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Are incentives against economic justice?

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Abstract

We consider the problem of fairly allocating a social endowment of indivisible goods and money when the domain of admissible preferences contains, but is not restricted to, quasi-linear preferences. We analyze the manipulability of the Generalized Money Rawlsian Fair (GMRF) solutions [Alkan A., Demange G., Gale D., 1991. Fair allocation of indivisible goods and criteria of justice. Econometrica 59, 1023-1039]. These solutions are envy-free, efficient, and satisfy a strong form of solidarity when budget changes. (i) We show that the Nash and strong Nash equilibrium correspondences of the "preference revelation game form" associated with each GMRF solution coincide with the no-envy solution. Even though each agent has an incentive to lie if the others truthfully report their preferences, in equilibrium, no agent prefers another agent's allotment to hers according to her true preferences; moreover, in equilibrium, efficiency is preserved according to agents' true preferences. (ii) As a corollary, we show that the GMRF solutions "naturally implement" the no-envy solution in Nash and strong Nash equilibria.

JEL classification: D63, C72.
Keywords: efficiency; fairness; Generalized Money Rawlsian Fair solutions; implementation; indivisible goods; manipulation; mechanism design; no-envy.

1 Introduction

The literature on fair allocation of indivisible goods and money has identified normatively compelling "solutions," i.e., systematic ways of selecting allocations for each possible configuration of preferences. Unfortunately, none of these solutions, in fact, no solution satisfying the basic requirement of "no-envy" in this environment, can provide agents with incentives to truthfully reveal their preferences (Tadenuma and Thomson, 1995). In light of this impossibility, we investigate the consequences of manipulation for a central family of solutions: the Generalized Money Rawlsian Fair (GMRF) solutions (Alkan et al., 1991). To each solution we associate a preference revelation "game form" in which each agent's

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strategy space is the set of admissible preferences and the outcome function is the solution itself (an adjustment is necessary here, for the GMRF solutions are not single-valued; see Subsection 2.2 for details). Our main result, Theorem 1, states that both the Nash and strong Nash equilibrium correspondences of the game form associated with each GMRF solution coincide with the "no-envy solution." This result allows us to precisely assess the consequences of agents' strategic behavior. In equilibrium, and regardless of the fact that all agents may lie, the outcomes that ensue from the manipulation of these solutions are envy-free and efficient with respect to agents' true preferences. Moreover, interpreted as an "implementation" result, Theorem 1 allows us to identify natural mechanisms that doubly implement the no-envy solution.

Let us describe our environment more precisely. We consider the problem of fairly allocating a social endowment of objects (indivisible goods) and an amount (possibly negative) of a perfectly divisible good, which we refer to as "money," when the domain of admissible preferences contains, but is not restricted to, quasi-linear preferences (Svensson, 1983; Maskin, 1987; Alkan et al., 1991). There are as many agents as objects and each agent must receive an object. Individual consumptions of money should add up to a given amount, which we refer to as the "budget."

An allocation is envy-free if no agent prefers the allotment of any other agent to hers (Foley, 1967; Varian, 1974). A central solution, known as the Money Rawlsian Fair solution, selects for each preference profile the allocations that maximize across agents the minimal individual consumption of money among the envy-free allocations (Alkan et al., 1991). More generally, one may want to "recalibrate" this solution and assign different importance to money depending upon which object it is associated with (Alkan et al., 1991). For instance, consider the allocation of tasks and salary among workers. On the one hand, a manager may want to choose an envy-free allocation, for these allocations provide equal opportunity to the workers, maintain harmony in the workplace, and may reduce the likelihood of a discrimination complain. On the other hand, the manager may be interested in favoring the workers who perform certain tasks that she considers more valuable for her business (agents may not internalize this asymmetry in their preferences). To this purpose, consider a list of increasing functions, one for each object, that recalibrate the value of the respective consumptions of money. Then for each preference profile, select the allocations that maximize across agents the minimal individual recalibrated consumption of money among the envy-free allocations. The resulting solution, which we refer to as a GMRF solution, is parameterized by the family of calibration functions.

The GMRF solutions share several desirable properties. Each of them selects for each preference profile a unique allocation up to Pareto indifference (Alkan et al., 1991). In our context, envy-free allocations are efficient (Svensson, 1983). Thus, so are the GMRF solutions. Additionally, they satisfy a strong form of solidarity when budget changes: each agent benefits if the budget increases, and symmetrically, each agent contributes in welfare terms if the budget decreases (Alkan et al., 1991).[1]

[1]These solutions also achieve some form of "democratic equality." In Rawls's Theory of Justice, this principle of distributive justice calls for the minimal welfare among the agents, measured by means of

We investigate how manipulable the GMRF solutions are, under the assumption of perfect information. Consider a GMRF solution, say C. It is well known that no selection from the solution that associates with each preference profile its set of envy-free allocations, i.e., the no-envy solution, is strategy-proof (Tadenuma and Thomson, 1995).[2] Thus, no selection from C is strategy-proof. However, this only means that an agent may have the incentive to lie if all the other agents report their true preferences. Thus, it is natural to investigate what the consequences of all agents behaving strategically are. Our main result, Theorem 1, answers this question. It states that the outcomes from the manipulation, both individual and coalitional, of C are exactly the envy-free allocations for the true preferences. More precisely, for each preference profile, each Nash equilibrium outcome of the manipulation game induced by (the preference revelation game form associated with) C and the profile is envy-free, and thus, efficient for that profile. In fact, each envy-free allocation for the profile is a strong Nash equilibrium outcome of the manipulation game induced by C and the profile. Even though the allocations selected by C are not necessarily attained, the basic objectives of fairness and efficiency survive. Thus, incentives are against the truthful revelation of preferences, but not against our basic criteria of economic justice.

Let us emphasize that the domain we consider contains, but is not restricted to, quasi-linear preferences. Let C be a subsolution of the no-envy solution and i an agent. When preferences are quasi-linear, one can identify agent i's possibilities to manipulate C by tracking the changes on the set of envy-free allocations that are induced by a change in her preferences (c.f., Beviá, 2010; Āzacis, 2008). This technique is not available when preferences belong to our general domain, however. We overcome this issue by developing an alternative approach that relies on a deeper knowledge of the solutions under consideration. Indeed, suppose that C is a GMRF solution. Consider the manipulation game induced by C at a preference profile. Suppose that for some announcement of preferences, the outcome is z. Let z' be an allocation that agent i prefers to z. We identify situations in which agent i can change her report in order to obtain z', as follows. First, we develop a characterization of the C-optimal allocations in terms of a binary relation on the set of objects that we associate with each feasible allocation at a preference profile (Section 3). Then, we identify situations in which, by changing her report, agent i guarantees that she consume z'_i at each allocation whose induced binary relation for the new profile satisfies the conditions that characterize it as C-optimal (Section 4).

Our results have consequences for the "implementation" of the no-envy solution (see Jackson, 2001, for a survey on implementation literature). First, we identify natural game forms that "doubly" implement it: Theorem 1 implies that the game form associated with each GMRF solution doubly implements the no-envy solution, in Nash and strong Nash

some index, to be maximized (Rawls, 1972). In contrast, for a GMRF solution, the minimal recalibrated consumption of money is maximized among all the envy-free allocations. There is a formal connection, however. For each GMRF solution, there is a representation of preferences for which the recommended allocations for all possible budgets are the envy-free allocations that maximize the minimal utility across agents with respect to it (Velez, 2009b).

[2] Strategy-proofness, also referred to as "incentive compatibility," requires that no agent benefit by misreporting her preferences.

equilibria. Second, we show, in Section 5, that this implementation result still holds when agents are required to report quasi-linear preferences (their true preferences may not be quasi-linear). Thus, we identify a family of game forms with "simpler" strategy spaces that doubly implement the no-envy solution.

A variation of our model is when consumptions of money are required to be negative, as in the allocation of the rooms and the division of the rent among housemates (Su, 1999; Brams and Kilgour, 2001). If no further restriction on preferences is assumed, then envy-free allocations may not exist. In Section 6, we identify a rectangular sub-domain of preferences in which envy-free allocations with negative consumptions of money always exist and to which our results concerning the manipulation of the GMRF solutions extend.

The manipulation of solutions was first studied in the context of exchange economies. This literature was motivated by the impossibility to have individually rational, efficient, and strategy-proof solutions (Hurwicz, 1972).[3] It concluded that the outcomes from the manipulation of each individually rational and efficient solution contains the Walrasian solution (Lindahl solution in the public good case), but may not be efficient (Hurwicz, 1972; Thomson, 1979; Otani and Sicilian, 1982; Thomson, 1984). Similar results hold for the manipulation of the Shapley value (Thomson, 1988) and the manipulation of cooperative bargaining solutions (applied to the associated utility possibility sets) in private and public good economies (Sobel, 1981, 2001; Kıbrıs, 2002). For the distribution of a collectively owned bundle of infinitely divisible commodities the conclusions parallel the ones for exchange economies. Here a notion of fairness plays the role of individual rationality, and the Walrasian solution operated from equal endowments plays the role of the Walrasian solution (Thomson, 1987). In contrast to the results in this literature, in the games we analyze equilibrium outcomes are always efficient for the true preferences.[4]

The manipulation of solutions has also been studied for the allocation of one object and an amount of money among n agents ($n-1$ agents receive a null object). The main result here is that the outcomes from the manipulation of each selection from the no-envy solution satisfying a mild condition coincide with the ones selected by the no-envy solution (Tadenuma and Thomson, 1995).[5] This result extends to the n-object and n-agent case in the quasi-linear domain (Beviá, 2010). In such an environment (i.e., preferences are quasi-linear and there are n-objects and n-agents), the preference revelation game form associated with a solution introduced by Abdulkadiroğlu et al. (2004), implements the no-envy solution in Nash and strong Nash equilibria (Āzacis, 2008).[6] Let us emphasize that the aforementioned results hinge on one of two assumptions: (i) there is only one object, or (ii) preferences are quasi-linear. We do not impose any of these restrictions.

The remainder of this paper is organized as follows. Section 2 presents the model.

[3] Hurwicz (1972) results are for the two-agent and two-good case. Serizawa (2002) extended this impossibility to the n-agent and l-good case.

[4] See Velez and Thomson (2009) for a mechanism whose manipulation does not lead to a loss of efficiency in the context of the division of a bundle of infinitely divisible commodities.

[5] Our main theorem restricted to the two-agent case is a consequence of this result.

[6] See Thomson (2005) for the implementation of the envy-free solution in the context of the allocation of a bundle of infinitely divisible commodities.

Section 3 introduces the GMRF solutions. Section 4 analyzes the manipulability of the GMRF solutions. Section 5 studies the implementation of the no-envy solution. Section 6 discusses extensions of our results. The Appendix contains proofs.

2 The Model

2.1 Environment and axioms

We consider the problem of allocating a finite set of objects A and an amount $M \in \mathbb{R}$ of an infinitely divisible good, which we refer to as "money," among a group of agents N. We assume that the number of agents and objects are equal, i.e., $|N| = |A|$. Generic objects are denoted α and β. Agents consume bundles in $\mathbb{R} \times A$.[7] The generic consumption bundle is (x_α, α). The domain of preferences on $\mathbb{R} \times A$ is $\boldsymbol{\mathcal{R}}$. Agent i's generic preference is R_i and the generic preference profile is $R \equiv (R_i)_{i \in N}$. As usual, I_i and P_i are the symmetric and asymmetric parts of R_i, respectively. We assume that preferences satisfy two properties:

• **Money-monotonicity**: for each $\alpha \in A$ and each $\{x_\alpha, x'_\alpha\} \subseteq \mathbb{R}$ such that $x'_\alpha > x_\alpha$, $(x'_\alpha, \alpha) \, P_i \, (x_\alpha, \alpha)$.[8]

• **No object is infinitely better than any other**: for each $\{\alpha, \beta\} \subseteq A$ and each $x_\beta \in \mathbb{R}$, there is $x_\alpha \in \mathbb{R}$ such that $(x_\beta, \beta) \, I_i \, (x_\alpha, \alpha)$.

Our domain contains, but is not restricted to, the preferences that satisfy the following invariance property with respect to translation of consumptions of money. A preference $R_i \in \mathcal{R}$ is **quasi-linear** if for each $\{\alpha, \beta\} \subseteq A$, each $\{x_\alpha, x_\beta\} \subseteq \mathbb{R}$ such that $(x_\alpha, \alpha) \, R_i \, (x_\beta, \beta)$, and each $\Delta \in \mathbb{R}$, $(x_\alpha + \Delta, \alpha) \, R_i \, (x_\beta + \Delta, \beta)$. We denote the sub-domain of quasi-linear preferences $\boldsymbol{\mathcal{Q}}$. In Section 5 we investigate agents' incentives for the revelation of preferences if their reports are restricted to be quasi-linear ("true" preferences may not be quasi-linear, however).

We assume that each agent receives one object and some amount of money. An **allocation** is a pair $z \equiv (x, \mu) \in \mathbb{R}^A \times A^N$ such that $\sum_{\alpha \in A} x_\alpha = M$ and $\mu : N \to B$ is a bijection. The consumption of money associated with object α at z is x_α. **Agent i's allotment at z** is $z_i \equiv (x_{\mu(i)}, \mu(i))$. Let $\boldsymbol{Z}$ be the set of all allocations. Agent i's preferences R_i induce preferences on Z, which for convenience we also denote R_i, as follows: for each $\{z, z'\} \subseteq Z$, $z' \, R_i \, z$ if and only if $z'_i \, R_i \, z_i$.

Let $R \in \mathcal{R}^N$. For each $i \in N$ and each $R'_i \in \mathcal{R}$, the profile (R_{-i}, R'_i) is obtained from R by replacing R_i by R'_i. For each $K \subseteq N$, R_K is the subprofile $(R_i)_{i \in K}$.

2.1.1 Solutions

We are interested in systematic ways of selecting allocations for each possible configuration of preferences; a **solution**, generically denoted C, associates with each $R \in \mathcal{R}^N$ a non-

[7] See Section 6 for an extension of our results when consumptions of money are required to be negative as in the allocation of rooms and rent among housemates who collectively lease a house.

[8] Money-monotonicity implies continuity, i.e., weak upper and lower contour sets are closed in the product topology on $\mathbb{R} \times A$ induced by the Euclidean and discrete topologies.

empty subset of allocations $C(R) \subseteq Z$. A **selection** from a solution C, generically denoted c, is a function that associates with each $R \in \mathcal{R}^N$ an element of $C(R)$. We write $c \in C$.

2.1.2 Fairness and efficiency

We consider two properties of allocations. First is no-envy. Let $R \in \mathcal{R}^N$. An allocation $z \in Z$ is **envy-free for $\boldsymbol{R}$** if for each $\{i,j\} \subseteq N$, $z_i \, R_i \, z_j$. The no-envy test provides an ordinal and operational notion of fairness (Foley, 1967; Varian, 1974). It has played a central role in the study of fairness issues in resource allocation problems (see Thomson, 2010, for a survey). Second is efficiency. As usual, an allocation $z \in Z$ is **efficient for $\boldsymbol{R}$** if there is no $z' \in Z$ such that for each $i \in N$, $z'_i \, R_i \, z_i$ and for at least one $j \in N$, $z'_j \, P_j \, z_j$. Let $\boldsymbol{F(R)}$ and $\boldsymbol{P(R)}$ be the sets of *envy-free* and *efficient* allocations for R, respectively. It is well known that under our assumptions, the set $F(R)$ is non-empty (Alkan et al., 1991; Velez, 2009a); moreover, since there are as many agents as objects, then $F(R) \subseteq P(R)$ (Svensson, 1983). We refer to the solution that associates with each $R \in \mathcal{R}^N$ the set $F(R)$ as the *no-envy* solution.

2.2 Manipulation of a solution

Let C be a solution. What are the outcomes that ensue if agents manipulate C? On the one hand, if the solution is single-valued, it is straightforward to precisely formulate this question. Consider the game form in which each agent's strategy space is the domain of admissible preferences and the outcome function is C itself. For each preference profile, the outcomes form the manipulation of C are the equilibrium outcomes of the game induced by the game form associated with C and the profile. On the other hand, if C is multivalued, it is not clear when an agent will want to change her report given the other agents' reports. This is so even if C is essentially single-valued, i.e., for each problem, each agent is indifferent among all the C-optimal allocations. The main issue here is that when an agent envisions the outcomes attainable by a change in her report, these outcomes are welfare-equivalent according to her reported preferences; however, they may not be welfare-equivalent according to her true preferences (Thomson, 1979, 1984).

Further assumptions to solve this indeterminacy are necessary in order to establish how manipulable C is. We follow Velez and Thomson (2009) who, in a similar environment, show that any speculation about which assumption is more appropriate can be bypassed. Their construction is as follows. Assume that each agent reports not only preferences but also a consumption bundle. If the reported list of bundles is one of the C-optimal allocations for the announced preferences, then it is the outcome of the allocation process (thus, the bundle reported by an agent can be interpreted as the bundle she requests). Otherwise, there is a selection $c \in C$ that determines this outcome. One can evaluate the equilibrium correspondence of this game form associated with C and parameterized by the selection c. If the resulting correspondence happens not to depend on c, then one can regard these outcomes as the ones resulting from the manipulation of C.[9]

[9] Alternative approaches to this problem have been studied in the literature. Thomson (1979, 1984,

Let us formally define the game form associated with a solution and one of its selections. Let $\mathcal{D} \subseteq \mathcal{R}$ be a sub-domain of preferences, C a solution, and $c \in C$. The **game form** $\boldsymbol{\langle S(\mathcal{D})^N, C^c \rangle}$ is defined as follows:

- Each agent's strategy space is $S(\mathcal{D}) \equiv \mathcal{D} \times (\mathbb{R} \times A)$.
- Given strategy profile $(R, z) \equiv (R_i, z_i)_{i \in N} \in S(\mathcal{D})^N$, the outcome is

$$C^c(R, z) \equiv \begin{cases} z & \text{if } z \in C(R) \\ c(R) & \text{otherwise.} \end{cases}$$

For each $R^0 \in \mathcal{R}^N$, $\boldsymbol{\langle S(\mathcal{D})^N, C^c, R^0 \rangle}$ is the game induced by the game form $\langle S(\mathcal{D})^N, C^c \rangle$ and preferences R^0.

A **Nash equilibrium of** $\boldsymbol{\langle S(\mathcal{D})^N, C^c, R^0 \rangle}$ is a strategy profile $(R, z) \in S(\mathcal{D})^N$, such that for each $i \in N$ and each $(R'_i, z'_i) \in S(\mathcal{D})$, $C^c(R, z) \, R_i^0 \, C^c(R_{-i}, R'_i, z_{-i}, z'_i)$. For each game $\langle S(\mathcal{D})^N, C^c, R^0 \rangle$ the set of **Nash equilibria** is $\boldsymbol{\mathcal{N}\langle S(\mathcal{D})^N, C^c, R^0 \rangle}$ and the set of **Nash equilibrium outcomes** is $\boldsymbol{\mathcal{O}\langle S(\mathcal{D})^N, C^c, R^0 \rangle}$. If for any two selections of C, c and c', $\mathcal{O}\langle S(\mathcal{D})^N, C^c, R^0 \rangle = \mathcal{O}\langle S(\mathcal{D})^N, C^{c'}, R^0 \rangle$, then we drop this parameter from our notation and denote this common set by $\boldsymbol{\mathcal{O}\langle S(\mathcal{D})^N, C, R^0 \rangle}$.

We also consider coalitional manipulation. A **strong Nash equilibrium of** $\boldsymbol{\langle S(\mathcal{D})^N, C^c, R^0 \rangle}$ is a strategy profile $(R, z) \in S(\mathcal{D})^N$, such that for each $N' \subseteq N$ and each $(R'_{N'}, z'_{N'}) \in S(\mathcal{D})^{N'}$, if there is $i \in N'$ such that $C^c(R_{-N'}, R'_{N'}, z_{-N'}, z'_{N'}) \, P_i^0 \, C^c(R, z)$, then there is $j \in N'$ such that $C^c(R, z) \, P_j^0 \, C^c(R_{-N'}, R'_{N'}, z_{-N'}, z'_{N'})$. For each game $\langle S(\mathcal{D})^N, C^c, R^0 \rangle$ the set of **strong Nash equilibria** is $\boldsymbol{\mathcal{N}^*\langle S(\mathcal{D})^N, C^c, R^0 \rangle}$ and the set of **strong Nash equilibrium outcomes** is $\boldsymbol{\mathcal{O}^*\langle S(\mathcal{D})^N, C, R^0 \rangle}$. If for any two selections of C, c and c', $\mathcal{O}^*\langle S(\mathcal{D})^N, C^c, R^0 \rangle = \mathcal{O}^*\langle S(\mathcal{D})^N, C^{c'}, R^0 \rangle$, then we drop this parameter from our notation and denote this common set by $\boldsymbol{\mathcal{O}^*\langle S(\mathcal{D})^N, C, R^0 \rangle}$.

3 Generalized Money Rawlsian Fair solutions

A remarkable feature of our environment is that *envy-free* and *efficient* allocations always exist (Alkan et al., 1991; Velez, 2009a). In fact, generically, there is a continuum of allocations satisfying these properties. Thus, it is not only possible, but also desirable to include additional criteria in the selection of a "fairest" allocation. For this purpose Alkan et al. (1991) propose to select for each preference profile the allocations that maximize across agents the minimal individual consumption of money among all *envy-free* allocations: the Money Rawlsian Fair solution.[10]

1987, 1988), Tadenuma and Thomson (1995), and Beviá (2010) develop a solution concept that parallels Nash equilibrium and applies to "games" with multiple outcomes. Kıbrıs (2002) studies the manipulation of arbitrary selections from the solution.

[10] A dual approach to maximizing the minimum consumption of money among all *envy-free* allocations is that of minimizing the maximal consumption of money among all *envy-free* allocations. All our results extend to this dual construction. See Section 6 for details.

Let us formalize this definition. The **Money Rawlsian Fair (MRF) solution** associates with each $R \in \mathcal{R}^N$ the set:

$$\boldsymbol{\Re(R)} \equiv \underset{(x,\mu)\in F(R)}{\arg\max} \left\{ \min_{\alpha \in A} x_\alpha \right\}.$$

Since $F(R)$ is compact, then $\Re(R)$ is non-empty.[11]

By selecting a Money Rawlsian Fair allocation one guarantees that the minimal consumption of money at the allocation is maximal among all the available *envy-free* allocations. More generally, one may want to assign different importance to the consumptions of money associated with the various objects and maximize the minimal adjusted consumption of money (Alkan et al., 1991). To formalize this idea, let $\mathcal{I}$ be the space of real-valued, continuous, and monotone increasing functions defined on $\mathbb{R}$ and $f \equiv (f_\alpha)_{\alpha\in A} \in \mathcal{I}^A$. The **Generalized Money Rawlsian Fair (GMRF) solution with respect to $\boldsymbol{f}$** associates with each $R \in \mathcal{R}^N$ the set:

$$\boldsymbol{\Re_f(R)} \equiv \underset{(x,\mu)\in F(R)}{\arg\max} \left\{ \min_{\alpha \in A} f_\alpha(x_\alpha) \right\}.$$

Since $F(R)$ is compact, then $\Re_f(R)$ is non-empty.[12]

Let us remark that the MRF solution is the GMRF solution with respect to the family of identity functions.

We refer to the allocations selected by a GMRF solution for a profile R as the GMRF allocations for R. The following example illustrates that, in some cases, depending on the structure of the family of functions f, the description of the GMRF allocations with respect to f can be simplified.

Example 1. Let $\alpha \in A$. The **$\boldsymbol{\alpha}$-money maximal fair solution** associates with each $R \in \mathcal{R}^N$ the set:

$$\boldsymbol{\Re_\alpha(R)} \equiv \underset{(x,\mu)\in F(R)}{\arg\max} \{x_\alpha\}.$$

One can easily see that for each family of functions $f \in \mathcal{I}^A$ such that for each $\delta \in A \setminus \{\alpha\}$, $\sup \text{Range}(f_\alpha) \leq \inf \text{Range}(f_\delta)$, $\Re_\alpha(R)$ coincides with the set $\Re_f(R)$.[13] $\square$

We now develop a mathematical characterization of the GMRF allocations, which allows us to understand, in the next section, how agents manipulate the GMRF solutions. Our characterization generalizes Alkan's (1994) characterization of the MRF allocations.

Let $R \in \mathcal{R}^N$ and $z \in Z$. First, we associate with R and z a binary relation on the set of objects A. Consider the following directed graph, whose node set is the set of objects A, associated with R and z: there is an arc from object α to object β if the agent who receives

[11] Since preferences are continuous, then $F(R)$ is closed. Now, since at each $z \in Z$, at least, one agent consumes no less than $\frac{M}{|N|}$, then $F(R)$ is bounded, and thus compact.

[12] See footnote 11.

[13] For instance, $f_\alpha : \mathbb{R} \to]-\infty, 0[$ and for each $\delta \in A \setminus \{\alpha\}$, $f_\alpha : \mathbb{R} \to]0, +\infty[$.

object α at z is indifferent, under R, between her consumption at z and the consumption of the agent who receives object β at z. Object α dominates object β in terms of the binary relation associated with R and z if and only if there is a path of these arcs "flowing" from α to β. Formally, this binary relation on A, denoted $\boldsymbol{\succeq(R,z)}$, is defined as follows: for each pair $\{\alpha, \beta\} \subseteq A$, $\alpha \succeq(R,z)\, \beta$ if and only if there is a set of objects $\{\alpha_0, \ldots, \alpha_T\} \subseteq A$ such that $\alpha_0 = \alpha$, $\alpha_T = \beta$, and

$$(x_{\alpha_0}, \alpha_0)\, I_{\mu^{-1}(\alpha_0)}\, (x_{\alpha_1}, \alpha_1) \ldots (x_{\alpha_{T-1}}, \alpha_{T-1})\, I_{\mu^{-1}(\alpha_{T-1})}\, (x_{\alpha_T}, \alpha_T).$$

We now characterize, with a simple test, the GMRF allocations with respect to a family of functions f. These allocations are the *envy-free* allocations for which each object dominates, in terms of the binary relation induced by preferences and the allocation, one of the objects that is received by an agent whose f-transformed consumption of money is minimal (Figure 1 illustrates the test applied to MRF allocations). The following proposition formalizes this result. We sketch the proof below.

Proposition 1. Let $f \in \mathcal{I}^A$, $R \in \mathcal{R}^N$, and $z \in Z$. Then $z \in \Re_f(R)$ if and only if $z \in F(R)$ and for each $\alpha \in A$, there is $\beta \in \arg\min_{\delta \in A} f_\delta(x_\delta)$ such that $\alpha \succeq(R,z)\, \beta$.

First, we sketch the proof of sufficiency of the conditions in Proposition 1. Let f, R, and z be as in the statement of the proposition. Suppose that $z \in F(R)$ and for each $\alpha \in A$, there is $\beta \in \arg\min_{\delta \in A} f_\delta(x_\delta)$ such that $\alpha \succeq(R,z)\, \beta$. We argue that $z \in \Re_f(R)$. Let $z' \equiv (x', \mu') \in F(R)$. One can easily prove that for each $\{\alpha, \beta\} \subseteq A$, if $\alpha \succeq(R,z)\, \beta$ and $x'_\beta > x_\beta$, then $x'_\alpha > x_\alpha$ (see Alkan et al., 1991, Lemma 3). Thus, if z were not an element of $\Re_f(R)$, feasibility would be violated for each element of $\Re_f(R)$.

We now sketch the proof of necessity of the conditions in Proposition 1. The argument closely follows that of Lemma 1 in Alkan (1994). Let f and R be as in the statement of the proposition. Suppose that $z \equiv (x, \mu) \in \Re_f(R)$. Let G be the set of objects that do not dominate, in terms of $\succeq(R,z)$, any of the objects in $\arg\min_{\delta \in A} f_\delta(x_\delta)$, and $N(G)$ the set of agents who receive objects in G at z. Similarly, let $N(A \setminus G)$ be the set of agents who receive objects in $A \setminus G$ at z. Necessity in the proposition requires that G be empty. If this were not true, then each agent in $N(G)$ would prefer her consumption to the consumption of each agent in $N(A \setminus G)$. Thus, there would be an amount of money $\varepsilon > 0$ that we would be able to extract, in the aggregate, from the agents in $N(G)$, and distribute it among the agents in $N(A \setminus G)$, without creating any envy (see Perturbation Lemma in Alkan et al., 1991 and Money Monotonicity Theorem in Alkan, 1994). Since, $\arg\min_{\delta \in A} f_\delta(x_\delta) \subseteq A \setminus G$, then z would not be an allocation that maximizes the minimal individual f-transformed consumption of money across agents among all *envy-free* allocations for R. This would contradict $z \in \Re_f(R)$.

Proposition 1 plays an important role in our analysis of the manipulation of the GMRF solutions. Consider such a solution, say $\Re_f$. The crucial step in our analysis of the preference revelation game form associated with $\Re_f$ identifies situations in which an agent, say i, can benefit by changing her report. Consider a profile of announcements, say $(R, \widetilde{z})$, and let z be its associated outcome. Let z' be an allocation that agent i prefers to z. Agent i can benefit by changing her report whenever there are preferences R'_i such that her consumption at each allocation in $\Re_f(R_{-i}, R'_i)$ is z'_i. Proposition 1 translates the problem of determining whether z' belongs to $\Re_f(R_{-i}, R'_i)$ into the analysis of the binary relation $\succeq(R_{-i}, R'_i, z')$. Thus, the problem of determining if agent i can manipulate $\Re_f$ is mathematically equivalent to the one of determining the way in which $\succeq(R_{-i}, R'_i, z')$ changes with R'_i. We successfully deal with this task in Section 4.

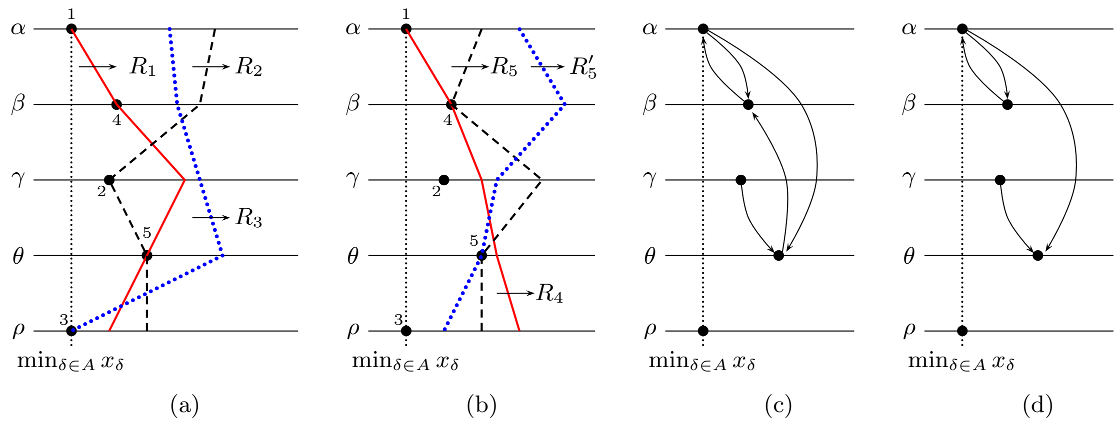

Figure 1: Verifying whether $z \in \Re(R)$ or $z \notin \Re(R)$. Let $N \equiv \{1, 2, 3, 4, 5\}$ and $A \equiv \{\alpha, \beta, \gamma, \theta, \rho\}$. Panels (a) to (d) display the consumption space $\mathbb{R} \times A$ (for a range of consumptions of money); each point x_α on the axis corresponding to object α, represents bundle (x_α, α). Let $R \in \mathcal{R}^N$, $M \in \mathbb{R}$, and $z \equiv (x, \mu) \in Z$. Panels (a) and (b) display the consumption of each agent at z as a black dot with the identity of the agent next to it. Panel (a) also displays agents 1, 2, and 3's "indifference curves" through their respective allotment at z, i.e., bundles that are indifferent for the agent are joined by a line. Panel (b) displays agents 4 and 5's indifference curves at their respective allotment at z. Panel (b) also displays alternative preferences for agent 5, i.e., R'_5. To check whether $z \in \Re(R)$ or not one has to: (i) verify that $z \in F(R)$; and (ii) construct arrows from each agent's allotment to the other bundles at the allocation for which the agent is indifferent to her own consumption; then, verify that from each consumption bundle at the allocation there is a "path of arrows" which "flows" from the reference bundle to one of the bundles with minimal consumption of money. If z passes both tests, then $z \in \Re(R)$. If at least one of these two tests fails, then $z \notin \Re(R)$. Panel (c) displays this construction for z at profile R. Observe that z passes both tests. Thus, $z \in \Re(R)$. Panel (d) displays this construction for (R_{-5}, R'_5). Observe that z passes test (i), but not test (ii): there is no path of arrows flowing from the bundles with objects γ and θ to one of the bundles with minimal consumption of money. Thus, $z \notin \Re(R_{-5}, R'_5)$.

4 Manipulation of the GMRF solutions

In this section we study the manipulation of the GMRF solutions. Our main theorem characterizes the Nash and strong Nash equilibrium outcome correspondences of the game form associated with each $\Re_f$ and each of its selections on the domain $\mathcal{R}$. Each of these correspondences coincides with the solution that associates with each profile its set of *envy-free* allocations.

Theorem 1. Let $f \in \mathcal{I}^A$ and $r \in \Re_f$. For each $R^0 \in \mathcal{R}^N$,

$$\mathcal{O}\langle S(\mathcal{R})^N, \Re_f^r, R^0\rangle = \mathcal{O}^*\langle S(\mathcal{R})^N, \Re_f^r, R^0\rangle = F(R^0).$$

Given a GMRF solution, the outcomes from its manipulation are independent of the selection that completes the allocation process whenever the solution selects multiple outcomes for a reported profile. Thus, we can drop this parameter from our notation and conclude that for each preference profile the outcomes that result from the manipulation of a GMRF solution are the *envy-free* allocations for that profile. Since in our environment *envy-free* allocations are also *efficient* (Svensson, 1983), then we conclude that these outcomes are also *efficient*.

Corollary 1. Let $f \in \mathcal{I}^A$. For each $R^0 \in \mathcal{R}^N$,

$$\mathcal{O}\langle S(\mathcal{R})^N, \Re_f, R^0\rangle = \mathcal{O}^*\langle S(\mathcal{R})^N, \Re_f, R^0\rangle = F(R^0) \subseteq P(R^0).$$

Theorem 1 and Corollary 1 allow us to precisely assess the extent to which an agent, or a group of agents, can manipulate a GMRF solution. In equilibrium, and regardless of the fact that all agents may lie, the outcomes that ensue from the manipulation of these solutions are *envy-free* and *efficient* with respect to their true preferences. Thus, incentives are against the truthful revelation of preferences, but not against our basic criteria of economic justice.

Section 5 discusses the interpretation of Theorem 1 as an implementation result. Section 6 discusses its extension to situations in which the individual maximum consumption of money is restricted, as in the allocation of rooms and the division of the rent among housemates. Section 6 also discusses the extension of this theorem to a family of solutions defined by a dual construction to that of the GMRF solutions. For each preference profile, these dual solutions minimize the maximal "recalibrated" consumption of money among the *envy-free* allocations.

The proof of Theorem 1 follows from three lemmas. Let f, r, and R^0 be as in the statement of the theorem. A preliminary result, Lemma 1, states that if $(R, z^*) \in \mathcal{N}\langle S(\mathcal{D})^N, \Re_f, R^0\rangle$, $z \equiv \Re_f^r(R, z^*)$, and the strategy spaces are rich enough, then for any pair of objects $\{\alpha, \beta\}$, $\alpha \succeq(R, z)\, \beta$.[14] We provide here some intuition for the proof. Consider an agent, say i, and suppose that her object at z is β. Let G be the set of objects that do not dominate β in terms of the binary relation $\succeq(R, z)$, and $N(G)$ the set of agents who receive objects in G at z. The lemma states that G is empty. Indeed, we prove that if G is nonempty, then agent i has a profitable deviation at (R, z^*). Let $N(A \setminus G)$ be the set of agents who receive objects in $A \setminus G$ at z. Intuitively, if the consumptions of money of the members of $N(A \setminus G)$ slightly increase, then no envy is induced among the members of $N(G)$ with respect to their reported preferences. Thus, one can extract, in the aggregate, an amount of money ε from the members of $N(G)$, and distribute it among the members of $N(A \setminus G)$ without generating envy with respect to their reported preferences. We prove, and it turns out to be a delicate argument, that agent i can enforce such a redistribution by changing her preferences and moreover, that she benefits from the change. This last

[14] The lemma requires that $\mathcal{D}$ contain the quasi-linear domain. Instead, one could replace this assumption with the requirement that for each list of bundles $L \equiv (x_\alpha, \alpha)_{\alpha \in A}$, there be a preference in $\mathcal{D}$ for whom the bundles in L are all indifferent. The same notion of richness applies to Lemmas 2 and 3.

step involves identifying a specific "reallocation" z^ε and a preference $R'_i \in \mathcal{Q}$ such that: (i) agent i receives object β at z^ε, (ii) part of the structure of $\succeq(R,z)$ is preserved in $\succeq(R_{-i}, R'_i, z^\varepsilon)$ in such a way that $z^\varepsilon \in \Re_f(R_{-i}, R'_i)$, and (iii) for each $z' \in \Re_f(R_{-i}, R'_i)$, $z'_i = z^\varepsilon_i$. Since $i \in N(A \setminus G)$, then independently of the selector r, agent i benefits by reporting R'_i instead of R_i. The proof is in the Appendix.

Lemma 1. Let $\mathcal{D} \subseteq \mathcal{R}$ be such that $\mathcal{D} \supseteq \mathcal{Q}$, $f \in \mathcal{I}^A$, and $r \in \Re_f$. For each $R^0 \in \mathcal{R}^N$, if $(R, z^*) \in \mathcal{N}\langle S(\mathcal{D})^N, \Re^r_f, R^0\rangle$ then for each $\{\alpha, \beta\} \in A$, $\alpha \succeq(R, \Re^r_f(R, z^*))\, \beta$.

Lemma 1 considerably advances our understanding of equilibrium behavior in the manipulation of the GMRF solutions. In particular, it implies the following property. Let $\mathcal{D}$, f, r, and R^0 be as in the statement of the lemma. Let $(R, z^*) \in \mathcal{N}\langle S(\mathcal{D})^N, \Re^r_f, R^0\rangle$ and $z \equiv \Re^r_f(R, z^*)$. Then, for each pair of agents, say $\{i, k\}$, it is possible to reshuffle consumptions at z so that agent i receives z_k without generating envy, with respect to the reported preferences, among the agents different from agent i.

Our second result, Lemma 2, states that each outcome in $\mathcal{O}\langle S(\mathcal{D})^N, \Re^r_f, R^0\rangle$ is *envy-free* for R^0 whenever strategy spaces are rich enough.[15] Let $(R, z^*) \in \mathcal{N}\langle S(\mathcal{D})^N, \Re^r_f, R^0\rangle$ and $z \equiv \Re^r_f(R, z^*)$. We prove that if an agent, say i, prefers another agent's consumption at z to hers, then she can benefit by changing her report at (R, z^*). Indeed, under the hypothesis that for some agent, say k, $z_k\, P^0_i\, z_i$, we exhibit $R'_i \in \mathcal{Q}$ and $\hat{z} \in \Re_f(R_{-i}, R'_i, z^*)$ such that $\hat{z}_i\, P^0_i\, z_i$ and at each allocation in $\Re_f(R_{-i}, R'_i, z^*)$, agent i receives $\hat{z}_i$. Thus, independently of the selector r, agent i benefits by reporting R'_i instead of R_i. Here, as in the proof of Lemma 1, the delicate argument is the construction of R'_i and $\hat{z}$. From Lemma 1, it is possible to reshuffle consumptions at z so agent i receives z_k without generating envy, with respect to the reported preferences, among the agents different from agent i. Let z' be such a reallocation. Then agent i prefers z'_i to z_i. We prove that there are R'_i and $\hat{z}$ such that (i) $\hat{z}_i$ is arbitrarily close to z'_i (actually, equal to z'_i for some configurations of R and z), (ii) part of the structure of $\succeq(R, z')$ is preserved in $\succeq(R_{-i}, R'_i, \hat{z})$ in such a way that $\hat{z} \in \Re_f(R_{-i}, R'_i)$, and (iii) for each $z'' \in \Re_f(R_{-i}, R'_i)$, $z''_i = \hat{z}_i$. The proof is in the Appendix.

Lemma 2. Let $\mathcal{D} \subseteq \mathcal{R}$ be such that $\mathcal{D} \supseteq \mathcal{Q}$, $f \in \mathcal{I}^A$, and $r \in \Re_f$. For each $R^0 \in \mathcal{R}^N$,

$$\mathcal{O}\langle S(\mathcal{D})^N, \Re^r_f, R^0\rangle \subseteq F(R^0).$$

Finally our third result, Lemma 3, states that if strategy spaces are rich enough, then the Nash and strong Nash equilibrium outcome correspondences of the game form associated with each GMRF solution coincide with the *no-envy* solution.[16] The proof is in the Appendix.

Lemma 3. Let $\mathcal{D} \subseteq \mathcal{R}$ be such that $\mathcal{D} \supseteq \mathcal{Q}$, $f \in \mathcal{I}^A$, and $r \in \Re_f$. For each $R^0 \in \mathcal{R}^N$,

$$\mathcal{O}\langle S(\mathcal{D})^N, \Re^r_f, R^0\rangle = \mathcal{O}^*\langle S(\mathcal{D})^N, \Re^r_f, R^0\rangle = F(R^0).$$

[15] See Footnote 14.
[16] See Footnote 14.

The proof of Theorem 1 is a straightforward application of Lemma 3 in which the domain of admissible preferences is $\mathcal{D} = \mathcal{R}$.

5 Implementation of the no-envy solution

Given a solution C one can ask whether there exists a game form whose equilibrium correspondence coincides with C. Such a game form is said to implement C. Among all game forms, the ones associated with a solution are of particular interest. One could think of them as the allocation mechanisms that naturally emerge in an arbitration process (Sobel, 1981). That is, an arbitrator first identifies the desirable outcomes with no knowledge of preferences. Then, she asks agents to report preferences in order to select one of the desirable allocations for the particular situation. Thus, a follow up question is whether there exists a solution whose induced manipulation game forms implement C.

Formally, let C and B be two solutions, $\mathcal{D} \subseteq \mathcal{R}$, and $\mathcal{D}' \subseteq \mathcal{R}$. We say that $\langle \boldsymbol{S(\mathcal{D}')^N, B} \rangle$ **implements** $\boldsymbol{C}$ **in Nash equilibria on** $\boldsymbol{\mathcal{D}}$ if for each $R^0 \in \mathcal{D}$ and each $b \in B$, $\mathcal{O}\langle S(\mathcal{D}')^N, B^b, R^0 \rangle = C(R^0)$. The solution $\boldsymbol{B}$ **naturally implements** $\boldsymbol{C}$ **in Nash equilibria on** $\boldsymbol{\mathcal{D}}$ if $\langle S(\mathcal{D})^N, B \rangle$ implements C in Nash equilibria on $\mathcal{D}$.[17]

We say that $\langle \boldsymbol{S(\mathcal{D}')^N, B} \rangle$ **implements** $\boldsymbol{C}$ **in strong Nash equilibria on** $\boldsymbol{\mathcal{D}}$ if for each $R^0 \in \mathcal{D}$ and each $b \in B$, $\mathcal{O}^*\langle S(\mathcal{D}')^N, B^b, R^0 \rangle = C(R^0)$. The solution $\boldsymbol{B}$ **naturally implements** $\boldsymbol{C}$ **in strong Nash equilibria on** $\boldsymbol{\mathcal{D}}$ if $\langle S(\mathcal{D})^N, B \rangle$ implements C in strong Nash equilibria on $\mathcal{D}$.

We now show that our results have implications for the implementation of the *no-envy* solution. First is the natural implementation of F by the GMRF solutions.

Corollary 2. For each $f \in \mathcal{I}^A$, $\Re_f$ naturally implements F in Nash and strong Nash equilibria on $\mathcal{R}$.

Corollary 2 follows from Theorem 1. We omit the proof.

The following corollary states that for the purpose of implementing F, one can reduce strategy spaces to the sub-domain of quasi-linear preferences times the consumption space.

Corollary 3. Let $f \in \mathcal{I}^A$. For each $\mathcal{D} \subseteq \mathcal{R}$, $\langle S(\mathcal{Q})^N, \Re_f \rangle$ implements F in Nash and strong Nash equilibria on $\mathcal{D}$.

Corollary 2 follows from Lemma 3. We omit the proof.

One can think that a game form with simple strategy spaces is more "realistic." For instance, it is more realistic to imagine agents reporting elements of a finite dimensional space, as is the case in game form $\langle S(\mathcal{Q})^N, \Re^r \rangle$ for some $r \in \Re$. Another goal that one achieves with the reduction of strategy spaces is that the "complexity" of the mechanism may be reduced.

[17] Here "natural implementation" refers to the allocation by means of a game form in which agents have the opportunity to report their true preferences (of course, they may choose not to do it) and whose outcome function is justified by the normative appeal of the solution that induces it. Alternatively, in the context of exchange economies, Saijo et al. (1996) use the expression "natural implementation" to refer to the implementation by means of abstract game forms that satisfy a list of "simplicity" requirements.

In contrast to the unrestricted domain of preferences, given reports $(R, z) \in S(\mathcal{Q})^N$, there is a polynomially bounded algorithm to calculate $\Re^r(R, z)$ (Aragones, 1995). We emphasize though that since agents do not have the opportunity to "tell the truth," the outcomes of this game cannot be interpreted as the outcomes resulting from agents' manipulation.

6 Discussion

In this section we discuss the extension of our results. First, we consider the case in which individual consumptions of money are bounded. We concentrate on the requirement that consumptions of money be negative. A similar argument applies to other upper bounds on individual consumptions of money.

If consumptions of money are required to be negative, then one can interpret our model as pertaining to the allocation of rooms and the division of the rent among housemates (Su, 1999; Brams and Kilgour, 2001). We first observe that the set of *envy-free* allocations in which individual consumptions of money are negative may be empty in our domain $\mathcal{R}$ (this impossibility remains even if we restrict attention to quasi-linear preferences). This is the case when preferences coincide in that some rooms are considerably "inferior" to the other. Thus, the agents who receive such rooms must be compensated for no-envy to be satisfied.

Suppose that the rooms in a house do not differ "too much." Then, this restriction should be formalized and imposed in the agents' domain of admissible preferences. For instance, let $M < 0$ be the rent to collect in a particular application. Suppose that preferences are such that receiving a room for free is preferred to receiving any other room and paying an equal share of the rent:

$$\mathcal{D}^* \equiv \left\{ R_0 \in \mathcal{R} : \text{for each } \{\alpha, \beta\} \subseteq A,\ (0, \alpha)\, P_0 \left(\frac{1}{n} M, \beta \right) \right\}.$$

One can easily see that if preferences belong to $\mathcal{D}^*$, then at each *envy-free* allocation each agent contributes to the rent, i.e., has a negative consumption of money. Thus, all *envy-free* allocations are "interior" and all of our proofs can be easily modified to deliver a parallel result to Theorem 1 on this domain.

A dual construction to the GMRF allocations is that in which, instead of maximizing the minimal recalibrated consumption of money among the *envy-free* allocations, one minimizes the maximal recalibrated consumption of money among the *envy-free* allocations. Let us formally define this solution. The **Generalized Minmax Fair (GMF) solution with respect to $\boldsymbol{f}$** associates with each $R \in \mathcal{R}^N$ the set:

$$\mathbf{Minmax}_{\boldsymbol{f}}(\boldsymbol{R}) \equiv \underset{(x,\mu) \in F(R)}{\arg\min} \left\{ \max_{\beta \in A} f_\beta(x_\beta) \right\}.$$

Consider the GMF solution that selects for each preference profile, the allocations that minimize across agents the maximal individual consumption of money among the *envy-free* allocations. This solution selects allocations that respect the requirement that consumptions of money be negative whenever possible. Thus, it provides an attractive alternative to the GMRF solutions when it is desirable that consumptions of money be negative.

Using similar techniques to the ones developed above, one can characterize the GMF allocations with respect to a family of functions f: they are the *envy-free* allocations for which each object is dominated, in terms of the binary relation induced by preferences and the allocation, by one of the objects that is received by an agent whose f-transformed consumption of money is maximal. Moreover, one can characterize the Nash and Strong Nash equilibrium outcome correspondences of the game form associated with each Minmax_f and each of its selections on the domain $\mathcal{R}$. Each of these correspondences coincides with the *no-envy* solution.[18] Thus, a parallel statement to Theorem 1 holds for the family of the GMF solutions.

7 Appendix

Resources and population are fixed in our model. However, in the course of the proof of Lemmas 1, 2, and 3 we need to consider subproblems in which a subset of resources has to be allocated among a subgroup of agents. Thus, in order to facilitate our exposition, we introduce a variable-population and variable-resource environment. We consider problems in which a subset of objects, say $B \subsetneq A$, and some amount of money $m \in \mathbb{R}$ have to be distributed among the members of a subgroup of agents $K \subsetneq N$ such that $|K| = |B|$. For each $B \subsetneq A$, the domain of preferences on $\mathbb{R} \times B$ satisfying *money-monotonicity* and *no object is infinitely better than another* is $\mathcal{R}(B)$.[19] Let $K \subsetneq N$. An **economy** with agent set K is a triple $e \equiv (B, R, m)$, where $B \subsetneq A$ is such that $|B| = |K|$, $R \equiv (R_i)_{i \in K} \in \mathcal{R}(B)^K$, and $m \in \mathbb{R}$. The set of economies with agent set K is $\boldsymbol{\mathcal{E}^K}$. For each $x \in \mathbb{R}^B$, let $\boldsymbol{\overline{x}_B} \equiv \sum_{\beta \in B} x_\beta$. Let $K \subsetneq N$ and $e \equiv (B, R, m) \in \mathcal{E}^K$. An **allocation for $\boldsymbol{e}$** is a pair $z \equiv (x, \mu) \in \mathbb{R}^B \times B^K$ such that $\overline{x}_B = m$ and $\mu : K \to B$ is a bijection. The set of allocations for e is $Z(e)$. An allocation $z \in Z(e)$ is **envy-free for $\boldsymbol{e}$** if for each $\{i, j\} \subsetneq K$, $z_i \, R_i \, z_j$. The set of *envy-free* allocations for e is $\boldsymbol{F(e)}$.

We now define the GMRF allocations for the variable-population and variable-resource domain. Let $K \subsetneq N$, $B \subsetneq A$ be such that $|K| = |B|$, $f \equiv (f_\beta)_{\beta \in B} \in \mathcal{I}^B$, and $e \equiv (B, R, m) \in \mathcal{E}^K$. The set of **Generalized Money Rawlsian Fair (GMRF) allocations with respect to $\boldsymbol{f}$ for $\boldsymbol{e}$** is: $\boldsymbol{\Re_f(e)} \equiv \arg\max \left\{ \min_{\beta \in B} f_\beta(x_\beta) : (x, \mu) \in F(e) \right\}$.

Let $K \subsetneq N$, $e \equiv (B, R, m) \in \mathcal{E}^K$, and $z \equiv (x, \mu) \in Z(e)$. The binary relation on B, denoted $\boldsymbol{\succeq(R, z)}$, is defined as follows: for each pair $\{\alpha, \beta\} \subsetneq B$, $\alpha \succeq(R, z) \, \beta$ if and only if there are $\{\beta_0, \ldots, \beta_T\} \subsetneq B$ such that $\beta_0 = \alpha$, $\beta_T = \beta$, and

$$(x_{\beta_0}, \beta_0) \, I_{\mu^{-1}(\beta_0)} \, (x_{\beta_1}, \beta_1) \ldots (x_{\beta_{T-1}}, \beta_{T-1}) \, I_{\mu^{-1}(\beta_{T-1})} \, (x_{\beta_T}, \beta_T).$$

[18] The GMF allocations satisfy the properties stated in Proposition 3. Nevertheless, the characterization of the outcomes from the manipulation of the GMF solutions involves a non-symmetric construction to the one that proves Theorem 1.

[19] See Section 2.1.

The following lemma states two basic properties of $\succeq(R,z)$: (i) this relation is reflexive and transitive, and (ii) the statement $\alpha \succeq(R,z)\beta$ is invariant under changes in the preferences of the agent who receives object β at z. We omit the straightforward proof.

Lemma 4. Let $K \subseteq N$, $B \subseteq A$ such that $|B| = |K|$, and $e \equiv (B, R, m) \in \mathcal{E}^K$. For each $z \equiv (x,\mu) \in Z(e)$, $\succeq(R,z)$ is reflexive and transitive. Moreover, for each $i \in K$ and each $R'_i \in \mathcal{R}$, if $\alpha \in B$ is such that $\alpha \succeq(R,z)\,\mu(i)$, then $\alpha \succeq(R_{-i}, R'_i, z)\,\mu(i)$.

Proposition 1 extends to the variable-population and variable-resource domain. We omit the proof.

Proposition 2. Let $K \subseteq N$, $B \subseteq A$ be such that $|B| = |K|$, $f \in \mathcal{I}^B$, $e \equiv (B, R, m) \in \mathcal{E}^K$, and $z \in Z(e)$. Then $z \in \Re_f(e)$ if and only if $z \in F(e)$ and for each $\alpha \in B$, there is $\beta \in \arg\min_{\delta \in B} f_\delta(x_\delta)$ such that $\alpha \succeq(R,z)\,\beta$.

The following proposition states two properties of the GMRF allocations that are used in the sequel. We refer the reader to Alkan et al. (1991, Theorem 6) for a proof.[20] Our conventions for vector inequalities are as follows. Let $B \subseteq A$. For each $\{x, x'\} \subseteq \mathbb{R}^B$ we write $x' \geq x$ if for each $\beta \in B$, $x'_\beta \geq x_\beta$; we write $x' \gg x$ if for each $\beta \in B$, $x'_\beta > x_\beta$.

Proposition 3. Let $K \subseteq N$, $B \subseteq A$ such that $|B| = |K|$, and $e \equiv (B, R, m) \in \mathcal{E}^K$. For each $z \equiv (x,\mu) \in \Re_f(e)$ and each $z' \equiv (x',\mu') \in \Re_f(e)$, $x = x'$ and for each $i \in K$, $z'\, I_i\, z$. For each $\varepsilon \in \mathbb{R}_{++}$, each $z \equiv (x,\mu) \in \Re_f(e)$, and each $z' \equiv (x',\mu') \in \Re_f(B, R, m+\varepsilon)$, $x' \gg x$ and for each $i \in K$, $z'\, P_i\, z$.

We now present the proof of the results of Section 4.

Proof of Lemma 1. Let $\mathcal{D}$, f, and r be as in the statement of the lemma. Let $R^0 \in \mathcal{R}^N$. Suppose that $(R, z^*) \in \mathcal{N}\langle S(\mathcal{D})^N, \Re^r_f, R^0\rangle$. Let $z \equiv (x,\mu) \equiv \Re^r_f(R, z^*)$. We prove that for each $\{\alpha, \beta\} \in A$, $\alpha \succeq(R,z)\,\beta$. Let $\beta \in A$. Suppose by means of contradiction that $\{\alpha \in A : \text{it is not the case that } \alpha \succeq(R,z)\,\beta\}$ is non-empty. Let $i \equiv \mu^{-1}(\beta)$. We prove that there is a profitable deviation for agent i. We proceed in four steps. In Steps 1, 2, and 3 we construct an allocation z^ε such that: (i) $z^\varepsilon\, P^0_i\, z$ and (ii) there is $R'_i \in \mathcal{Q}$ such that for each $\hat{z} \in \Re_f(R_{-i}, R'_i)$, $\hat{z}_i = z^\varepsilon_i$. In Step 4 we conclude that (R, z^*) is not an equilibrium.

Step 1: Defining sets H and G. Let $H \equiv \{\alpha \in A : \text{it is not the case that } \alpha \succeq (R,z)\,\beta\}$ and $G \equiv \{\delta \in A : \delta \succeq(R,z)\,\beta\}$. Clearly, $H = A \setminus G$ and $H \neq \emptyset$. Since $\succeq(R,z)$ is reflexive, then $\mu(i) = \beta \in G$. Thus, $G \neq \emptyset$.

Step 2: Constructing allocation z^ε. Let $N(G) \equiv \mu^{-1}(G)$, $N(H) \equiv \mu^{-1}(H)$, $z|_G \equiv (x|_G, \mu|_{N(G)})$, $z|_H \equiv (x|_H, \mu|_{N(H)})$, $R|_{N(G)} \equiv R_{N(G)}|_{\mathbb{R}\times G}$, and $R|_{N(H)} \equiv R_{N(H)}|_{\mathbb{R}\times H}$.

• **Defining ε (an amount to be collected and redistributed).** We claim that for each $\alpha \in H$ and each $\delta \in G$, $(x_\alpha, \alpha)\, P_{\mu^{-1}(\alpha)}\, (x_\delta, \delta)$. Suppose by means of contradiction that there are $\alpha \in H$ and $\delta \in G$, such that $(x_\delta, \delta)\, R_{\mu^{-1}(\alpha)}\, (x_\alpha, \alpha)$. Since $z \in \Re_f(R) \subseteq F(R)$, then $(x_\alpha, \alpha)\, R_{\mu^{-1}(\alpha)}\, (x_\delta, \delta)$. Thus, $(x_\alpha, \alpha)\, I_{\mu^{-1}(\alpha)}\, (x_\delta, \delta)$ and consequently, $\alpha \succeq (R,z)\,\delta$. Since $\delta \in G$, then $\delta \succeq (R,z)\,\beta$. By Lemma 4, $\alpha \succeq (R,z)\,\beta$ and thus, $\alpha \in G$. This is a

[20] One can also easily prove Proposition 3 as a corollary of Proposition 2.

contradiction. By continuity of preferences, there is $\varepsilon \in \mathbb{R}_{++}$ such that for each $\alpha \in H$ and each $\delta \in G$, $(x_\alpha - \varepsilon, \alpha)\, P_{\mu^{-1}(\alpha)}\, (x_\delta + \varepsilon, \delta)$.

• **Collecting $\boldsymbol{\varepsilon}$ from $\boldsymbol{N(H)}$:** Since $z \in F(R)$, then $z|_H \in F(H, R|_{N(H)}, \overline{x}_H)$. We claim that for each $\alpha \in H$ there is $\rho \in H \cap \arg\min_{\delta \in A} f_\delta(x_\delta)$ such that $\alpha \succeq (R|_{N(H)}, z|_{N(H)})\, \rho$.[21] Since $z \in \Re_f(R)$, then by Proposition 2 there is $\rho \in \arg\min_{\delta \in A} f_\delta(x_\delta)$ such that $\alpha \succeq (R, z)\, \rho$. That is, there are $\{\beta_0, \ldots, \beta_T\} \subseteq A$ such that $\beta_0 = \alpha$, $\beta_T = \rho$, and $(x_{\beta_0}, \beta_0)\, I_{\mu^{-1}(\beta_0)} \ldots I_{\mu^{-1}(\beta_{T-1})}\, (x_{\beta_T}, \beta_T)$. We prove that $\{\beta_0, \ldots, \beta_T\} \subseteq H$. Suppose by means of contradiction that for some t, $\beta_t \in G$. Thus, $\beta_t \succeq (R, z)\, \beta$. By Lemma 4, $\alpha \succeq (R, z)\, \beta$ and thus, $\alpha \in G$. This is a contradiction. Thus, $\alpha \succeq (R|_{N(H)}, z|_{N(H)})\, \rho$. By Proposition 2, $z|_H \in \Re_f(H, R|_{N(H)}, \overline{x}_H)$. Let $\boldsymbol{y \equiv (u, \sigma)} \in \Re_f(H, R|_{N(H)}, \overline{x}_H - \varepsilon)$.

• **Distributing $\boldsymbol{\varepsilon}$ among $\boldsymbol{N(G)}$:** Since $z \in F(R)$, then $z|_G \in F(G, R|_{N(G)}, \overline{x}_G)$. Thus, under preferences $R|_{N(G)}$ no agent in $N(G) \setminus \{i\}$ envies any other agent at $z|_G$. We claim that for each $\delta \in G$, $\delta \succeq (R|_{N(G)}, z|_G)\, \mu(i) = \beta$. Let $\delta \in G$. Since $\delta \succeq (R, z)\, \beta$, then there are $\{\beta_0, \ldots, \beta_T\} \subseteq A$ such that $\beta_0 = \delta$, $\beta_T = \beta$, and

$$(x_{\beta_0}, \beta_0)\, I_{\mu^{-1}(\beta_0)}\, (x_{\beta_1}, \beta_1) \ldots (x_{\beta_{T-1}}, \beta_{T-1})\, I_{\mu^{-1}(\beta_{T-1})}\, (x_{\beta_T}, \beta_T).$$

Thus, $\{\beta_0, \ldots, \beta_T\} \subseteq G$. Consequently, $\delta \succeq (R|_{N(G)}, z|_G)\, \beta$. We claim that there is $\boldsymbol{v \equiv (w, \lambda)} \in Z(G, R_{N(G)}, \overline{x}_G + \varepsilon)$ such that:

(1) $w \gg x|_G$.

(2) For each $j \in N(G) \setminus \{i\}$ and each $l \in N(G)$, $v_j\, R_j\, v_l$.

(3) $\lambda(i) = \mu|_G(i)$.

(4) For each $\delta \in G$, $\delta \succeq (R|_{N(G)}, v)\, \lambda(i)$.

Let $\widehat{R}_i \in \mathcal{R}(G)$ be a preference relation such that for each $\delta \in G \setminus \{\mu(i)\}$, $(x_{\mu(i)}, \mu(i))\, \widehat{P}_i\, (x_\delta + \varepsilon, \delta)$. Let $R|_{N(G) \setminus \{i\}}$ be the profile obtained from $R|_{N(G)}$ by removing i's preferences. Since For each $j \in N(G) \setminus \{i\}$ and each $l \in N(G)$, $z_j\, R_j\, z_l$, then $z|_G \in F(G, R|_{N(G) \setminus \{i\}}, \widehat{R}_i, m)$. Let $\delta \in G$. Since $\delta \succeq (R|_{N(G)}, z|_G)\, \mu(i)$, then by Lemma 4, $\delta \succeq (R|_{N(G) \setminus \{i\}}, \widehat{R}_i, z|_G)\, \mu(i)$. By Proposition 2, $z|_G \in \Re_{\mu(i)}(G, R|_{N(G) \setminus \{i\}}, \widehat{R}_i, m)$. Let $v \equiv (w, \lambda) \in \Re_{\mu(i)}(G, R|_{N(G) \setminus \{i\}}, \widehat{R}_i, m + \varepsilon)$. By Proposition 3, $w \gg x|_G$. Since $\Re_{\mu(i)}(G, R|_{N(G) \setminus \{i\}}, \widehat{R}_i, m + \varepsilon) \subseteq F(G, R|_{N(G) \setminus \{i\}}, \widehat{R}_i, m + \varepsilon)$, then for each $j \in N(G) \setminus \{i\}$ and each $l \in N(G)$, $v_j\, R_j\, v_l$. Let $\delta \in G \setminus \{\mu(i)\}$. Since $(x_{\mu(i)}, \mu(i))\, \widehat{P}_i\, (x_\delta + \varepsilon, \delta)$ and $w \gg x|_G$, then $(x_{\mu(i)} - \varepsilon, \mu(i))\, \widehat{P}_i\, (w_\delta, \delta)$. Since $v \in F(G, R|_{N(G) \setminus \{i\}}, \hat{R}_i, m + \varepsilon)$, then $\lambda(i) = \mu(i)$. Finally, let $\delta \in G$. Since $v \in \Re_{\mu^\varepsilon(i)}(G, R|_{N(G) \setminus \{i\}}, \widehat{R}_i, m + \varepsilon)$, then by Proposition 2, $\delta \succeq (R|_{N(G) \setminus \{i\}}, \widehat{R}_i, v)\, \lambda(i)$. By Lemma 4, $\delta \succeq (R|_{N(G)}, v)\, \lambda(i)$.

• **Defining $\boldsymbol{z^\varepsilon}$:** Let x^ε be the vector obtained by concatenating u and w, i.e., the vector defined by: $x^\varepsilon|_H \equiv u$ and $x^\varepsilon|_G \equiv w$. Let μ^ε be the bijection that coincides with σ on H and with λ on G. Let $z^\varepsilon \equiv (x^\varepsilon, \mu^\varepsilon)$.

• **Claim 1: For each $\boldsymbol{\alpha \in H}$, $\boldsymbol{x^\varepsilon_\alpha \in (x_\alpha - \varepsilon, x_\alpha)}$ and for each $\boldsymbol{\delta \in G}$, $\boldsymbol{x^\varepsilon_\delta \in (x_\delta, x_\delta + \varepsilon)}$.** Let $\alpha \in H$. By Proposition 3, $x|_H \gg u$. Thus, $x^\varepsilon_\alpha < x_\alpha$. Now, since $\overline{u}_H = \overline{x}_H - \varepsilon$, then $x^\varepsilon_\alpha > x_\alpha - \varepsilon$. A symmetric argument shows that for each $\delta \in G$,

[21] This proves, in particular, that $H \cap \arg\min_{\delta \in A} f_\delta(x_\delta) \neq \emptyset$.

$x^\varepsilon_\delta \in (x_\delta, x_\delta + \varepsilon)$.

Step 3: There is $R'_i \in \mathcal{Q}$ such that for each $\hat{z} \in \Re_f(R_{-i}, R'_i, z^*)$, $\hat{z}_i = z^\varepsilon_i$.

• **Defining R'_i:** Fix $\rho \in \arg\min_{\alpha\in H} f_\alpha(w_\alpha)$. Let $R'_i \in \mathcal{Q}$ be such that $(x^\varepsilon_\beta, \beta)\, I'_i\, (x^\varepsilon_\rho, \rho)$ and for each $\delta \in A \setminus \{\beta, \rho\}$, $(x^\varepsilon_\beta, \beta)\, P'_i\, (x_\delta + \varepsilon, \delta)$.

• **Claim 2: $z^\varepsilon \in F(R_{-i}, R'_i)$.** Since $\overline{x}^\varepsilon_A = \overline{u}_H + \overline{w}_G = M$, then $z^\varepsilon \in Z(A, R_{-i}, R'_i, M)$. Since for each $\alpha \in A$, $x_\alpha + \varepsilon \geq x^\varepsilon_\alpha$, then for each $j \in N$, $z^\varepsilon_i\, R'_i\, z^\varepsilon_j$. It remains to prove that for each $j \in N \setminus \{i\}$ and each $l \in N$, $z^\varepsilon_j\, R_j\, z^\varepsilon_l$. There are four cases.

Case 1: $\{j, l\} \subseteq N(H)$. Since $y \in \Re_f\left(H, R|_{N(H)}, \overline{x}_H - \varepsilon\right)$ then $y \in F\left(H, R|_{N(H)}, \overline{x}_H - \varepsilon\right)$. Thus, $z^\varepsilon_j = y_j\, R_j\, y_l = z^\varepsilon_l$.

Case 2: $\{j, l\} \subseteq N(G)$. By property (2) of v and since $j \neq i$, $v_j\, R_j\, v_l$. Thus, $z^\varepsilon_j = v_j\, R_j\, v_l = z^\varepsilon_l$.

Case 3: $j \in N(H)$ and $l \subseteq N(G)$. By the same argument as in Case 1, $z^\varepsilon_j = y_j\, R_j\, (u_{\mu(j)}, \mu(j))$. Since $u_{\mu(j)} > x_{\mu(j)} - \varepsilon$, then $(u_{\mu(j)}, \mu(j))\, P_j\, (x_{\mu(j)} - \varepsilon, \mu(j))$. Let $\delta \equiv \mu^\varepsilon(l)$. Thus, $\delta \in G$. Recall that $(x_{\mu(j)} - \varepsilon, \mu(j))\, P_j\, (x_\delta + \varepsilon, \delta)$. Since $x_\delta + \varepsilon > w_\delta$, then $(x_\delta + \varepsilon, \delta)\, P_j\, (w_\delta, \delta) = z^\varepsilon_l$. Altogether, $z^\varepsilon_j\, P_j\, z^\varepsilon_l$.

Case 4: $j \subseteq N(G) \setminus \{i\}$ and $l \in N(H)$. By property (2) of v and since $j \neq i$, then $z^\varepsilon_j = v_j\, R_j\, (w_{\mu(j)}, \mu(j))$. Since $w \gg x|_G$, then $(w_{\mu(j)}, \mu(j))\, P_j\, (x_{\mu(j)}, \mu(j))$. Since $z \in \Re_f(R)$, then $(x_{\mu(j)}, \mu(j))\, R_j\, (x_{\mu^\varepsilon(l)}, \mu^\varepsilon(l))$. Now, since $x|_H \gg u$, then $(x_{\mu^\varepsilon(l)}, \mu^\varepsilon(l))\, P_j\, y_l = z^\varepsilon_l$. Altogether, $z^\varepsilon_j\, P_j\, z^\varepsilon_l$.

• **Claim 3: For each $\alpha \in A$, there is $\eta \in \arg\min_{\delta\in A} f_\delta(x^\varepsilon_\delta)$ such that $\alpha \succeq(R_{-i}, R'_i, z^\varepsilon)\, \eta$.** First, since $H \cap \arg\min_{\delta\in A} x_\delta \neq \emptyset$, $x|_H \gg u$, $w \gg x|_G$, and $f \in \mathcal{I}^A$, then $\arg\min_{\delta\in H} f_\delta(u_\delta) = \arg\min_{\delta\in A} f_\delta(x^\varepsilon_\delta)$. There are two cases.

Case 1: $\alpha \in H$. Since $y \in \Re_f(A\setminus G, R|_{N(H)}, \overline{x}_H - \varepsilon)$, then there is $\eta \in \arg\min_{\delta\in H} f_\delta(u_\delta)$ such that $\alpha \succeq(R|_{N(H)}, y)\, \eta$. Thus, $\alpha \succeq(R_{-i}, R'_i, z^\varepsilon)\, \eta$.

Case 2: $\alpha \in G$. By property (4) of v, $\alpha \succeq(R|_G, v)\, \lambda(i)$. Thus, $\alpha \succeq(R, z^\varepsilon)\, \lambda(i)$. By Lemma 4, $\alpha \succeq(R_{-i}, R'_i, z^\varepsilon)\, \lambda(i)$. Now, recall from the definition of R'_i that $z^\varepsilon_i\, I'_i\, (x^\varepsilon_\rho, \rho)$ where $\rho \in \arg\min_{\delta\in H} f_\delta(u_\delta)$. Thus, $\lambda(i) \succeq(R_{-i}, R'_i, z^\varepsilon)\, \rho$. By Lemma 4, $\alpha \succeq(R_{-i}, R'_i, z^\varepsilon)\, \rho$.

• **Claim 4: $z^\varepsilon \in \Re_f(R_{-i}, R'_i)$.** Since $z^\varepsilon \in F(R_{-i}, R'_i)$ and for each $\alpha \in A$ there is $\eta \in \arg\min_{\delta\in A} f_\delta(x^\varepsilon_\delta)$ such that $\alpha \succeq(R_{-i}, R'_i, z^\varepsilon)\, \eta$, then by Proposition 2, $z^\varepsilon \in \Re_f(R_{-i}, R'_i)$.

• **Claim 5: For each $\hat{z} \in \Re_f(R_{-i}, R'_i)$, $\hat{z}_i = z^\varepsilon_i$.** Let $\hat{z} \equiv (\hat{x}, \hat{\mu}) \in \Re_f(R_{-i}, R'_i)$. By Proposition 3, $\hat{x} = x$ and $\hat{z}_i\, I'_i\, z^\varepsilon_i$. By definition of R'_i, for each $\alpha \in A \setminus \{\beta, \rho\}$, $z^\varepsilon_i = (w_\beta, \beta)\, P'_i\, (x_\alpha + \varepsilon, \alpha)\, P'_i\, (x^\varepsilon_\alpha, \alpha)$. Thus, $\hat{z}_i = z^\varepsilon_i$ or $\hat{z}_i = (x^\varepsilon_\rho, \rho)$. Recall from Case 3 in the proof that $z^\varepsilon \in F(R_{-i}, R'_i)$, that for each $j \in N(H)$ and each $l \subseteq N(G)$, $z^\varepsilon_j\, P_j\, z^\varepsilon_l$. Since $N_{\hat{z}z} = N$, then for each $j \in N(H)$, $\hat{\mu}(j) \in H$. Now, since $\hat{\mu}$ is a bijection and $|N(H)| = |H|$, then $\hat{\mu}(N(H)) = H$. Since $i \in N(G)$ and $\rho \in H$, then $\hat{\mu}(i) \neq \rho$. Thus, $\hat{z}_i = z^\varepsilon_i$.

Step 4: Concluding that $(R, z) \notin \mathcal{N}\langle S(\mathcal{D})^N, \Re^r_f, R^0\rangle$.

Since $z^\varepsilon_i = (x^\varepsilon_\beta, \beta)$ and $x^\varepsilon_\beta = w_\beta > x_\beta$, then $z^\varepsilon\, P^0_i\, z$. Since for each $\hat{z} \in \Re_f(R_{-i}, R'_i)$, $\hat{z}_i = z^\varepsilon_i$, then

$$\Re^r_f(R_{-i}, R'_i, z^*)\, P^0_i\, \Re^r_f(R, z^*).$$

Thus, $(R, z^*) \notin \mathcal{N}\langle S(\mathcal{D})^N, \Re^r_f, R^0\rangle$. This is a contradiction. □

Proof of Lemma 2. Let $\mathcal{D}$, f, and r be as in the statement of the lemma. Let $R^0 \in \mathcal{R}^N$.

We prove that $\mathcal{O}\langle S(\mathcal{D})^N, \Re_f^r, R^0\rangle \subseteq F(R^0)$. Let $(R, z^*) \in \mathcal{N}\langle S(\mathcal{D})^N, \Re_f^r, R^0\rangle$. We claim that $\Re_f^r(R, z^*) \in F(R^0)$. Suppose by means of contradiction that $\Re_f^r(R, z^*) \notin F(R^0)$. Let $z \equiv (x, \mu) \equiv \Re_f^r(R, z^*)$. Since $z \notin F(R^0)$, then there are $\{i, k\} \subseteq N$ such that $z_k \, P_i^0 \, z_i$. There are two cases:

Case 1: $\boldsymbol{\mu(k) \in \arg\min_{\alpha \in A} f_\alpha(x_\alpha)}$.

We prove that there is a profitable deviation for agent i. We proceed in four steps. In Steps 1, 2, and 3 we construct an allocation z' such that: (i) $z' \, P_i^0 \, z$ and (ii) there is $R_i' \in \mathcal{Q}$ such that for each $\hat{z} \in \Re_f(R_{-i}, R_i')$, $\hat{z}_i = z_i'$. In Step 4 we conclude that (R, z^*) is not an equilibrium.

Step 1: Constructing allocation $\boldsymbol{z'}$.

By Lemma 1, $\mu(k) \succeq(R, z) \, \mu(i)$. Thus, there are $\{\alpha_t\}_{t=0}^T \subseteq A$ such that $\alpha_0 = \mu(k)$, $\alpha_T = \mu(i)$, and $(x_{\alpha_0}, \alpha_0) \, I_{\mu^{-1}(\alpha_0)} \ldots I_{\mu^{-1}(\alpha_{T-1})} \, (x_{\alpha_T}, \alpha_T)$. Let $L \equiv \{\alpha_0, \ldots, \alpha_T\}$ and $O \equiv A \setminus L$. Let $z' = (x', \mu') \in Z(A, R, M)$ be the allocation defined as follows: (1) for each $j \in N$ such that $\mu(j) \in O$, $z_j' \equiv z_j$; (2) for each $j \in N$ such that there is $t \in \{0, \ldots, T-1\}$, such that $\mu(j) = \alpha_t$, let $z_j' \equiv (x_{\alpha_{t+1}}, \alpha_{t+1})$; and (3) let $z_i' \equiv (x_{\alpha_0}, \alpha_0)$. Observe that $x' = x$.

Step 2: Properties of allocation $\boldsymbol{z'}$.

• **Claim 1: For each $\boldsymbol{R_i' \in \mathcal{R}}$ and each $\boldsymbol{\alpha \in A}$, $\boldsymbol{\alpha \succeq(R_{-i}, R_i', z') \, \alpha_0}$.** Let $R_i' \in \mathcal{R}$. By the definition of z', $(x_{\alpha_T}, \alpha_T) \, I_{(\mu')^{-1}(\alpha_T)} \, (x_{\alpha_{T-1}}, \alpha_{T-1}) \ldots I_{(\mu')^{-1}(\alpha_1)} \, (x_{\alpha_0}, \alpha_0)$. Thus, if $\alpha \in L$, $\alpha \succeq(R_{-i}, R_i', z') \, \alpha_0$. Suppose now that $\alpha \in O$. Since $\alpha \succeq(R, z)\alpha_0$, then there are $\{\beta_t\}_{t=0}^{T'} \subseteq A$ such that $\beta_0 = \alpha$, $\beta_{T'} = \mu(i)$, and $(x_{\beta_0}, \beta_0) \, I_{\mu^{-1}(\beta_0)} \ldots I_{\mu^{-1}(\beta_{T'-1})} \, (x_{\beta_{T'}}, \beta_{T'})$. Let $t^* \equiv \min\{0 \leq t \leq T : \beta_t \in L\}$. The index t^* is well defined because $\beta_{T'} \in L$. Since $\beta_0 \in O$, then $t^* > 0$. Since for each $\beta \in O$, $\mu^{-1}(\beta) = (\mu')^{-1}(\beta)$, then $(x_{\beta_0}, \beta_0) \, I_{(\mu')^{-1}(\beta_0)} \ldots I_{(\mu')^{-1}(\beta_{t^*-1})} \, (x_{\beta_{t^*}}, \beta_{t^*})$. Thus, $\beta_0 \succeq (R_{-i}, R_i', z') \, \beta_{t^*}$. Since $\beta_{t^*} \succeq (R_{-i}, R_i', z') \, \alpha_0$, then by Lemma 4, $\alpha = \beta_0 \succeq(R_{-i}, R_i', z') \, \alpha_0$.

• **Claim 2: For each $\boldsymbol{j \subseteq N \setminus \{i\}}$ and each $\boldsymbol{l \in N}$, $\boldsymbol{z_j' \, R_j \, z_l'}$.** There are two subcases.

Subcase 1: $\mu(j) \in O$. Since $z_j' = z_j$, $z \in F(R)$, and $x' = x$, then for each $l \in N$, $z_j' \, R_j \, z_l'$.

Subcase 2: $\mu(j) \in L$. Then, there is $t \in \{0, \ldots, T-1\}$ such that $\mu(j) = \alpha_t$. Since $z_j' = (x_{\alpha_{t+1}}, \alpha_{t+1})$, $(x_{\alpha_t}, \alpha_t) \, I_j \, (x_{\alpha_{t+1}}, \alpha_{t+1})$, $z \in F(R)$, and $x' = x$, then for each $l \in N$, $z_j' \, R_j \, z_l$.

Step 3: There is $\boldsymbol{R_i' \in \mathcal{Q}}$ such that for each $\boldsymbol{\hat{z} \in \Re(R_{-i}, R_i')}$, $\boldsymbol{\hat{z}_i = z_i'}$.

• **Defining $\boldsymbol{R_i'}$:** Let $R_i' \in \mathcal{Q}$ be a quasi-linear preference such that for each $\alpha \in A \setminus \{\alpha_0\}$, $(x_{\alpha_0}, \alpha_0) = z_i' \, P_i' \, (x_\alpha', \alpha)$.

• **Claim 3: $\boldsymbol{z' \in \Re_f(R_{-i}, R_i')}$.** By Step 2, $z' \in F(R_{-i}, R_i')$ and for each $\alpha \in A$, $\alpha \succeq(R_{-i}, R_i', z') \, \alpha_0$. Since $\alpha_0 \in \arg\min_{\alpha \in A} f_\alpha(x_\alpha')$, then by Proposition 2, $z' \in \Re_f(R_{-i}, R_i')$.

• **Claim 4: For each $\boldsymbol{\hat{z} \in \Re_f(R_{-i}, R_i')}$, $\boldsymbol{\hat{z}_i = z_i'}$.** Let $\hat{z} \equiv (\hat{x}, \hat{\mu}) \in \Re_f(R_{-i}, R_i')$. By Proposition 3, $\hat{z}_i \, I_i' \, z_i'$. Since for each $\alpha \in A \setminus \{\alpha_0\}$, $z_i' \, P_i' \, (x_\alpha', \alpha)$, then $\hat{\mu}(i) = \mu'(i)$. Thus, $\hat{z}_i = z_i'$.

Step 4: Concluding that $\boldsymbol{(R, z^*) \notin \mathcal{N}\langle S(\mathcal{D})^N, \Re_f^r, R^0\rangle}$. Since $z_i' = z_k$, then

$z' \, P_i^0 \, z$. Since for each $\hat{z} \in \Re_f(R_{-i}, R'_i)$, $\hat{z}_i = z'_i$, then

$$\Re_f^r(R_{-i}, R'_i, z^*) \, P_i^0 \, \Re_f^r(R, z^*).$$

Thus, $(R, z^*) \notin \mathcal{N}\langle S(\mathcal{D})^N, \Re_f^r, R^0\rangle$. This is a contradiction.

Case 2: $\boldsymbol{\mu(k) \notin \arg\min_{\alpha \in A} f_\alpha(x_\alpha)}$.

We prove that there is a profitable deviation for agent i. We proceed in four steps. In Steps 1, 2, and 3 we construct an allocation z^ε such that: (i) $z^\varepsilon \, P_i^0 \, z$ and (ii) there is $R'_i \in \mathcal{Q}$ such that for each $\hat{z} \in \Re_f(R_{-i}, R'_i)$, $\hat{z}_i = z_i^\varepsilon$. In Step 4 we conclude that (R, z^*) is not an equilibrium. Let $z' \equiv (x', \mu') \in Z(e)$ be the allocation constructed in Case 1.

Step 1: Defining sets $\boldsymbol{H}$ and $\boldsymbol{G}$. Let $H \subseteq A$ be the set of objects α such that there are $\{\beta_0, \ldots, \beta_T\} \subseteq A \setminus \{\mu(k)\}$ such that $\beta_0 = \alpha$, $\beta_T \in \arg\min_{\alpha \in A} f_\alpha(x_\alpha)$, and

$$(x'_{\beta_0}, \beta_0) \, I_{(\mu')^{-1}(\beta_0)} \, \ldots I_{(\mu')^{-1}(\beta_{T-1})} \, (x'_{\beta_T}, \beta_T).$$

Since $\arg\min_{\alpha \in A} f_\alpha(x_\alpha) \neq \emptyset$ and $\mu(k) \notin \arg\min_{\alpha \in A} f_\alpha(x_\alpha)$, then $\arg\min_{\alpha \in A} f_\alpha(x_\alpha) \subseteq H$. Thus, $H \neq \emptyset$. Let $G \equiv A \setminus H$. Clearly, $\mu(k) \in G$. Thus, $G \neq \emptyset$.

Step 2: Constructing allocation $\boldsymbol{z^\varepsilon}$. Recall that $z'_i = z_k$. Let $N(H) \equiv (\mu')^{-1}(H)$, $N(G) \equiv (\mu')^{-1}(G)$, $R|_{N(H)} \equiv R_{N(H)}|_{\mathbb{R} \times H}$, $R|_{N(G)} \equiv R_{N(G)}|_{\mathbb{R} \times G}$, $z'|_H \equiv (x'|_H, \mu'|_{N(H)}) \in Z(H, R|_{N(H)}, \overline{x}'_H)$, and $z'|_G \equiv (x'|_G, \mu'|_{N(G)}) \in Z(G, R|_{N(G)}, \overline{x}'_G)$.

• **Defining $\boldsymbol{\varepsilon}$ (an amount to be collected and redistributed).** We claim that for each $j \in N(G) \setminus \{i\}$ and each $l \in N(H)$, $z'_j \, P_j \, z'_l$. Suppose by means of contradiction that there are $j \in N(G) \setminus \{i\}$ and $l \in N(H)$ such that $z'_l \, R_j \, z'_j$. Recall from Step 2 in Case 1 that $z'_j \, R_j \, z'_l$. Thus, $z'_j \, I_j \, z'_l$. Thus, $\mu'(j) \in H$ and therefore $j \in N(H)$. This is a contradiction. By continuity of preferences, there is $\varepsilon \in \mathbb{R}_{++}$ such that for each $j \in N(G) \setminus \{i\}$ and each $l \in N(H)$, $(x_{\mu'(j)} - \varepsilon, \mu'(j)) \, P_j \, (x_{\mu'(l)} + \varepsilon, \mu'(l))$. Since $\arg\min_{\alpha \in A} f_\delta(x'_\alpha) \subseteq H$, then ε can be chosen small enough such that $\min_{\alpha \in H} f_\alpha(x'_\alpha + \varepsilon) < \min_{\delta \in G} f_\delta(x'_\delta - \varepsilon)$. Moreover, since $(x'_{\mu(k)}, \mu(k)) \, P_i^0 \, (x'_{\mu(i)}, \mu(i))$, then ε can be chosen small enough such that $(x'_{\mu(k)} - \varepsilon, \mu(k)) \, P_i^0 \, (x'_{\mu(i)}, \mu(i)) = z_i$.

• **Distributing $\boldsymbol{\varepsilon}$ among $\boldsymbol{N(H)}$.** Recall from Step 2 in Case 1 that for each $j \subseteq N \setminus \{i\}$ and each $l \in N$, $z'_j \, R_j \, z'_l$. Thus, $z'|_H \in F(H, R|_{N(H)}, \overline{x}'_H)$. By definition of H, for each $\alpha \in H$, there are $\{\beta_0, \ldots, \beta_{T'}\} \subseteq H$ such that $\beta_0 = \alpha$, $\beta_{T'} \in \arg\min_{\delta \in A} f_\delta(x_\delta)$, and

$$(x'_{\beta_0}, \beta_0) \, I_{(\mu')^{-1}(\beta_0)} \, \ldots I_{(\mu')^{-1}(\beta_{T'-1})} \, (x'_{\beta_{T'}}, \beta_{T'}).$$

Thus, for each $\alpha \in H$ there is $\beta \in \arg\min_{\delta \in H} f_\delta(x_\delta)$ such that $\alpha \succeq (R|_H, z'|_H) \, \beta$. By Proposition 2, $z'|_H \in \Re_f(H, R|_{N(H)}, \overline{x}'_H)$. Let $\boldsymbol{y \equiv (u, \sigma)} \in \Re_f(H, R|_{N(H)}, \overline{x}'_H + \varepsilon)$.

• **Collecting $\boldsymbol{\varepsilon}$ from $\boldsymbol{N(G)}$.** We claim that for each $\beta \in G$, $\beta \succeq (R|_{N(G)}, z'|_{N(G)}) \, \mu(k)$. We prove that for each $\beta \in G$, there are $\{\beta_t\}_{t=0}^T \subseteq G$ such that $\beta_0 = \beta$, $\beta_T = \mu(k)$, and $(x'_{\beta_0}, \beta_0) \, I_{(\mu')^{-1}(\beta_0)} \, \ldots I_{(\mu')^{-1}(\beta_{T-1})} \, (x'_{\beta_T}, \beta_T)$. Since for each $\beta \in A$, and in particular for each $\beta \in G$, $\beta \succeq (R_{-i}, R'_i, z') \, \mu(k)$, then there are $\{\delta_t\}_{t=0}^T \subseteq A$ such that $\delta_0 = \beta$, $\delta_T = \mu(k)$, and $(x'_{\delta_0}, \delta_0) \, I_{(\mu')^{-1}(\delta_0)} \, \ldots I_{(\mu')^{-1}(\delta_{T-1})} \, (x'_{\delta_T}, \delta_T)$. We claim that $\{\delta_t\}_{t=0}^T \subseteq G$. Suppose by means of contradiction that there is $0 < t^* \leq T$ such that $\delta_{t^*} \in H$. Since

$(x'_{\delta_{t^*-1}}, \delta_{t^*-1})\, I_{(\mu')^{-1}(\delta_{t^*-1})}\, (x'_{\delta_{t^*}}, \delta_{t^*})$, then $\delta_{t^*-1} \in H$. The recursive argument shows that $\beta = \delta_0 \in H$. This is a contradiction. Now, recall from Step 2 in Case 1 that for each $j \subseteq N \setminus \{i\}$ and each $l \in N$, $z'_j \, R_j \, z'_l$.

We claim that there is $\boldsymbol{v \equiv (w, \lambda)} \in (G, R|_{N(G)}, \overline{x}'_G - \varepsilon)$ such that:

(1) $x'|_{N(G)} \gg w$.

(2) For each $k \in N(G) \setminus \{i\}$, and each $l \in N(G) \setminus \{i\}$, $v_k \, R_k \, v_l$.

(3) $\lambda(i) = \mu'(i) = \mu(k)$.

(4) For each $\beta \in G$, $\beta \succeq (R'|_{N(G)}, v) \lambda(i)$.

Let $\widehat{R}_i \in \mathcal{R}(G)$ be a preference such that for each $\delta \in G \setminus \{\mu'(i)\}$, $(x_{\mu'(i)} - \varepsilon, \mu'(i))\, \widehat{P}_i\, (x'_\delta + \varepsilon, \delta)$. Let $R|_{N(G)\setminus\{-i\}}$ be the profile obtained from $R|_{N(G)}$ by removing agent i's preferences. Since For each $j \in N(G) \setminus \{i\}$ and each $l \in N(G)$, $z'_j \, R_j \, z'_l$, then $z'|_G \in F(G, R|_{N(G)\setminus\{-i\}}, \widehat{R}_i, m)$. Let $\delta \in G$. Since $\delta \succeq (R|_{N(G)}, z'|_G)\, \mu'(i)$, then by Lemma 4, $\delta \succeq (R|_{N(G)\setminus\{-i\}}, \widehat{R}_i, z')\, \mu'(i)$. Thus, $z'|_G \in \Re_{\mu(i)}(G, R|_{N(G)\setminus\{-i\}}, \widehat{R}_i, m)$. Let $v \in \Re_{\mu(i)}(G, R|_{N(G)\setminus\{-i\}}, \widehat{R}_i, m - \varepsilon)$. A similar argument to the one in Distributing ε among $N(G)$ in Lemma 1, shows that v satisfies conditions (1) to (4).

• **Defining $\boldsymbol{z^\varepsilon}$.** Let x^ε be the vector obtained by concatenating u and w, i.e., the vector defined by: $x^\varepsilon|_H \equiv u$ and $x^\varepsilon|_G \equiv w$. Let μ^ε be the bijection that coincides with σ on $N(H)$ and with λ on $N(G)$. Let $z^\varepsilon \equiv (x^\varepsilon, \mu^\varepsilon)$.

• **Claim 5: For each $\boldsymbol{\alpha \in H}$, $\boldsymbol{x^\varepsilon_\alpha \in (x'_\alpha, x'_\alpha + \varepsilon)}$ and for each $\boldsymbol{\delta \in G}$, $\boldsymbol{x^\varepsilon_\delta \in (x'_\delta - \varepsilon, x'_\delta)}$.** Let $\alpha \in H$. By Proposition 3, $u \gg x'|_H$. Thus, $x^\varepsilon_\alpha > x'_\alpha$. Now, since $\overline{u}_H = \overline{x}'_H + \varepsilon$, then $x^\varepsilon_\alpha < x'_\alpha + \varepsilon$. A symmetric argument shows that for each $\delta \in G$, $x^\varepsilon_\delta \in (x'_\delta - \varepsilon, x'_\delta)$.

Step 3: There is $\boldsymbol{R'_i \in \mathcal{Q}}$ such that for each $\boldsymbol{\hat{z} \in \Re_f(R_{-i}, R'_i)}$, $\boldsymbol{\hat{z}_i = z^\varepsilon_i}$.

Since $\min_{\alpha \in H} f_\alpha(x'_\alpha + \varepsilon) < \min_{\delta \in G} f_\delta(x'_\delta - \varepsilon)$, then $\arg\min_{\alpha \in A} f_\alpha(x^\varepsilon_\alpha) \subseteq H$. Fix $\rho \in \arg\min_{\alpha \in A} f_\delta(x^\varepsilon_\alpha) \subseteq H$. Let $R'_i \in \mathcal{Q}$ be a quasi-linear preference such that $(x^\varepsilon_\rho, \rho)\, I'_i\, (x^\varepsilon_{\mu(k)}, \mu(k))$ and for each $\delta \in A \setminus \{\mu(k), \rho\}$, $(x^\varepsilon_{\mu(k)}, \mu(k))\, P'_i\, (x^\varepsilon_\delta, \delta)$.

A similar argument to the one in Step 2 in the proof of Lemma 1 shows that $z^\varepsilon \in \Re_f(R_{-i}, R'_i)$ and for each $\hat{z} \in \Re_f(R_{-i}, R'_i)$, $\hat{z}_i = z^\varepsilon_i$.

Step 4: Concluding that $\boldsymbol{(R, z^*) \notin \mathcal{N}\langle S(\mathcal{D})^N, \Re^r_f, R^0\rangle}$. Since $z^\varepsilon_i = (x^\varepsilon_{\mu(k)}, \mu(k))$ and $x^\varepsilon_{\mu(k)} > x'_{\mu(k)} - \varepsilon$, then $z^\varepsilon_i \, P^0_i \, z_i$. Since for each $\hat{z} \in \Re_f(R_{-i}, R'_i)$, $\hat{z}_i = z^\varepsilon_i$, then $\Re^r_f(R_{-i}, R'_i, z^*)\, P^0_i\, \Re^r_f(R, z^*)$. Thus, $(R, z^*) \notin \mathcal{N}\langle S(\mathcal{D})^N, \Re^r_f, R^0\rangle$. This is a contradiction. □

Proof of Lemma 3. Let $\mathcal{D}$, f, and r be as in the statement of the lemma. Let $R^0 \in \mathcal{R}^N$. By Lemma 2, $\mathcal{O}\langle S(\mathcal{D})^N, \Re^r_f, R^0\rangle \subseteq F(R^0)$. Since $\mathcal{O}^*\langle S(\mathcal{D})^N, \Re^r_f, R^0\rangle \subseteq \mathcal{O}\langle S(\mathcal{D})^N, \Re^r_f, R^0\rangle$, then it is left to prove that $F(R^0) \subseteq \mathcal{O}^*\langle S(\mathcal{D})^N, \Re^r_f, R^0\rangle$. Let $z \equiv (x, \mu) \in F(R^0)$. Let $R \in \mathcal{Q}^N$ be such that for each $\{i, j\} \subseteq N$, $z_i \, I_i \, z_j$. We claim that $(R, z) \in \mathcal{N}^*\langle S(\mathcal{D})^N, \Re^r_f, R^0\rangle$. Suppose that there is $N' \subseteq N$, $i \in N'$ and $(R'_{N'}, z'_{N'}) \in \mathcal{D} \times (\mathbb{R} \times A)$ such that

$$\Re^r_f(R_{-N'}, R'_{N'}, z_{-N'}, z'_{N'})\, P^0_i\, \Re^r_f(R, z).$$

Let $y \equiv (u, \sigma) \equiv \Re^r_f(R_{-N'}, R'_{N'}, z_{-N'}, z'_{N'})$. Since $z \in F(R^0)$, then $u_{\sigma(i)} > x_{\sigma(i)}$. Since

$y \in \Re_f(R_{-N'}, R'_{N'}) \subseteq F(R_{-N'}, R'_{N'})$, then by definition of R, for each $k \in N \setminus N'$, $u_{\sigma(k)} > x_{\sigma(k)}$. Since $\overline{x}_A = \overline{u}_A$, then there is $j \in N'$ such that $u_{\mu(k)} < x_{\mu(k)}$. Since $z \in F(R^0)$, then $\Re^r_f(R, z)\, P^0_j\, \Re^r_f(R_{-N'}, R'_{N'}, z_{-N'}, z'_{N'})$. Thus, $(R, z) \in \mathcal{N}^*\langle S(\mathcal{D})^N, \Re^r_f, R^0\rangle$. Since $z = \Re^r_f(R, z)$, then $z \in \mathcal{O}^*\langle S(\mathcal{D})^N, \Re^r_f, R^0\rangle$. □

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