{v_2^*} (v_3^* - v) dF(v) + \int_{v_2^*}^{v_3^*} (v_3^* - v) dF(v)^2. \end{aligned}$$

The above equations hold with equality when $v_i^* \leq 1$. On the right side of the third equation, the first part is the revenue bidder 3 receives when the other two bidders do not participate in the auction. The second part is the revenue he receives when the highest bid of the other two is less than v_2^*. This happens when bidder 2 does not participate in the auction. The third part is the revenue when the others' highest bid is greater than v_2^* and less than v_3^*.

When $F(v)$ is concave, we cannot have such an equilibrium. To see this, from the first two equations, we have $v_1^* F(v_2^*)F(v_3^*) > v_2^* F(v_1^*)F(v_3^*)$; i.e., we have $\frac{F(v_2^*)}{v_2^*} > \frac{F(v_1^*)}{v_1^*}$ with $v_2^* > v_1^*$, which cannot be true when $F(v)$ is concave.

When $F(v)$ is strictly convex, from the first two equations, we treat v_3^* as a constant. Then it seems as if bidder 1 and bidder 2 possess participation costs $\frac{c_1}{F(v_3^*)}$. We know there is an equilibrium in which $v_1^* < v_2^*$ and the equilibrium is a function of v_3^*. Inserting into the third equation, we can get v_3^*. In particular, when $v_3^* > 1$, bidder 3 never participates in the auction.

Case 2: $v_1^* < v_3^* < v_2^*$. Then we have

$$\begin{aligned}
c_1 &= v_1^* F(v_2^*)F(v_3^*), \\
c_3 &\geq v_3^* F(v_1^*)F(v_2^*) + F(v_2^*)\int_{v_1^*}^{v_3^*}(v_3^* - v)dF(v), \\
c_2 &\geq v_2^* F(v_1^*)F(v_3^*) + F(v_3^*)\int_{v_1^*}^{v_3^*}(v_2^* - v)dF(v) + \int_{v_3^*}^{v_2^*}(v_2^* - v)dF(v)^2.
\end{aligned}$$

When $F(v)$ is concave, from the first and third equation above, we have $v_1^* F(v_2^*)F(v_3^*) > v_2^* F(v_1^*)F(v_3^*)$, which again cannot be true for $v_1^* < v_2^*$. So $v_1^* = v_3^*$. The problem can be reduced to the type symmetric equilibrium. When $F(v)$ is strictly convex, we can treat v_1^* in the second and third equation as constant. From the discussion in Section 3, we know that when $c_3 - c_1$ is sufficiently small, there exists an equilibrium in which $v_2^* < v_3^*$. A limiting case is when $c_3 = c_1$. As Tan and Yilankaya (2006) point out, when $F(v)$ is strictly convex but not log-concave, there may exist equilibria with three or more cutoff points.

Case 3: $v_3^* < v_1^* < v_2^*$. The discussion for this is similar to Case 2.

Summarizing our discussion above and the results we obtain in Sections 3 and 4, we have the following proposition:

Proposition 6 *For the independent private values economy with two groups and three bidders, when $F(v)$ is concave, we only have the unique type-symmetric monotonic equilibrium. When $F(v)$ is strictly convex, type-asymmetric equilibria exist.*

4.3 Bidders with Different Valuation Distributions

We consider an economy where bidders have different valuation distributions $F_1(v)$ and $F_2(v)$. Here $F_i(v)$ is the probability that bidder i's valuation is less than or equal to v, and $i = 1, 2$. Tan and Yilankaya (2006) considered a similar economic environment where there are two groups of bidders with different valuation distributions but the same participation costs. Miralles (2006) studied equilibrium behavior when bidders' valuation distributions can be ordered in a first order stochastic dominance ranking but still retain the same participation cost.

Here, we allow both valuation distribution functions and participation costs of bidders to be different. Again, we assume $c_1 < c_2$ and use x and y to refer the cutoff points used by bidder 1 and 2, respectively. We want to investigate the existence of equilibria and equilibria behavior.

To find a monotonic equilibrium, we need to consider the following two equations:

$$c_1 = xF_2(y)$$

$$c_2 \geq xF_1(x) + \int_x^y F_1(v)dv.$$

Again, the first equation implicitly defines x as a decreasing function of y, denoted by $x(y)$. We then have $\frac{dx}{dy} = -\frac{xf_2(y)}{F_2(y)}$. Also we know $x(y)$ has a fixed point $v_1^s \neq 0$ determined by $c_1 = v_1^s F_2(v_1^s)$. Since $x(y)$ is monotonically decreasing, we have $x < v_1^s$ and $y > v_1^s$.

Inserting $x(y)$ into the second equation and letting $\lambda(y) = xF_1(x) + \int_x^y F_1(v)dv$ with $x < y$, we have

$$\lambda'(y) = F_1(y) + xf_1(x)\frac{dx}{dy} = \frac{F_1(y)F_2(y) - x^2 f_1(x)f_2(y)}{F_2(y)}.$$

When $F_1(v)$ and $F_2(v)$ are both concave, we have

$$\lambda'(y) > \frac{F_1(y)F_2(y) - xyf_1(x)f_2(y)}{F_2(y)} > \frac{F_1(y)F_2(y) - F_1(x)F_2(y)}{F_2(y)} > 0,$$

which indicates that $\lambda(y)$ is a monotonically increasing function.

For the existence of neg-monotonic equilibrium, we consider the following two equations:

$$c_2 = yF_1(x)$$

$$c_1 \geq yF_2(y) + \int_y^x F_2(v)dv.$$

From the first equation we have $y = \frac{c_2}{F_1(x)}$. Inserting it into the right side of the second equation and letting $\phi(x) = yF_2(y) + \int_y^x F_2(v)dv$ with $x \geq y$, by the same reason as before, we have $\phi'(x) > 0$ when both $F_1(v), F_2(v)$ are concave. $y = \frac{c_2}{F_1(x)}$ also has a fixed point v_2^s determined by $c_2 = v_2^s F_1(v_2^s)$. Since $x(y)$ is monotonically decreasing, we have $y < v_2^s$ and $x > v_2^s$.

We then have the following proposition:

Proposition 7 (Existence and Uniqueness Theorem) For a two-bidder economy with different continuously differentiable distribution functions $F_1(v)$ and $F_2(v)$ and different costs $c_1 < c_2$, we have the following results:

(1) There always exists an equilibrium (v_1^*, v_2^*).

(2) Suppose $F_1(v)$ and $F_2(v)$ are both concave and $F_1(v) < F_2(v)$ for all $v \in (0,1)$. Then there exists a unique equilibrium that is monotonic.

(3) Suppose $F_1(v)$ and $F_2(v)$ are both concave and $F_1(v) > F_2(v)$ for all $v \in (0,1)$. Let v_1^s and v_2^s satisfy $c_1 = v_1^s F_2(v_1^s)$ and $c_2 = v_2^s F_1(v_2^s)$, respectively. Then, we have

i) If $v_1^s < v_2^s$, there is a unique equilibrium that is monotonic;

ii) If $v_1^s > v_2^s$, there is a unique equilibrium that is neg-monotonic, satisfying $v_1^* > v_2^*$;

iii) If $v_1^s = v_2^s = v^s$, there is a unique equilibrium that is a special neg-monotonic equilibrium, satisfying $v_1^* = v_2^* = v^s$.

Remark 6 $F_1(v) < F_2(v)$ for all $v \in [0,1]$ means that bidder 1 is a strong bidder in the sense that there is a high probability that his valuation is higher than bidder 2's valuation. A higher valuation together with a smaller participation cost makes bidder 1 more likely to participate in the auction; i.e., he is more likely to choose a lower cutoff point. However when $F_1(v) > F_2(v)$ for all $v \in (0,1)$, bidders with higher participation costs may have lower or identical cutoff points even though their participation costs are higher.

When $F_1(v) > F_2(v)$ for all $v \in (0,1)$, then, for each given value v, the probability that bidder 2 does not participate in the auction is less than that of bidder 1. Thus, bidder 2 has an advantage in winning the bid and a disadvantage in the participation cost. When the advantage can overbid the disadvantage, bidder 2 has a lower cutoff point, rather than a higher one, resulting in the nonexistence of a monotonic equilibrium. We can also interpret this in another way. $F_1(v) > F_2(v)$ implies that $F_1(v)$ is more concave than $F_2(v)$ and that bidder 1 is more risk averse than bidder 2. This reduces his entrance probability by leading him to choose a higher cutoff point.

Remark 7 Unlike the results obtained in Section 3, (3.iii) shows that when bidders' distribution functions are different, $v_1^* = v_2^*$ can be an equilibrium although bidders' participation costs are different. That is, we have a special neg-monotonic equilibrium with $v_1^* = v_2^*$ even when $c_1 < c_2$. When bidders' distribution functions are the same, as in Section 3, this is impossible.

Thus, when bidders have different distributions on valuations, some of the previous results no longer hold true. The distributions of valuations have substantial effects on types of equilibria.

4.4 Positive Lower Bound of Supports

The support of valuations also affects the existence of equilibria. When the lower bound of the support of the valuation is not zero, there may be an equilibrium in which one bidder always participates in the auction and the other never participates in the auction.

Suppose the support of the distribution function $F(v)$ is $[v_l, v_h]$. There are six cases for consideration in studying equilibrium behavior of bidders:

Case 1. $v_h < c_1 < c_2$. It is clear both bidders never participate in the auction.

Case 2. $v_l < c_1 < v_h < c_2$. Bidder 2 never participates in the auction. Bidder 1 participates in the auction if $v_1 \geq c_1$ and does not otherwise.

Case 3. $c_1 < v_l < v_h < c_2$. Bidder 2 never participates, and bidder 1 always participates.

Case 4. $v_l < c_1 < c_2 < v_h$. The analysis and results are the same as those in Section 3 that deals with the special case where $v_l = 0$ and $v_h = 1$.

Case 5. $c_1 < v_l < c_2 < v_h$. We can have the following equilibrium: Bidder 1 always enters, and bidder 2 never enters. For this to be an equilibrium, we need $v_h - v_l < c_2$; that is, the maximum revenue bidder 2 gets from participating in the auction must be smaller than his participation costs. When $c_2 \leq v_h - v_l$, bidder 2 will choose a cutoff point $v_2^* \in [c_2, v_h]$. If there is an equilibrium in which bidder 1 never participates, then bidder 2 uses $v_2^* = c_2$. To have such an equilibrium, we need

$$v_h F(c_2) + \int_{c_2}^{v_h} (v_h - v) dF(v) = c_2 F(c_2) + \int_{c_2}^{v_h} F(v) dv < c_1.$$

A sufficient condition for this is $v_h + c_2 F(c_2) < c_1 + c_2$.

Case 6. $c_1 < c_2 < v_l < v_h$. We can have the following equilibrium: Bidder 1 always participates in the auction, and bidder 2 never participates in the auction. For this to be an equilibrium, we need $v_h - v_l < c_2$. Bidder 2 always participates in the auction, and bidder 1 never participates in the auction. For this to be an equilibrium, we need $v_h - v_l < c_1$. When both bidders choose a cutoff point inside the support of valuations, we can use the same analysis as in Section 3 to investigate the equilibrium behavior.

5 Conclusion

This paper investigates equilibria of second price auctions when bidders have private valuations and different participation costs that are common knowledge. We identify two types of equilibria: monotonic and neg-monotonic equilibria. We show that there always exists an equilibrium that is monotonic, and further that, it is unique when $F(v)$ is concave or when $F(v)$ is strictly convex with additional restrictions.

We also consider the existence of neg-monotonic equilibria. We show that when the distribution function of valuation is strictly convex and when the difference of participation costs is sufficiently small, there is a neg-monotonic equilibrium. One policy implication is that one may solve multiple equilibria by eliminating neg-monotonic equilibria through differentiating participation costs significantly. We also show that when the difference in participation costs goes to zero, the monotonic equilibria of concave valuation distribution converge to the symmetric equilibrium, while the monotonic equilibrium of convex valuation distributions converges to the asymmetric equilibrium. This is contradictory to our common intuition.

We provide some comparative static analysis. we show that the cutoff point is increasing in one's own participation cost but is decreasing in the opponents' participation costs. We also show that as the number of bidders increases, the cutoff points for all bidders will increase. This is consistent with the idea that more potential bidders will increase competition among bidders and will thus reduce the expected payoff of each buyer, with the natural consequence of reduced buyer participation.

We also consider some extensions of our basic model. We discuss equilibrium behavior for the economic environment with two types of bidders, and get similar results. However, when bidders are allowed to have different valuation distribution functions, some of the results for the basic model are no longer true. We also extend the basic model to the one with a positive lower bound of the support. In this case, we may have an equilibrium in which some bidders always enter the auction.

Appendix: Proofs

Proof of Lemma 1:

Suppose not. All bidders never participate in the auction (i.e., $v_i^* > 1$ for all bidders i). When bidder n knows the other $n-1$ bidders will not participate in the auction regardless of their valuations, then bidder n participates in the auction when his value is greater than or equal to his participation cost. Then we have $v_n^* = c_n \leq 1$, a contradiction.

Proof of Lemma 2:

First note that $v_1^s < v_2^s$ by monotonicity of $vF(v)$. When bidder 2 chooses never to participate, then $v_1^* = c_1 < v_1^s$ and $v_2^* > 1$. The above lemma holds obviously.

Now suppose $v_2^* \leq 1$. We have

$$v_1^* F(v_1^*) + \int_{v_1^*}^{v_2^*} F(v)dv = c_2 = v_2^s F(v_2^s).$$

Since $v_1^* F(v_1^*) + \int_{v_1^*}^{v_2^*} F(v)dv = v_2^* F(v_2^*) - \int_{v_1^*}^{v_2^*} vf(v)dv$, we have

$$v_2^* F(v_2^*) - \int_{v_1^*}^{v_2^*} vf(v)dv = v_2^s F(v_2^s).$$

Then $v_2^s F(v_2^s) < v_2^* F(v_2^*)$. We must have $v_2^s < v_2^*$ by the monotonicity of $vf(v)$. Also, since we have $v_2^* > v_1^s$ and $c_1 = v_1^* F(v_2^*) = v_1^s F(v_1^s)$, for this equation to be true, we must have $v_1^* < v_1^s$. Otherwise we have $v_1^* F(v_2^*) > v_1^s F(v_1^s)$, a contradiction. So $v_1^* < v_1^s$. Thus, we prove $v_2^* > v_2^s > v_1^s > v_1^*$.

Proof of Proposition 1:

The proof of Proposition 1 is based on the following fives lemmas (from Lemma 3 to Lemma 7).

Lemma 3 *For the economic environment with two bidders, there always exists an equilibrium that is monotonic; i.e., for $c_2 > c_1$, there exists a cutoff point vector (v_1^*, v_2^*) such that $v_2^* > v_1^*$.*

Proof. When $c_1 F(c_1) + \int_{c_1}^{1} F(v)dv < c_2$, as we discussed above, bidder 2 will never participate in the auction and thus $v_1^* = c_1$ and $v_2^* > 1$ constitute a monotonic equilibrium. Now we consider the case of $c_1 F(c_1) + \int_{c_1}^{1} F(v)dv \geq c_2$.

Given that the point v_1^s determined by $c_1 = v_1^s F(v_1^s)$, we have $x < v_1^s$ and $y > v_1^s$ by noting that $y = y(x)$ is a decreasing function. Since $h(c_1) = c_1 F(c_1) + \int_{c_1}^{1} F(v)dv - c_2 \geq 0$ and $h(v_1^s) = c_1 - c_2 < 0$, there exists a $v_1^* \in [c_1, v_1^s)$ such that $h(v_1^*) = 0$. Thus, $v_1^* < v_1^s$ and $v_2^* = y(v_1^*) > v_1^s$ constitute a monotonic equilibrium. ■

Lemma 4 *If $F(v)$ is concave, there is a unique monotonic equilibrium.*

Proof. Since $F(v)$ is concave, we have $F(v) \geq vF'(v) = vf(v)$ for any point $v \in [0,1]$, and by noting $y > x$, we have

$$\lambda'(y) = F(y) - \frac{x^2}{F(y)} f(y)f(x) > F(y) - \frac{F(x)xf(y)}{F(y)} > F(y) - \frac{F(x)yf(y)}{F(y)} \geq F(y) - F(x) > 0,$$

which indicates that $\lambda(y)$ is monotonically increasing. First consider the case where $\lambda(1) = c_1F(c_1) + \int_{c_1}^1 F(v)dv \geq c_2$. Since $\lambda(v_1^s) - c_2 = c_1 - c_2 < 0$, then, by monotonicity and continuity of λ and $x(y)$, $y = v_2^* \in (v_1^s, 1]$ is uniquely determined by $\lambda(y) - c_2 = 0$, as is $x = v_1^* < v_1^s$. Thus, the monotonic equilibrium is unique. Now suppose $\lambda(1) < c_2$. Then bidder 2 will never participate in the auction; thus $x = v_1^* = c_1$ and $v_2^* > 1$ will again be the unique monotonic equilibrium. ∎

Lemma 5 *If $F(v)$ is concave, there is no neg-monotonic equilibrium, and thus the equilibrium is unique and monotonic.*

Proof. We first prove there is no neg-monotonic equilibrium in which $v_1^* > 1$. To see this, notice that $v_1^* > 1$ requires $c_1 > c_2F(c_2) + \int_{c_2}^1 F(v)dv$. However when $F(v)$ is concave, we have

$$c_1 > c_2F(c_2) + \int_{c_2}^1 F(v)dv \geq c_2F(c_2) + (1-c_2)F(c_2) = F(c_2) \geq c_2$$

by noting that $F(c_2) \geq c_2$ since $F(c) = F(c \times 1 + (1-c)0) \geq cF(1) + (1-c)F(0) = c$. But this contradicts the fact that $c_1 < c_2$.

We now show that there does not exist any neg-monotonic equilibrium with $v_1^* \leq 1$ either. Suppose not. We then have $v_2^* < v_1^*$ and $\frac{F(v_1^*)}{v_1^*} > \frac{F(v_2^*)}{v_2^*}$, which contradicts the fact that $\frac{F(v)}{v}$ is a non-increasing function when $F(v)$ is a concave function. Thus, there does not exist any neg-monotonic equilibrium in either case. Consequently, by Lemma 4, the equilibrium is unique, which is monotonic. ∎

Lemma 6 *Suppose $F(v)$ is strictly convex and $\frac{f(v)}{F(v)^2}$ is non-increasing. Then, there is a unique monotonic equilibrium.*

Proof. Notice that $\lambda'(y)$ can be written as

$$\lambda'(y) = F(y) - \frac{x^2}{F(y)} f(y)f(x) = F(y)[1 - \frac{x^2 f(x)f(y)}{F(y)^2}].$$

Since $\frac{x^2 f(x)f(y)}{F(y)^2}$ is a decreasing function in y by noting $f(v)$ is an increasing function by strict convexity of $F(v)$ and $x = \frac{c_1}{F(y)}$, $1 - \frac{x^2 f(x)f(y)}{F(y)^2}$ is an increasing function in y, as is $\lambda'(y)$. Thus, there is at most one $y = y_0$, if any, satisfying $\lambda'(y_0) = 0$. Also, notice that, when $x = y = v_1^s$,

$$\lambda'(v_1^s) = F(v_1^s) - \frac{v_1^{s2}}{F(v_1^s)} f(v_1^s) f(v_1^s) < 0$$

by the strict convexity of $F(v)$. Then $\lambda(y)$ either decreases over the entire interval $[v_1^s, 1]$ (in this case $y_0 > 1$) or decreases first over $[v_1^s, y_0]$ and then increases over $[y_0, 1]$ if $y_0 \leq 1$. If $\lambda(y)$ decreases over the entire interval $[v_1^s, 1]$, then $\lambda(y) < c_2$ for all $y \in [v_1^s, 1]$, which means bidder 2 never participates in the auction. Thus we have a unique monotonic equilibrium with $v_1^* = c_1$ and $v_2^* > 1$. On the other hand, if $y_0 \leq 1$, $\lambda(y)$ decreases first over $[v_1^s, y_0]$ and then increases over $[y_0, 1]$. Thus $\lambda(y) = c_2 > c_1$ has at most one solution v_2^*. If the solution exists, we have a unique monotonic equilibrium with $v_1^* \leq 1$ and $v_2^* \leq 1$; otherwise the unique monotonic equilibrium is given by $v_1^* = c_1$ and $v_2^* > 1$. ■

Lemma 7 *Suppose $F(v)$ is strictly convex. There exists a neg-monotonic equilibrium when $c_1 = c_m$ and at least two neg-monotonic equilibria when $c_1 > c_1$. There is no neg-monotonic equilibrium when $c_1 < c_m$.*

Proof. Since

$$\phi'(x) = F(x) + y(x) f(y(x)) y'(x)$$

and

$$y'(x) = -\frac{y f(x)}{F(x)},$$

we have

$$\phi'(v_2^s) = F(v_2^s) - v_2^s f(v_2^s) \frac{v_2^s f(v_2^s)}{F(v_2^s)} = \frac{F^2(v_2^s) - (v_2^s f(v_2^s))^2}{F(v_2^s)} < 0$$

by noting that $v_2^s f(v_2^s) > F(v_2^s)$ by $F(v) < v f(v)$ for all $v \in [c_2, 1]$ and $v_2^s \geq c_2$, which indicates that $\phi(x)$ is decreasing at $x = x_2^s$. Then $\phi(x)$ has a minimum value $c_m < c_2$ in the interval $[v_2^s, 1]$ since $\phi(v_2^s) = c_2$. Let $\phi(x_m) = c_m$.

When $c_1 < c_m$, we have $\phi(x) > c_1$ in the interval $[v_2^s, 1]$. Thus, there is no neg-monotonic equilibrium with $v_1^* \leq 1$ since the set $\{x | \phi(x) = c_1, v_2^s \leq x \leq 1\}$ is empty. On the other hand, since $\phi(1) = c_2 F(c_2) + \int_{c_2}^1 F(v) dv \geq c_m > c_1$, we do not have a neg-monotonic equilibrium at which bidder 1 never participates so that $v_1^* > 1$ is not an equilibrium strategy for bidder 1.

When $c_1 = c_m$, since $\phi(x_m) = c_m$, then $x = x_m, y = c_2 / F(x_m)$ is the unique neg-monotonic equilibrium. Note that when $c_1 = c_m$ we do not have an equilibrium in which bidder 1 never participates since $\phi(1) \geq c_m = c_1$.

When $c_m < c_1 < c_2$, we have at least two neg-monotonic equilibria. To see this, first notice that there exists an $x_1 \in (v_2^s, x_m)$ such that $\phi(x_1) = c_1$ by the continuity of $\phi(x)$ and

$\phi(x_m) = c_m < c_1$, $\phi(v_2^s) = c_2 > c_1$. If $\phi(1) < c_1$, we have a neg-monotonic equilibrium at which bidder 1 never participates and bidder 2's equilibrium strategy is $v_2^* = c_2$. Otherwise if we have $\phi(1) \geq c_1$, we can also find an $x_2 \in (x_m, 1]$ such that $\phi(x_2) = c_1$ by the continuity of $\phi(x)$ on $x_2 \in (x_m, 1]$, $\phi(1) > c_1$ and $\phi(x_m) = c_m < c_1$. Then $(x_1, c_2/F(x_1))$ and $(x_2, c_2/F(x_2))$ will be two neg-monotonic equilibria. ■

Proof of Proposition 2:

2.(1): When $F(v)$ is concave, the monotonic equilibrium and symmetric equilibrium are both unique, so we have the result.

2.(2) From the proof of Lemma 6, we know that, when $F(v)$ is strictly convex and $\frac{f(v)}{F(v)^2}$ is a non-increasing function of v, there is at most one y_0 such that $\lambda'(y_0) = 0$; $\lambda(y)$ either decreases over the entire interval $[v_1^s, 1]$ or decreases first over $[v_1^s, y_0]$ and then increases over $[y_0, 1]$ if $y_0 \leq 1$. Thus, there is a unique monotonic equilibrium, which is either given by $v_1^* = c_1$ and $v_2^* > 1$ when $\lambda(y)$ and c_2 have no intersection, or given by (v_1^*, v_2^*) with $v_1^* < v_1^s < y_0 < v_2^* \leq 1$ when $\lambda(y)$ and c_2 have an intersection. Here v_2^* is determined by $\lambda(v_2^*) = c_2$ and $v_1^* = c_1/F(v_2^*)$. Thus, from Figure 1, one can see that, when $c_2 \to c_1$, we have an equilibrium given by an asymmetric equilibrium $(v_1^{*\prime}, v_2^{*\prime})$ with $v_1^{*\prime} < v_1^s < y_0 < v_2^{*\prime} \leq 1$, where $v_2^{*\prime}$ is determined by $\lambda(v_2^{*\prime}) = c_1$ and $v_1^{*\prime} = c_1/F(v_2^{*\prime})$. So the unique monotonic equilibrium converges to an asymmetric equilibrium.

2.(3) When $F(v)$ is strictly convex and $c_2 - c_1$ is sufficiently small, there are two neg-monotonic equilibria (x_1, y_1) and (x_2, y_2) with $y_1 = y(x_1)$, $y_2 = y(x_2)$, and $y_1 < y_2 < v_2^s < x_1 < x_m < x_2 < x_0$ as we showed in Lemma 7. Thus, from Figure 2, as $c_1 \to c_2$, the neg-monotonic equilibrium (x_1, y_1) converges to the symmetric equilibrium (v_2^s, v_2^s), and the other neg-monotonic equilibrium (x_2, y_2) converges to the asymmetric equilibrium (x_0, y_0).

Proof of Proposition 3:

First consider a change in c_1. Taking derivatives with respect to c_1 on both sides of (1) and (3), we have

$$v_1^* f(v_2^*) \frac{\partial v_2^*}{\partial c_1} + F(v_2^*) \frac{\partial v_1^*}{\partial c_1} = 1,$$

$$F(v_2^*) \frac{\partial v_2^*}{\partial c_1} + v_1^* f(v_1^*) \frac{\partial v_1^*}{\partial c_1} = 0.$$

Solving for $\frac{\partial v_1^*}{\partial c_1}$ and $\frac{\partial v_2^*}{\partial c_1}$, we have

$$\frac{\partial v_1^*}{\partial c_1} = \frac{F(v_2^*)}{F(v_2^*)^2 - v_1^{*2} f(v_1^*) f(v_2^*)},$$

$$\frac{\partial v_2^*}{\partial c_1} = -\frac{v_1^* f(v_1^*)}{F(v_2^*)^2 - v_1^{*2} f(v_1^*) f(v_2^*)}.$$

Note that, since $v_1^* < v_2^*$ and $F(v)$ is concave, we have

$$F(v_2^*)^2 - v_1^{*2} f(v_1^*) f(v_2^*) > F(v_2^*)^2 - v_1^* v_2^* f(v_1^*) f(v_2^*) > 0,$$

and thus we have $\frac{\partial v_1^*}{\partial c_1} > 0$ and $\frac{\partial v_2^*}{\partial c_1} < 0$.

We now consider the change in c_2. Taking derivatives with respect to c_2 on both sides of (1) and (3) and solving for $\frac{\partial v_1^*}{\partial c_2}$, $\frac{\partial v_2^*}{\partial c_2}$, we have

$$\frac{\partial v_1^*}{\partial c_2} = -\frac{v_1^* f(v_2^*)}{F(v_2^*)^2 - v_1^{*2} f(v_1^*) f(v_2^*)} < 0,$$

$$\frac{\partial v_2^*}{\partial c_2} = \frac{F(v_2^*)}{F(v_2^*)^2 - v_1^{*2} f(v_1^*) f(v_2^*)} > 0$$

by noting that $F(v_2^*)^2 - v_1^{*2} f(v_1^*) f(v_2^*) > 0$.

Proof of Proposition 4:

The proof of Proposition 4 consists of the following lemmas:

Lemma 8 *For the economic environment with with two types of bidders, there always exists a type-symmetric equilibrium that is monotonic; i.e., for $c_2 > c_1$, there exists a cutoff point vector (v_1^*, v_2^*) such that $v_2^* > v_1^*$.*

Proof. Consider the following cutoff point reaction equations

$$c_1 = xF(x)^{n_1 - 1} F(y)^{n_2} \tag{18}$$

$$c_2 = xF(x)^{n_1} F(y)^{n_2 - 1} + F(y)^{n_2 - 1} \int_x^y F(v)^{n_1} dv \tag{19}$$

with $x < y$. From (18),

$$\frac{dx}{dy} = -\frac{n_2 x f(y) F(x)}{F(y)[F(x) + (n_1 - 1) x f(x)]} < 0,$$

which indicates that x is a decreasing function of y.

Given v_1' determined by $c_1 = v_1' F(v_1')^{n_1}$, when $v_1' F(v_1')^{n_1} + \int_{v_1'}^1 F(v)^{n_1} dv < c_2$, bidder 2 will never participate in the auction and thus $v_1^* = c_1$ and $v_2^* > 1$ constitute a monotonic equilibrium. So we only need to consider the case of $v_2^* \leq 1$.

From (18), given v_1^s determined by $c_1 = v_1^s F(v_1^s)^{n_1 - 1} F(v_1^s)^{n_2}$, we have $x < v_1^s$ and $y > v_1^s$ by noting that $y = y(x)$ is a decreasing function. Also , by definition, we have $v_1' < v_1^s$.

Let

$$h(x) = xF(x)^{n_1} F(y(x))^{n_2 - 1} + F(y(x))^{n_2 - 1} \int_x^{y(x)} F(v)^{n_1} dv - c_2.$$

Since $h(v_1') = v_1' F(v_1')^{n_1} + \int_{v_1'}^1 F(v)^{n_1} dv - c_2 \geq 0$ and $h(v_1^s) = c_1 - c_2 < 0$, there exists a $v_1^* \in [v_1', v_1^s)$ such that $h(v_1^*) = 0$. Thus, $v_1^* < v_1^s$ and $v_2^* = y(v_1^*) > v_1^s$ constitute a monotonic equilibrium. ■

Lemma 9 *When $F(v)$ is a concave distribution function, there is a unique type-symmetric monotonic equilibrium.*

Proof. Let

$$\begin{aligned}\lambda(y) &= xF(x)^{n_1} F(y)^{n_2-1} + F(y)^{n_2-1} \int_x^y F(v)^{n_1} dv \\ &= F(y)^{n_2-1}(xF(x)^{n_1} + \int_x^y F(v)^{n_1} dv).\end{aligned}$$

We have

$$\begin{aligned}\lambda'(y) &= (n_2-1)F(y)^{n_2-2} f(y)(xF(x)^{n_1} + \int_x^y F(v)^{n_1} dv) \\ &\quad + F(y)^{n_2-1}[F(y)^{n_1} + n_1 x f(x) F(x)^{n_1-1} \frac{dx}{dy}], \\ &= F(y)^{n_2-2}[(n_2-1)f(y) \int_x^y F(v)^{n_1} dv + F(y)^{n_1+1} \\ &\quad + (n_2-1)f(y)xF(x)^{n_1} + n_1 F(y) x F(x)^{n_1-1} f(x) \frac{dx}{dy}].\end{aligned}$$

Inserting $\frac{dx}{dy}$ into $\lambda'(y)$ and rearranging the terms, we have

$$\begin{aligned}\lambda'(y) &= F(y)^{n_2-2}\{(n_2-1)f(y) \int_x^y F(v)^{n_1} dv + F(y)^{n_1+1} \\ &\quad - xf(y)F(x)^{n_1}[\frac{n_1 n_2}{(n_1-1) + \frac{F(x)}{xf(x)}} - (n_2-1)]\} \\ &= F(y)^{n_2-2} f(y)\{(n_2-1) \int_x^y F(v)^{n_1} dv + \frac{F(y)^{n_1+1}}{f(y)} \\ &\quad - xF(x)^{n_1}[\frac{n_1 n_2}{(n_1-1) + \frac{F(x)}{xf(x)}} - (n_2-1)]\}.\end{aligned}$$

For (18), when $y = x = v_1^s$ determined by $c_1 = v_1^s F(v_1^s)^{n_1-1} F(v_1^s)^{n_2}$, we have

$$\lambda'(v_1^s) = F(v_1^s)^{n_2-2}\{F(v_1^s)^{n_1+1} - v_1^s f(v_1^s) F(v_1^s)^{n_1}[\frac{n_1 n_2}{(n_1-1) + \frac{F(v_1^s)}{v_1^s f(v_1^s)}} - (n_2-1)]\}.$$

When $F(v)$ is a concave distribution function, we have $xf(x) \leq F(x)$ and $\frac{n_1 n_2}{(n_1-1)+\frac{F(x)}{xf(x)}} - (n_2-1) < 1$ for all x. Thus, for $y > x$

$$\lambda'(y) > F(y)^{n_2-2}[(n_2-1)f(y) \int_x^y F(v)^{n_1} dv + F(y)^{n_1+1} - yf(y)F(y)^{n_1}] > 0.$$

So $\lambda(y)$ is an increasing function of y when $y > v_1^s$. Then $y = v_2^* > v_1^s$ can be uniquely determined by $\lambda(y) = c_2$. This together with $v_2^* = x(v_2^*) < v_1^s$ constitutes a monotonic equilibrium. If for all $y \in (v_1^s, 1]$ we have $\lambda(y) < c_2$, then bidder 2 will never participate; i.e., $v_2^* > 1$ and $v_2^* < 1$ as determined by $c_1 = v_1^* F(v_1^*)^{n_1 - 1}$ compose a unique monotonic equilibrium. In either case we only have one monotonic equilibrium. ■

Lemma 10 *When $F(v)$ is concave, there is no type-symmetric neg-monotonic equilibrium.*

Proof. We only need to show that there is no neg-monotonic equilibrium in which $v_1^* > 1$. The case where there is no neg-monotonic equilibrium with $v_1^* \leq 1$ is the same as in Lemma 5. Suppose not. We then have

$$c_1 > v_2' F(v_2')^{n_2} + \int_{v_2'}^{1} F(v)^{n_2} dv \geq F(v_2')^{n_2} = \frac{c_2 F(v_2')}{v_2'} \geq c_2$$

by noting that $F(v_2') \geq v_2'$ since $F(v) = F(v \times 1 + (1 - v)0) \geq vF(1) + (1 - v)F(0) = v$. But this contradicts the fact that $c_1 < c_2$. ■

Lemma 11 *Suppose $F(v)$ is strictly convex. If $\frac{f(v)}{F(v)^2}$ and $\frac{F(v)}{vf(v)}$ are non-increasing functions, then there is a unique type-symmetric monotonic equilibrium.*

Proof. When $F(v)$ is strictly convex, $f(v)$ is an increasing function and $vf(v) > F(v)$ for all v. Then we have $\lambda'(v_1^s) < 0$. Let

$$\lambda'(y) = F(y)^{n_2 - 2} f(y) \{(n_2 - 1) \int_x^y F(v)^{n_1} dv + \frac{F(y)^{n_1 + 1}}{f(y)} - xF(x)^{n_1} [\frac{n_1 n_2}{(n_1 - 1) + \frac{F(x)}{xf(x)}} - (n_2 - 1)]\} = 0.$$

Then we have

$$F(y)^{n_2 - 2} f(y)(n_2 - 1) \int_x^y F(v)^{n_1} dv + F(y)^{n_1 + n_2 - 1} [1 - \frac{f(y) x F(x)^{n_1}}{F(y)^{n_1 + 1}} (\frac{n_1 n_2}{(n_1 - 1) + \frac{F(x)}{xf(x)}} - (n_2 - 1))] = 0.$$

When $\frac{f(v)}{F(v)^2}$ is non-increasing, then $\frac{f(y)}{F(y)^{n_1 + 1}}$ is decreasing in y. Since $\frac{F(x)}{xf(x)}$ is non-increasing and $x(y)$ is decreasing, $\frac{F(x)}{xf(x)}$ is non-decreasing in y, and thus $xF(x)^{n_1} [\frac{n_1 n_2}{(n_1 - 1) + \frac{F(x)}{xf(x)}} - (n_2 - 1)]$ is decreasing in y.

Thus, there exists at most one $y \in (v_1^s, 1]$ such that $\lambda'(y) = 0$. For the same reason as in the proof of Lemma 6, we only have one unique monotonic equilibrium. ■

Remark 8 When $n_1 = n_2 = 1$, $xF(x)^{n_1} [\frac{n_1 n_2}{(n_1 - 1) + \frac{F(x)}{xf(x)}} - (n_2 - 1)]$ can be simplified to $x^2 f(x)$, which is a decreasing function of y; thus the second condition in the above lemma is redundant.

Lemma 12 *When $F(v)$ is strictly convex, there exists a unique type-symmetric neg-monotonic equilibrium when $c_1 = c_m$ and at least two type-symmetric neg-monotonic equilibria when $c_m < c_1 < c_2$. There is no type-symmetric neg-monotonic equilibrium when $c_1 < c_m$.*

Proof. Consider $\phi(x) = y(x)F(y(x))^{n_2}F(x)^{n_1-1} + F(x)^{n_1-1}\int_{y(x)}^{x} F(v)^{n_2}dv$, where $y(x)$ is defined by $c_2 = yF(y)^{n_2-1}F(x)^{n_1}$. We have

$$y'(x) = -\frac{n_1 y f(x)F(y)}{F(x)[F(y)+(n_2-1)yf(y)]},$$

and

$$\begin{aligned}\phi'(x) &= F(x)^{n_1-2}\{(n_1-1)f(x)\int_y^x F(v)^{n_2}dv + F(x)^{n_1+1}\\ &\quad -yf(x)F(y)^{n_2}[\frac{n_1n_2}{(n_2-1)+\frac{F(y)}{yf(y)}} - (n_1-1)]\}\\ &= F(x)^{n_1-2}f(x)\{(n_1-1)\int_y^x F(v)^{n_2}dv + \frac{F(x)^{n_2+1}}{f(x)}\\ &\quad -yF(y)^{n_2}[\frac{n_1n_2}{(n_2-1)+\frac{F(y)}{yf(y)}} - (n_1-1)]\}.\end{aligned}$$

When $x = y = v_2^s$, we have

$$\phi'(v_2^s) = F(v_2^s)^{n_1-2}f(v_2^s)\{\frac{F(v_2^s)^{n_2+1}}{f(v_2^s)} - v_2^sF(v_2^s)^{n_2}[\frac{n_1n_2}{(n_2-1)+\frac{F(v_2^s)}{v_2^sf(v_2^s)}} - (n_1-1)]\}.$$

Since $v_2^sf(v_2^s) > F(v_2^s)$ by the strict convexity of $F(v)$, we have $\phi'(v_2^s) < 0$, which indicates that $\phi(x)$ is decreasing at $x = v_2^s$. Thus $\phi(x)$ has a minimum value $c_m < c_2$ in the interval $[v_2^s, 1]$ since $\phi(v_2^s) = c_2$. Let $\phi(x_m) = c_m$.

When $c_1 < c_m$, we have $\phi(x) > c_1$ for $x \in [v_2^s, 1]$. However, for us to have a neg-monotonic equilibrium, we need $\phi(x) \le c_1$. Therefore we do not have type-symmetric neg-monotonic equilibria.

When $c_1 = c_m$, since $\phi(x_m) = c_m$, then (x, y) is the unique neg-monotonic equilibrium, where $x = x_m$ and y is determined by $c_2 = yF(y)^{n_2-1}F(x_m)^{n_1}$. Also note that when $c_1 = c_m$, we do have a neg-monotonic equilibrium in which bidder 1 never participates since $\phi(1) \ge c_1$.

When $c_m < c_1 < c_2$, we have at least two type-symmetric neg-monotonic equilibria. Indeed, since $\phi(x_m) = c_m < c_1$ and $\phi(v_2^s) = c_2 > c_1$, there is an x_1 such that $\phi(x_1) = c_1$. On the other hand, when $\phi(1) < c_1$, we have a neg-monotonic equilibrium in which bidder 1 never participates. When $\phi(1) \ge c_1$, we can find $x_2 \in (x_m, 1]$ such that $\phi(x_2) = c_1$ since $\phi(1) \ge c_1$ and $\phi(x_m) = c_m < c_1$. Thus we can find at least two type-symmetric neg-monotonic equilibria. ■

Remark 9 The condition that $F(v)$ is concave in Lemma 9 can be weakened to $F(v) \ge vf(v)$ for all $v \in [c_1, 1]$ and the condition that $F(v)$ is strictly convex in Lemma 12 can be weakened to $F(v) < vf(v)$ for all $v \in [c_2, 1]$.

Proof of Proposition 5:

5.(1):

Taking derivatives with respect to c_1 on both sides of (15) and (17) and making simplifications, we have

$$\begin{aligned} 1 &= n_1 v_1^{*n_1-1} v_2^{*n_2} \frac{\partial v_1^*}{\partial c_1} + n_2 v_2^{*n_2-1} v_1^{*n_1} \frac{\partial v_2^*}{\partial c_1}, \\ 0 &= n_1 v_1^{*n_1} v_2^* \frac{\partial v_1^*}{\partial c_1} + [\frac{n_1(n_1-1)}{n_1+1} v_1^{*n_1+1} + \frac{n_1+n_2}{n_1+1} v_2^{*n_1+1}] \frac{\partial v_2^*}{\partial c_1}. \end{aligned}$$

Solving for $\frac{\partial v_1^*}{\partial c_1}$ gives us

$$\frac{\partial v_1^*}{\partial c_1} = \frac{n_1(n_2-1)v_1^{*n_1+1} + (n_1+n_2)v_2^{*n_1+1}}{n_1(n_1+n_2)(v_1^{*n_1-1} v_2^{*n_2}(v_2^{*n_1+1} - v_1^{*n_1+1}))} > 0$$

by $v_2^* > v_1^*$. Then we have

$$\frac{\partial v_2^*}{\partial c_1} = -\frac{n_1 v_1^{*n_1} v_2^*}{\frac{n_1(n_1-1)}{n_1+1} v_1^{*n_1+1} + \frac{n_1+n_2}{n_1+1} v_2^{*n_1+1}} \frac{\partial v_1^*}{\partial c_1} < 0.$$

Now taking derivatives with respect to c_2 on both sides of (15) and (17) and making simplifications, we have

$$\begin{aligned} 0 &= n_1 v_2^* \frac{\partial v_1^*}{\partial c_2} + n_2 v_1^* \frac{\partial v_2^*}{\partial c_2}, \\ 1 &= \frac{v_2^{*n_2-2}}{n_1+1} [n_1(n_1+1) v_1^{*n_1} v_2^* \frac{\partial v_1^*}{\partial c_2} + (n_1(n_2-1)v_1^{*n_1+1} + (n_1+n_2)v_2^{*n_1+1}) \frac{\partial v_2^*}{\partial c_2}]. \end{aligned}$$

Solving for $\frac{\partial v_2^*}{\partial c_2}$, we have

$$\frac{\partial v_2^*}{\partial c_2} = \frac{n_1+1}{(n_1+n_2)(v_2^{*n_1+1} - v_1^{*n_1+1}) v_2^{*n_2-2}} > 0$$

by $v_2^* > v_1^*$. Thus we have

$$\frac{\partial v_1^*}{\partial c_2} = -\frac{n_2 v_1^*}{n_1 v_2^*} \frac{\partial v_2^*}{\partial c_2} < 0.$$

5.(2):

Note that (15) and (17) implicitly define v_1^* and v_2^* as functions of n_1 and n_2, denoted by $v_1^* = v_1^*(n_1, n_2)$ and $v_2^* = v_2^*(n_1, n_2)$. Let

$$\mathbf{V}_\mathbf{n}^* = \begin{pmatrix} \frac{\partial v_1^*}{\partial n_1} & \frac{\partial v_1^*}{\partial n_2} \\ \frac{\partial v_2^*}{\partial n_1} & \frac{\partial v_2^*}{\partial n_2} \end{pmatrix}.$$

Taking logs of both equations and defining

$$H(n_1, n_2; v_1^*, v_2^*) = \begin{cases} H_1(n_1, n_2; v_1^*, v_2^*) = n_1 \ln(v_1^*) + n_2 \ln(v_2^*) - \ln(c_1) \\ H_2(n_1, n_2; v_1^*, v_2^*) = (n_1+n_2) \ln(v_2^*) - \ln(n_1+1) + \ln(1 + n_1(\frac{v_1^*}{v_2^*})^{n_1+1}) - \ln(c_2), \end{cases}$$

we have

$$\mathbf{H_{v^*}} = \begin{pmatrix} \frac{\partial H_1}{\partial v_1^*} & \frac{\partial H_1}{\partial v_2^*} \\ \frac{\partial H_1}{\partial v_1^*} & \frac{\partial H_1}{\partial v_2^*} \end{pmatrix} = \begin{pmatrix} \frac{n_1}{v_1^*} & \frac{n_2}{v_2^*} \\ \frac{n_1(1+n_1)(\frac{v_1^*}{v_2^*})^{n_1}\frac{1}{v_2^*}}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}} & \frac{n_1+n_2}{v_2^*} + \frac{n_1(n_1+1)(\frac{v_1^*}{v_2^*})^{n_1}(-\frac{v_1^*}{(v_2^*)^2})}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}} \end{pmatrix}.$$

Since

$$\det(H_{v^*}) = \frac{n_1}{v_1^*}(\frac{n_1+n_2}{v_2^*} + \frac{n_1(n_1+1)(\frac{v_1^*}{v_2^*})^{n_1}(-\frac{v_1^*}{(v_2^*)^2})}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}}) - \frac{n_2}{v_2^*}\frac{n_1(1+n_1)(\frac{v_1^*}{v_2^*})^{n_1}\frac{1}{v_2^*}}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}}$$

$$= \frac{n_1}{v_1^*}\frac{n_1+n_2}{v_2^*} - \frac{n_1(n_1+1)(\frac{v_1^*}{v_2^*})^{n_1}}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}}\frac{n_1}{(v_2^*)^2} - \frac{n_1(1+n_1)(\frac{v_1^*}{v_2^*})^{n_1}}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}}\frac{n_2}{v_2^2}$$

$$= \frac{n_1}{v_1^*}\frac{n_1+n_2}{v_2^*} - \frac{n_1(1+n_1)(\frac{v_1^*}{v_2^*})^{n_1}}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}}\frac{n_2+n_1}{(v_2^*)^2}$$

$$= \frac{n_1(n_1+n_2)}{v_2^*}(\frac{1}{v_1^*} - \frac{(1+n_1)(\frac{v_1^*}{v_2^*})^{n_1}}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}}\frac{1}{v_2^*})$$

$$= \frac{n_1(n_1+n_2)}{v_1^* v_2^*}\frac{1-(\frac{v_1^*}{v_2^*})^{n_1+1}}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}}$$

and $v_1^* < v_2^*$, at the equilibrium (v_1^*, v_2^*), we have $\det(H_{v^*}) > 0$. Then, by the Implicit Function Theorem, we have

$$V_n^* = -H_{v^*}^{-1}H_n = \begin{pmatrix} \frac{\partial v_1^*}{\partial n_1} & \frac{\partial v_1^*}{\partial n_2} \\ \frac{\partial v_2^*}{\partial n_1} & \frac{\partial v_2^*}{\partial n_2} \end{pmatrix},$$

where

$$\mathbf{H_n} = \begin{pmatrix} \frac{\partial H_1}{\partial n_1} & \frac{\partial H_1}{\partial n_2} \\ \frac{\partial H_1}{\partial n_1} & \frac{\partial H_1}{\partial n_2} \end{pmatrix} = \begin{pmatrix} \ln(v_1^*) & \ln(v_2^*) \\ \ln(v_2^*) + \frac{(\frac{v_1^*}{v_2^*})^{n_1+1} + n_1(\frac{v_1^*}{v_2^*})^{n_1+1}(\ln(v_1^*)-\ln(v_2^*))}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}} - \frac{1}{1+n_1} & \ln(v_2^*) \end{pmatrix}$$

and

$$\mathbf{H_{v^*}^{-1}} = \frac{1}{\det(H_(v^*))}\begin{pmatrix} \frac{n_1+n_2}{v_2^*} + \frac{n_1(n_1+1)(\frac{v_1^*}{v_2^*})^{n_1}(-\frac{v_1^*}{(v_2^*)^2})}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}} & -\frac{n_2}{v_2^*} \\ -\frac{n_1(1+n_1)(\frac{v_1^*}{v_2^*})^{n_1}\frac{1}{v_2}}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}} & \frac{n_1}{v_1^*} \end{pmatrix}.$$

To determine the sign for each term in V_n^*, we ignore $\det(H_{v^*})$ which is positive.

The sign of $\frac{\partial v_1^*}{\partial n_1}$ is determined by the opposite sign of the following term:

$$(\frac{n_1+n_2}{v_2^*} + \frac{n_1(n_1+1)(\frac{v_1^*}{v_2^*})^{n_1}(-\frac{v_1^*}{(v_2^*)^2})}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}})\ln(v_1^*) - \frac{n_2}{v_2^*}\ln(v_2^*)$$

$$-\frac{n_2}{v_2^*}\frac{(\frac{v_1^*}{v_2^*})^{n_1+1}+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}(\ln(v_1^*)-\ln(v_2^*))}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}}+\frac{1}{1+n_1}\frac{n_2}{v_2^*}$$

which can be simplified to

$$\frac{1}{v_2^*(1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1})}[(n_1+n_2-n_1(\frac{v_1^*}{v_2^*})^{n_1+1})\ln(v_1)-n_2\ ln(v_2)+\frac{n_2}{1+n_1}(1-(\frac{v_1^*}{v_2^*})^{n_1+1})].$$

Let

$$f(v_1^*)=n_2(1-(\frac{v_1^*}{v_2^*})^{1+n1})-(1+n_1)n_2\ln(v_2^*)+\ln(v_1^*)[(1+n_1)(n_1+n_2)-(1+n_1)n_1(\frac{v_1^*}{v_2^*})^{n_1+1}]$$

be a function of v_1^*, where $0\le v_1^*\le v_2^*<1$.

From $f(v_2^*)=0$ and

$$f'(v_1^*)=-n_2(1+n_1)(\frac{v_1^*}{v_2^*})^{n_1}\frac{1}{v_2^*}+\frac{1}{v_1^*}(1+n_1)(n_1+n_2)-\frac{1}{v_2^*}(1+n_1)n_1(\frac{v_1^*}{v_2^*})^{n_1}$$

$$-(1+n_1)^2n_1(\frac{v_1^*}{v_2^*})^{n_1}\frac{1}{v_2^*}\ln(v_1^*)$$

$$=-(1+n_1)^2n_1(\frac{v_1^*}{v_2^*})^{n_1}\frac{1}{v_2^*}\ln(v_1^*)+(1+n_1)(n_1+n_2)(\frac{1}{v_1^*}-\frac{1}{v_2^*}(\frac{v_1^*}{v_2^*})^{n_1}),$$

we can see that both terms are positive when $0\le v_1^*\le v_2^*<1$. Thus, $f(v_1^*)$ is monotonic in the range $0\le v_1^*\le v_2^*<1$; hence $f(v_1^*)<0$. Then we have $-f(v_1^*)>0$, and thus $\frac{\partial v_1^*}{\partial n_1}>0$.

Now for $\frac{\partial v_2^*}{\partial n_1}$, we check the sign of

$$-\frac{n_1(1+n_1)(\frac{v_1^*}{v_2^*})^{n_1}\frac{1}{v_2^*}}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}}\ln(v_1^*)+\frac{n_1}{v_1^*}[\ln(v_2^*)+\frac{(\frac{v_1^*}{v_2^*})^{n_1+1}+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}(\ln(v_1^*)-\ln(v_2^*))}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}}-\frac{1}{1+n_1}],$$

which can be simplified to

$$-\frac{n_1}{v_1^*(1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1})}[(\frac{v_1^*}{v_2^*})^{n_1+1}\ln(v_1^*)-\ln(v_2^*)+\frac{1-(\frac{v_1^*}{v_2^*})^{n_1+1})}{1+n_1}].$$

Let

$$f(v_1^*)=(1-(\frac{v_1^*}{v_2^*})^{1+n_1})-(1+n_1)\ln(v_2^*)+(1+n_1)\ln(v_1^*)(\frac{v_1^*}{v_2^*})^{n_1+1}.$$

We can check that

$$f(v_2^*)=0,$$

and

$$f'(v_1^*)=-(n_1+1)(\frac{v_1^*}{v_2^*})^{n_1}\frac{1}{v_2^*}+\frac{1}{v_2^*}(\frac{v_1^*}{v_2^*})^{n_1}(1+n_1)+\ln(v_1^*)(\frac{v_1^*}{v_2^*})^{n_1}(1+n_1)^2\frac{1}{v_2^*}$$

$$=\ln(v_1^*)(\frac{v_1^*}{v_2^*})^{n_1}(1+n_1)^2\frac{1}{v_2^*}$$

is negative and it is monotonic over the domain. Thus $\frac{\partial v_2^*}{\partial n_1} > 0$.

Now for $\frac{\partial v_1^*}{\partial n_2}$, we check the sign of

$$-[(\frac{n_1+n_2}{v_2^*} + \frac{n_1(n_1+1)(\frac{v_1^*}{v_2^*})^{n_1}(-\frac{v_1^*}{(v_2^*)^2})}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}} - \frac{n_2}{v_2^*})\ln(v_2^*)].$$

It can be simplified to

$$-\frac{n_1}{v_2^*}\frac{1-(\frac{v_1^*}{v_2^*})^{n_1+1}}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}}\ln(v_2^*) > 0.$$

Thus, $\frac{\partial v_1^*}{\partial n_2} > 0$.

Now for $\frac{\partial v_2^*}{\partial n_2}$, we check the sign of

$$-[-\frac{n_1(1+n_1)(\frac{v_1^*}{v_2^*})^{n_1}\frac{1}{v_2^*}}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}} + \frac{n_1}{v_1^*}]\ln(v_2^*)$$

that can be simplified to

$$-\frac{n_1\ln(v_2^*)}{v_1^*}\frac{1-(\frac{v_1^*}{v_2^*})^{n_1+1}}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}} > 0.$$

Thus, $\frac{\partial v_2^*}{\partial n_2} > 0$.

Proof of Proposition 7:

7.(1) Suppose by contradiction that there does not exist any type of equilibrium. We then have no monotonic equilibrium. Thus, $\lambda(y) > c_2 = v_2^s F_1(v_2^s)$ for all $y \in [v_1^s, 1]$, and particularly, we have $\lambda(v_1^s) = v_1^s F_1(v_1^s) > v_2^s F_1(v_2^s)$. Then we have $\frac{v_1^s}{v_2^s} > \frac{F_1(v_2^s)}{F_1(v_1^s)}$. Since there is no neg-monotonic equilibrium either, we have $\phi(x) > c_1 = v_1^s F_2(v_1^s)$ for all $x \in [v_2^s, 1]$, and particularly, we have $\phi(v_2^s) = v_2^s F_2(v_2^s) > v_1^s F_2(v_2^s)$. Then we have $\frac{v_1^s}{v_2^s} < \frac{F_2(v_2^s)}{F_2(v_1^s)}$. Combining these two cases, we have

$$\frac{F_1(v_2^s)}{F_1(v_1^s)} < \frac{v_1^s}{v_2^s} < \frac{F_2(v_2^s)}{F_2(v_1^s)}.$$

Now we prove that these two inequalities cannot hold simultaneously. Indeed, if $v_1^s \le v_2^s$, then $1 \le \frac{F_1(v_2^s)}{F_1(v_1^s)} < \frac{v_1^s}{v_2^s} \le 1$, which is impossible. On the other hand, if $v_1^s > v_2^s$, $1 < \frac{v_1^s}{v_2^s} < \frac{F_2(v_2^s)}{F_2(v_1^s)} < 1$, which is also impossible. Thus, there must exist an equilibrium for any $F_1(v)$ and $F_2(v)$ under consideration.

7.(2) First note that $\lambda(v_1^s) = v_1^s F_1(v_1^s) < v_1^s F_2(v_2^s) = c_1 < c_2$ by $F_1(v) < F_2(v)$ and $\lambda(y)$ is monotonically increasing by the concavity of $F_1(v)$ and $F_2(v)$. Thus, if $\lambda(y) < c_2$ for all $y \in (v_1^s, 1]$, bidder 2 will never participate (i.e., $v_2^* > 1$), and bidder 1 uses $v_1^* = c_1$ as the cutoff point. Otherwise bidder 2 will use $v_2^* > v_1^s$ which is determined by $\lambda(y) = c_2$. Thus, in both cases, (v_1^*, v_2^*) compose a monotonic equilibrium. Since $\lambda(y)$ is monotonically increasing, such a monotonic equilibrium must be unique.

Finally, we show there does not exist any neg-monotonic equilibrium. To do so, we only need to focus on $\phi(x)$ with $x > v_2^s$. Since $\phi(v_2^s) = v_2^s F_2(v_2^s) > v_2^s F_1(v_1^s) = c_1$ and $\phi(x)$ is monotonic increasing, $\phi(x) > c_1$ for all $x \in (v_2^s, 1]$. Thus we do not have a neg-monotonic equilibrium. Hence, there is a unique equilibrium that is monotonic.

7.(3.i) Suppose $v_1^s < v_2^s$. We have $\phi(v_2^s) = v_2^s F_2(v_2^s) > v_1^s F_2(v_1^s) = c_1$. By $\phi'(x) > 0$ we have $\phi(x) > c_1$ for all $x \in (v_2^s, 1]$. Thus no neg-monotonic exists. Also, we have $\lambda(v_1^s) = v_1^s F_1(v_1^s) < v_2^s F_1(v_2^s) = c_2$. Then by the monotonicity of $\lambda(y)$, there is a unique equilibrium that is monotonic.

7.(3.ii) Suppose $v_1^s > v_2^s$. We have $\lambda(v_1^s) = v_1^s F_1(v_1^s) > v_1^s F_2(v_1^s) > v_2^s F_2(v_2^s) = c_2$. By $\lambda'(y) > 0$ we have $\lambda(y) > c_2$ for all $y \in (v_1^s, 1]$. So no monotonic equilibrium exists. On the other hand, we have $\phi(v_2^s) = v_2^s F_2(v_2^s) < v_1^s F_2(v_1^s) = c_1$. By $\phi'(x) > 0$, if for all $x \in (v_2^s, 1]$ we have $\phi(x) < c_1$, bidder 1 never participates in the auction (i.e., $v_1^* > 1$). Thus, $v_1^* > 1$ and $v_2^* = c_2$ will be the unique neg-monotonic equilibrium. Otherwise $v_2^* > v_2^s$ is uniquely determined by $\phi(x) = c_1$. Then $v_1^* < v_2^s$ and $v_2^* > v_2^s$ is the unique neg-monotonic equilibrium. Thus we have a unique equilibrium that is neg-monotonic.

7.(3.iii) Now suppose $v_1^s = v_2^s = v^s$. We then have $c_1 = v^s F_2(v^s)$ and $c_2 = v^s F_1(v^s)$. Then $\lambda(v^s) = v^s F_1(v^s) = c_2$ and $\phi(v^s) = v^s F_2(v^s) = c_1$. Thus $v_1^* = v_2^* = v^s$ is the equilibrium that is a special neg-monotonic equilibrium. The uniqueness comes from the monotonicity of $\lambda(y)$ and $\phi(x)$.

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