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Non-proportional inequality preservation in gains and losses

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Abstract

We characterize the family of claims-inequality and claims-order preserving continuous rules in the three-agent case for the problem of adjudicating conflicting claims. We show that there are infinitely many of such rules and provide a simple geometric construction that spans the whole family. Additionally, we prove that this family endowed with the partial order of Lorenz domination is a lattice that has maximal and minimal elements.

JEL classification: C71; D31; H24.

Keywords: claims problems; minimal award functions; proportional rule; Lorenz domination; inequality preservation.

1 Introduction

We consider the problem of distributing a resource among a group of agents who have conflicting claims on it. Such a problem is referred to in the

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literature as a claims problem. A typical situation is when a firm goes bankrupt and its liquidation value has to be divided among its creditors. A claims problem has the same mathematical structure a problem of collecting a prespecified amount of a resource from a group of agents whose individual endowments add up to at least that amount. Here, an illustration is a local government taxing incomes to cover the cost of some public project. However, we will use language that pertains to claims problems.[1]

A rule is a function that associates with each problem an "awards vector" for it. In our search for the "best" rule, we follow the axiomatic approach.[2] First, we require that inequalities in claims, under the Lorenz domination criterion, be reflected in awards and losses, i.e., the basic axiom of "claims-inequality preservation in gains and loses" (Hougaard and Thorlund-Petersen, 2001). More precisely, fix the endowment and suppose that two claims vectors, whose coordinates add up to the same amount, are related by Lorenz domination. We require that the awards and losses vectors associated with these claims vectors be also related by Lorenz domination (see Kasajima and Velez, 2009, for a discussion of these axioms).[3]

We also require that for each pair of agents, say i and j, such that agent i's claim is at least as large as agent j's claim, agent i's award and loss respectively be at least as large as agent j's award and loss, i.e., "claims-order preservation in gains and losses" (Aumann and Maschler, 1985). Finally, we require that small changes in the data of the problem have a small effect on the resulting awards vector, i.e., continuity. We refer to claims-inequality preservation in gains and losses and claims-order preservation in gains and losses as the "core axioms."

We restrict our attention to the three-agent case. Indeed, if there are more than three agents, it is already known that the core axioms and continuity single out the proportional rule (Hougaard and Østerdal, 2005; Kasajima and Velez, 2009).[4] It turns out that in the three agent case, in-

[1] Alternative names in the literature are "bankruptcy problem," "taxation problem," and "rationing problem."

[2] O'Neill (1982) pioneered the application of the axiomatic approach to this problem – see Moulin (2002) and Thomson (2003, 2008) for surveys.

[3] See Ju and Moreno-Ternero (2008, 2009) for the analysis of a related "inequality reduction" property.

[4] Hougaard and Østerdal (2005) assert that for more than two agents the proportional rule is the only rule satisfying these axioms. However, a step in their proof contains a subtle error. It fails in their Step 3. They write "let c''_k be the highest c_k for which there exists a pair (c'_k, c_k) satisfying $e_k(c'_k) = 0$, $e_k(c_k) = 0$, $e_k(\widehat{c}_k) > 0$, for all $c'_k < \widehat{c}_k < c_k$ (p.359 lines 9-10)." However, the variable c''_k in their statement may not exist, and in fact, examples can be constructed in which it does not exist. Thus their proof fails for any

finitely many rules satisfy the core axioms and continuity (the proportional rule is among them, however).[5] Our main result is a characterization of this family of rules.

The two-agent and three-agent cases are of particular interest, for real bankruptcy situations may well involve that few agents. In the two-agent case, the core axioms and continuity are very mild restrictions on rules. In fact, virtually all continuous rules satisfy them. The analysis of this case is straightforward. It is obvious that when there are only two agents, the award to an agent can be written as a function of her claim, the aggregate claim, and the endowment (being able to do so for the awards to the agents who have minimal and maximal claims will be critical to our characterization in the three-agent case). Thus, one can associate with each rule the so defined function. This simple observation allows one to easily characterize the family of rules. A direct application of the axioms yields the symmetry and monotonicity properties of the functions associated with rules that satisfy the axioms.[6]

By contrast, when there are three agents, the axioms significantly restrict rules. Here, the full characterization of the family of rules poses technical challenges. Nevertheless, we offer such a characterization (Theorem 1). Moreover, we provide a simple geometric construction that spans the whole family (Propositions 1 and 2). In Section 4, we reveal two interesting structural features of the family. First, it is closed under the so called "convexity" and "duality" operators. Second, if endowed with the partial order of Lorenz domination, this family is a lattice, and this lattice has maximal and minimal elements.

The proportional rule occupies a central role in the literature on claims problems. A basic axiom satisfied by the rule, "no advantageous transfer" (also known as "reallocation-proofness") requires that no group of agents receive more in the aggregate by transferring claims among themselves. This axiom singles out the proportional rule when there are at least three agents (Moulin 1985; Chun 1988; Ju et al. 2007; and Thomson 2008).[7] The core axioms imply a weak form of no advantageous transfer. Suppose that there are n agents. For each $k \leq n$, the axioms preclude manipulation among

number of agents greater than two.

[5]Thus, the assertion in Hougaard and Østerdal (2005) is incorrect for the three-agent case.

[6]The complete characterization of the family of two-agent rules satisfying these axioms is available from the authors upon request.

[7]See also Young (1988), de Frutos (1999), Ching and Kakkar (2001), and Chambers and Thomson (2002) for additional characterizations of the proportional rule.

coalitions composed of either the agents with the k highest claims, or the agents with the k lowest claims, when the members of a coalition transfer claims among themselves and the order of their claims remains the same. This weaker form of no advantageous transfer, "equal treatment of equals," and continuity single out the proportional rule when there are more than three agents (Kasajima and Velez, 2009). By contrast, our results reveal that in the three-agent case, the core axioms (which imply equal treatment of equals) and continuity together do not single out the proportional rule.

The remainder of the paper is organized as follows. Section 2 introduces the model. Section 3 presents our results. Section 4 further investigates the the structure of the family we characterize. Section 5 discusses our results. The Appendix contains proofs.

2 The model

Let $\boldsymbol{N} \equiv \{1, \ldots, n\}$ be a set of agents. A claims problem for N, or simply a **problem for $\boldsymbol{N}$**, is a pair (c, E) where $c = (c_i)_{i \in N} \in \mathbb{R}^N_+$ and $E \in \mathbb{R}_+$ are such that $\sum_{i \in N} c_i \geq E$. The coordinates of c are interpreted as **claims** the agents hold, and the number E as the amount available of an infinitely divisible resource, the **endowment**; this amount is not sufficient to honor all claims. The set of all problems is denoted $\boldsymbol{\mathcal{C}}$. An **awards vector for $(\boldsymbol{c}, \boldsymbol{E}) \in \boldsymbol{\mathcal{C}}$** is a vector $x \in \mathbb{R}^N_+$, such that $x \leq c$ and $\sum_{i \in N} x_i = E$.[8] A **rule**, denoted generically S, is a function that associates with each problem an awards vector for it.

For each $c \in \mathbb{R}^N_+$, let $\boldsymbol{\bar{c}} \equiv \sum_{i \in N} c_i$, $\mathbf{min}\, \boldsymbol{c} \equiv \min_{i \in N} c_i$, and $\mathbf{max}\, \boldsymbol{c} \equiv \max_{i \in N} c_i$. For each $s \in \mathbb{R}_+$, let $\boldsymbol{\mathcal{C}(s)} \equiv \{(c, E) \in \mathcal{C} : \bar{c} = s\}$.

A rule S is **continuous** if for each convergent sequence of problems $\{(c_k, E_k)\}_{k \in \mathbb{N}}$, the sequence $\{S(c_k, E_k)\}_{k \in \mathbb{N}}$ converges to $S(\lim_{k \to \infty}(c_k, E_k))$. A rule S is **claims-order preserving in gains (OPG)**, if for each $(c, E) \in \mathcal{C}$ and each $\{i, j\} \subseteq N$ such that $c_i \leq c_j$, $S_i(c, E) \leq S_j(c, E)$; it is **claims-order preserving in losses (OPL)** if for each $(c, E) \in \mathcal{C}$ and each $\{i, j\} \subseteq N$ such that $c_i \leq c_j$, $c_i - S_i(c, E) \leq c_j - S_j(c, E)$.[9]

For each $y \in \mathbb{R}^N_+$ and each $t \in \{1, \ldots, n\}$, let $y_{[t]}$ be the t-order statistic of y, i.e., the vector $(y_{[t]})_{t=1}^n$ is a permutation of the coordinates of y such that $y_{[1]} \leq y_{[2]} \leq \cdots \leq y_{[n]}$. Let $\{y, y'\} \subseteq \mathbb{R}^N_+$ be such that $\sum_{i \in N} y_i = \sum_{i \in N} y'_i$. Then, **$\boldsymbol{y}$ Lorenz dominates $\boldsymbol{y'}$**, denoted $y \succeq_L y'$, if for each $k \in \{1, \ldots, n\}$,

[8] Vector inequalities: $x \geq y$ allows x and y to be equal; $x \gneq y$ does not; $x > y$ means that each coordinate of x is greater than the corresponding coordinate of y.

[9] *OPG* and *OPL* are introduced by Aumann and Maschler (1985).

$\sum_{t=1}^{k} y_{[t]} \geq \sum_{t=1}^{k} y'_{[t]}$. A rule S is **claims-inequality preserving in gains (IPG)** if for each $(c, E) \in \mathcal{C}$ and each $c' \in \mathbb{R}_+^N$ such that $\bar{c} = \sum_{i \in N} c'_i$ and $c \succeq_L c'$, $S(c, E) \succeq_L S(c', E)$; it is **claims-inequality preserving in losses (IPL)** if for each $(c, E) \in \mathcal{C}$ and each $c' \in \mathbb{R}_+^N$ such that $\bar{c} = \sum_{i \in N} c'_i$ and $c \succeq_L c'$, $c - S(c, E) \succeq_L c' - S(c', E)$.[10]

Let S and S' be two rules. Then, $\boldsymbol{S}$ **Lorenz dominates** $\boldsymbol{S'}$, denoted $S \succeq_L S'$, if for each $(c, E) \in \mathcal{C}$, $S(c, E) \succeq_L S'(c, E)$.

We refer to *OPG*, *OPL*, *IPG*, and *IPL* as the **core axioms**.

We now introduce three rules that will be mentioned in the following sections. They are central in the literature.

Our first rule divides the endowment proportionally to claims.

Proportional rule, $\boldsymbol{P}$**:** For each $(c, E) \in \mathcal{C}$, $P(c, E) \equiv \lambda c$, where $\lambda \in \mathbb{R}_+$ is chosen such that $\sum_{i \in N} P_i(c, E) = E$.

Our second rule divides the endowment equally, subject to no agent receiving more than her claim.

Constrained equal awards rule, $\boldsymbol{CEA}$**:** For each $(c, E) \in \mathcal{C}$ and each $i \in N$, $CEA_i(c, E) \equiv \min\{c_i, \lambda\}$, where $\lambda \in \mathbb{R}_+$ is chosen such that $\sum_{i \in N} CEA_i(c, E) = E$.

Our final rule divides the aggregate loss (the difference $\sum_{i \in N} c_i - E$) equally, subject to no agent receiving a negative amount.

Constrained equal losses rule, $\boldsymbol{CEL}$**:** For each $(c, E) \in \mathcal{C}$ and each $i \in N$, $CEL_i(c, E) \equiv \max\{c_i - \lambda, 0\}$, where $\lambda \in \mathbb{R}_+$ is chosen such that $\sum_{i \in N} CEL_i(c, E) = E$.

3 Non-proportional inequality preserving rules

Lemma 1 states a critical structural property of each rule satisfying the *core axioms*: the award to a minimal claimant is a function of her claim, the aggregate claim, and the endowment. We call this function the **minimal award function** (associated with the rule). A symmetric statement is also true for the award to a maximal claimant, i.e., her award is a function of her claim, the aggregate claim, and the endowment. The lemma also lists four key properties satisfied by minimal award functions.[11]

Let $\boldsymbol{Y_n} \equiv \{(r, s, E) \in \mathbb{R}_+^3 : r \leq s/n \text{ and } s \geq E\}$.

[10] *IPG* and *IPL* are introduced by Hougaard and Thorlund-Petersen (2001).

[11] We state this lemma for $n \geq 2$ since these properties are crucial to the characterization of the family of rules satisfying the *core axioms* for the two-agent case and more than three agents. See Kasajima and Velez (2009) for more than three agents. The two-agent case is available from the authors upon request.

Lemma 1. Assume $n \geq 2$. If a rule S satisfies the *core axioms*, then there is $m : Y_n \to \mathbb{R}_+$ such that for each $(c, E) \in \mathcal{C}$ and each $i \in \arg\min_{k\in N} c_k$, $S_i(c, E) = m(\min c, \bar{c}, E)$, and for each $j \in \arg\max_{k\in N} c_k$, $S_j(c, E) = E - (n-1)\, m\Big(\frac{\bar{c}-\max c}{n-1}, \bar{c}, E\Big)$. Moreover, m satisfies the following properties:

P1: For each $(s, E) \in \mathbb{R}^2_+$ such that $s \geq E$, $m(\cdot, s, E) : [0, s/n] \to \mathbb{R}_+$ is a non-decreasing function such that $m(0, s, E) = 0$ and $m\,(s/n, s, E) = E/n$.

P2: For each $(s, E) \in \mathbb{R}^2_+$ such that $s \geq E$, and each $\{r, r'\} \subset [0, s/n]$ such that $r' \geq r$, $m(r', s, E) - m(r, s, E) \leq r' - r$.

P3: For each $(r, s, E) \in Y_n$,

$$m\bigg(\frac{s + (n-2)r}{2(n-1)}, s, E\bigg) = \frac{E}{2(n-1)} + \frac{n-2}{2(n-1)} m\,(r, s, E)\,. \tag{1}$$

P4: m is continuous in its first argument; moreover, it is continuous whenever S is.

We refer the reader to Kasajima and Velez (2009) for the proof.

Our main theorem is a characterization of the family of rules satisfying *continuity* and the *core axioms* when $n = 3$.

Theorem 1. Assume $n = 3$. A rule S is *continuous* and satisfies the *core axioms* if and only if there is a continuous function $m : Y_3 \to \mathbb{R}_+$ satisfying P1, P2, and P3 such that for each $(c, E) \in \mathcal{C}$ and each $i \in N$,

$$S_i(c, E) = \begin{cases} m(c_i, \overline{c}, E) & \text{if } c_i = \min c < \max c, \\ E - m(\min c, \overline{c}, E) - [E - 2m(\frac{\overline{c}-\max c}{2}, \overline{c}, E)] & \text{if } \min c < c_i < \max c, \\ E - 2m(\frac{\overline{c}-c_i}{2}, \overline{c}, E) & \text{if } c_i = \max c. \end{cases} \tag{2}$$

The proof is in the Appendix. Lemma 1 states that for each *continuous* rule that satisfies the *core axioms*, the associated minimal award function is continuous and satisfies P1, P2, and P3. Moreover, by the definition of a minimal award function, it is obvious that awards to minimal and maximal claimants are given by expression (2). By the feasibility constraint, the award to an intermediate claimant is also given by this expression. The surprising and non-trivial part of Theorem 1 is the converse. Given a continuous function $m : Y_3 \to \mathbb{R}_+$ that satisfies P1, P2, and P3, it is not even clear that expression (2) defines a rule. But, in fact, it not only does, but also it defines a *continuous* rule that satisfies the *core axioms*.

Our next task is to further understand the structure of the family of *continuous* rules that satisfy the *core axioms*. We do so by providing a mathematical characterization of all continuous functions $m : Y_3 \to \mathbb{R}_+$ satisfying P1, P2, and P3. We provide a simple geometric construction that spans the whole class. It follows from our construction that this class contains infinitely many functions.

Properties P1 and P2 are self explanatory. By contrast, it is not clear which functions also satisfy P3. Let $m : Y_3 \to \mathbb{R}_+$ be such a function. We show that P3 only imposes the following restriction on m, on the "initial portion" of its domain, i.e., the set $\boldsymbol{X} \equiv \{(r, s, E) \in \mathbb{R}^3_+ : r \leq s/4 \text{ and } s \geq E\}$: for each $(s, E) \in \mathbb{R}^2_+$ such that $s \geq E$, the function attains the proportional value at $(s/4, s, E)$. In fact, let f be a function defined on X that satisfies the restricted versions of P1 and P2 called p1 and p2 below, and such that for each $(s, E) \in \mathbb{R}^2_+$ such that $s \geq E$, attains the proportional value at $(s/4, s, E)$. Then, there is a unique continuous function $m : Y_3 \to \mathbb{R}_+$ that satisfies P1, P2, and P3 and coincides with f on X.

Let us first formally define the class of functions that parameterize our construction. An **initial minimal award function** is a continuous function $f : X \to \mathbb{R}_+$ that satisfies the following properties: for each $(s, E) \in \mathbb{R}^2_+$ such that $s \geq E$,

p1: $f(\cdot, s, E) : [0, s/4] \to \mathbb{R}_+$ is non-decreasing and $f(0, s, E) = 0$.

p2: For each $\{r, r'\} \subset [0, s/4]$ such that $r' \geq r$, $f(r', s, E) - f(r, s, E) \leq r' - r$.

a1: $f(s/4, s, E) = E/4$.

Let $\boldsymbol{\mathcal{F}}$ be the class of all initial minimal award functions. The following is an example of a function in this class.

Example 1. Ray-generated initial minimal award function. Let $\rho \in]0, 1[$. The ray-generated initial minimal award function associated with ρ is the function, $f_\rho : X \to \mathbb{R}_+$, defined by setting, for each $(r, s, E) \in X$,

$$f_\rho(r, s, E) \equiv \begin{cases} \rho r & \text{if } r \in [0, \frac{s-E}{4(1-\rho)}] \text{ and } E > \rho s, \\ r - \frac{s-E}{4} & \text{if } r \in]\frac{s-E}{4(1-\rho)}, \frac{s}{4}] \text{ and } E > \rho s, \\ \rho r & \text{if } r \in [0, \frac{E}{4\rho}] \text{ and } E \leq \rho s, \\ \frac{E}{4} & \text{if } r \in]\frac{E}{4\rho}, \frac{s}{4}] \text{ and } E \leq \rho s. \end{cases}$$

Figure 1 displays the graph of $f_\rho(\cdot, s, E)$ for different values of (s, E). One can easily verify that $f_\rho \in \mathcal{F}$. □

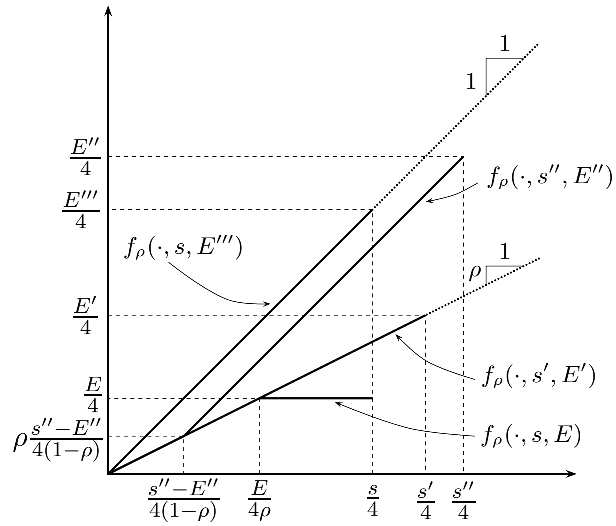

Figure 1: Ray-generated initial minimal award function (Example 1).

We now describe how to construct the unique continuous extension satisfying P1, P2, and P3 of each $f \in \mathcal{F}$. We denote it $\boldsymbol{m_f}$. Let $(s, E) \in \mathbb{R}^2_+$ be such that $s \geq E$. Obviously, $m_f(\cdot, s, E)$ and $f(\cdot, s, E)$ coincide on $[0, s/4]$. Now, on $]s/4, s/4+s/4^2]$, $m_f(\cdot, s, E)$ coincides with the function whose graph is obtained by homotheticly contracting $f(\cdot, s, E)$ to the interval $[0, s/4^2]$ and moving the origin to $(s/4, E/4)$. That is, for each $r \in]s/4, s/4 + s/4^2]$,

$$m_f(r, s, E) = \frac{E}{4} + \frac{1}{4} f\left(4\left(r - \frac{s}{4}\right), s, E\right).$$

Figure 2 illustrates this construction for f_ρ in Example 1. On $]s/4 + s/4^2, s/4 + s/4^2 + s/4^3]$, $m_f(\cdot, s, E)$ coincides with the function whose graph is obtained by homotheticly contracting $f(\cdot, s, E)$ to the interval $[0, s/4^3]$ and moving the origin to $(s/4 + s/4^2, E/4 + E/4^2)$. The construction is recursively completed by concatenating the functions whose graphs are the subsequent homothetic contractions of $f(\cdot, s, E)$. Since $\lim_{t\to\infty} \sum_{j=1}^{t} s/4^j = s/3$, this process completes the definition of $m_f(\cdot, s, E)$ for $[0, s/3[$. Finally, set $m_f(s/3, s, E)$ equal to $E/3$.

We now formally present our theorem. Let $\{a_t\}_{t\in\{0\}\cup\mathbb{N}}$ be the sequence defined as follows: for each $t \in \{0\} \cup \mathbb{N}$, $a_t \equiv \left(1 - 1/4^t\right)/3$.[12]

[12] For each $t \in \mathbb{N}$, $a_t \equiv \sum_{j=1}^{t} 1/4^j$, i.e., $\{a_t\}_{t\in\{0\}\cup\mathbb{N}}$ is the sequence of endpoints of intervals obtained by adding subsequent intervals of lengths equal to the powers of 1/4.

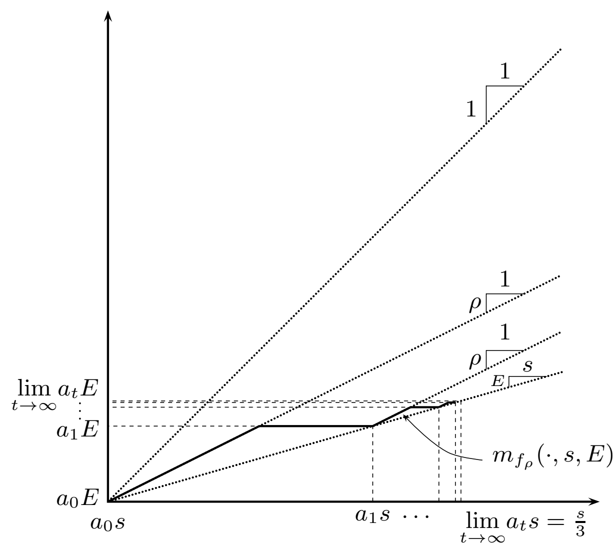

Figure 2: Minimal award function associated with f_ρ, m_{f_ρ}.

Proposition 1. Assume $n = 3$. A continuous function $m : Y_3 \to \mathbb{R}_+$ satisfies P1, P2, and P3 if and only if there is $f \in \mathcal{F}$ such that m coincides with the function defined by setting for each $(r, s, E) \in Y_3$,

$$m_f(r,s,E) \equiv \begin{cases} 0 & \text{if } r = 0, \\ a_t E + \frac{1}{4^t} f\left(4^t(r - a_t s), s, E\right) & \text{if there is } t \in \{0\} \cup \mathbb{N} \text{ s.t.} \\ & r \in]a_t s, a_{t+1} s], \\ \frac{E}{3} & \text{if } r = \frac{s}{3}. \end{cases}$$

Moreover, if a continuous function $m' : Y_3 \to \mathbb{R}_+$ satisfies P1, P2, P3, and coincides with m on X, then $m = m'$.

The proof is in the Appendix.

We can now provide an alternative characterization of the family of *continuous* rules that satisfy the *core axioms* in the three-agent case. For each $f \in \mathcal{F}$, let $\boldsymbol{S^f}$ be the rule defined by expression (2) for the minimal award function m_f. The following proposition is a consequence of Theorem 1 and Proposition 1. We omit the straightforward proof.

Proposition 2. Assume $n = 3$. A rule S is *continuous* and satisfies the *core axioms* if and only if there is $f \in \mathcal{F}$ such that $S = S^f$.

4 Structure

In this section we further investigate the structure of the family of *continuous* rules that satisfy the *core axioms*. We study its "convexity" and "duality" structure and the Lorenz ranking of rules in the family.

4.1 Convexity

Let S and S' be two rules and $\alpha \in [0,1]$. The **α-convex combination of S and S'**, denoted $\alpha S + (1-\alpha)S'$, is defined as follows: for each $(c,E) \in \mathcal{C}$, $\alpha S + (1-\alpha)S'(c,E) \equiv \alpha S(c,E) + (1-\alpha)S'(c,E)$.

The following lemma states that the family of *continuous* rules that satisfy the *core axioms* is "closed under the convexity operator." It also gives an expression that relates the minimal award functions of two rules and the minimal award function of their α-convex combination. We omit the straightforward proof.

Lemma 2. Assume $n = 3$. Let S and S' be two *continuous* rules that satisfy and the *core axioms*. Let m and m' be the minimal award functions associated with S and S', respectively. Then for each $\alpha \in [0,1]$, $\alpha S + (1-\alpha)S'$ is also *continuous* and satisfies the *core axioms*. Moreover, the minimal award function associated with $\alpha S + (1-\alpha)S'$ is $\alpha m + (1-\alpha)m'$.[13]

4.2 Duality

So far we have interpreted a claims problem as having to do with the distribution of an endowment among a group of claimants when the endowment is not enough to cover all claims. Alternatively, one can interpret a claims problem as pertaining to the distribution of some aggregate loss (aggregate claim minus the endowment) among the claimants (Thomson, 2008).[14] To a rule S one can associate its **dual**, denoted S^d, as the rule that associates for each $(c,E) \in \mathcal{C}$, the vector of awards $\boldsymbol{S^d(c,E)} \equiv c - S(c, \bar{c} - E)$ (Aumann and Maschler, 1985).

One can extend the duality notion to axioms. Two axioms are dual if whenever a rule S satisfies one of them, S^d satisfies the other. It is well known that *OPG* and *OPL* are dual. The following lemma states that so are *IPG* and *IPL*.

[13] Here $\alpha m + (1-\alpha)m'$ is the function defined as the pointwise convex combination of m and m'.

[14] We thank William Thomson, who suggested our investigation into the duality structure of the family of rules we characterize.

Lemma 3. *Claims-inequality preservation in awards* and *claims-inequality preservation in losses* are dual.

The proof is in the Appendix.

One can easily see that the dual of a *continuous* rule is *continuous*. Thus, the family of *continuous* rules that satisfies the *core axioms* is "closed under the duality operator." The following corollary formalizes this result.

Corollary 1. If a rule satisfies *continuity* and the *core axioms*, so does its dual.

We omit the straightforward proof.

We now investigate the relation between a *continuous* rule that satisfies the *core axioms* and its dual. Let S be such a rule. Let m and m^d be the minimal award functions associated with S and S^d, respectively. The following simple formula relates these minimal award functions: for each $(r, s, E) \in Y_3$,

$$m^d(r, s, E) = r - m(r, s, s - E).$$

Geometrically, one can interpret this relation as follows. For each (c, E), the function $m^d(\cdot, \bar{c}, E)$ is the symmetric image of $m(\cdot, \bar{c}, \bar{c} - E)$ with respect to the straight line of slope $\frac{1}{2}$.

4.3 Lattice structure

A recent literature has investigated the ranking of rules in terms of the partial order of Lorenz domination $\succeq_L$ (Moreno-Ternero and Villar, 2006; Bosmans and Lauwers, 2007; Thomson, 2007).[15] Here we study the Lorenz domination relation among *continuous* rules satisfying the *core axioms*. We show below that this family, when endowed with the partial order $\succeq_L$, is a lattice. Moreover, we show that it has maximal and minimal elements.[16]

Let S and S' be *continuous* rules that satisfy the *core axioms*. By Proposition 2, we know that there are $\{f, f'\} \subseteq \mathcal{F}$ such that $S = S^f$ and $S' = S^{f'}$. The following lemma states that the supremum and infimum of S and S' exist, and also it gives an explicit experesion for them.[17]

[15] One can easily see that $\succeq_L$ is reflexive, transitive, and anti-symmetric.

[16] In fact, our proof of Lemma 4 reveals that the family of continuous rules that satisfies the *core axioms* is a complete lattice, i.e., has the property that any subset has a supremum and an infimum. We thank Arthur de Avila and Jun Wako for suggesting our investigation into the lattice structure of this family of rules.

[17] The supremum and infimum in a lattice are commonly referred to as "join" and "meet," respectively.

Lemma 4. Assume $n = 3$. Let f and f' be two initial minimal award functions. Then, the least upper bound and the greatest lower bound of $\{S^f, S^{f'}\}$, under the partial order of Lorenz domination, in the set of *continuous* rules satisfying the *core axioms* exist. Moreover, they are $S^{\max\{f,f'\}}$ and $S^{\min\{f,f'\}}$, respectively.

The proof is in the Appendix.

We now show that there are Lorenz maximal and minimal elements in the family of *continuous* rules satisfying the *core axioms*. Lemma 4 allows us to identify them. Let us first introduce two additional initial minimal award functions.

Example 2. Constrained equal awards initial minimal award function. Let $f_{CEA} : X \to \mathbb{R}_+$ be the function defined as follows: for each $(r, s, E) \in X$,

$$f_{CEA}(r, s, E) \equiv \begin{cases} r & \text{if } r \in [0, E/4] \\ E/4 & \text{if } r \in]E/4, s/4]. \end{cases}$$

One easily sees that $f_{CEA} \in \mathcal{F}$. Let $(c, E) \in \mathcal{C}$ be such that $\min_{i \in N} c_i \leq E/4$. A special feature of f_{CEA} is that $f_{CEA}(\min c, \bar{c}, E)$ is the award to the minimal claimant recommended by CEA in (c, E). Figure 3 displays the graph of $f_{CEA}(\cdot, s, E)$ for different values of (s, E). □

Example 3. Constrained equal losses initial minimal award function. Let $f_{CEL} : X \to \mathbb{R}_+$ be the function defined as follows: for each $(r, s, E) \in X$,

$$f_{CEL}(r, s, E) \equiv \begin{cases} 0 & \text{if } r \in [0, \frac{s-E}{4}] \\ r - \frac{s-E}{4} & \text{if } r \in]\frac{s-E}{4}, \frac{s}{4}]. \end{cases}$$

One easily sees that $f_{CEL} \in \mathcal{F}$. Let $(c, E) \in \mathcal{C}$ be such that $\min_{i \in N} c_i \leq \frac{s-E}{4}$. A special feature of f_{CEL} is that $f_{CEL}(\min c, \bar{c}, E)$ is the award to the minimal claimant recommended by CEL in (c, E). □

We now study the rules associated with the initial minimal award functions f_{CEA} and f_{CEL}. These rules, $S^{f_{CEA}}$ and $S^{f_{CEL}}$, are respectively defined by expression (2) for the minimal award functions $m_{f_{CEA}}$ and $m_{f_{CEL}}$. The functions $m_{f_{CEA}}(\cdot_r, s, E)$ and $m_{f_{CEL}}(\cdot_r, s, E)$ are piecewise linear and interpolate the proportional values at $r = a_t s$ for each $t \in \{0\} \cup \mathbb{N}$. In each interval $[a_t s, a_{t+1} s]$, $m_{f_{CEA}}(\cdot_r, s, E)$ is composed of two pieces: a first segment of slope one and a second segment of slope zero. Symmetrically, in the

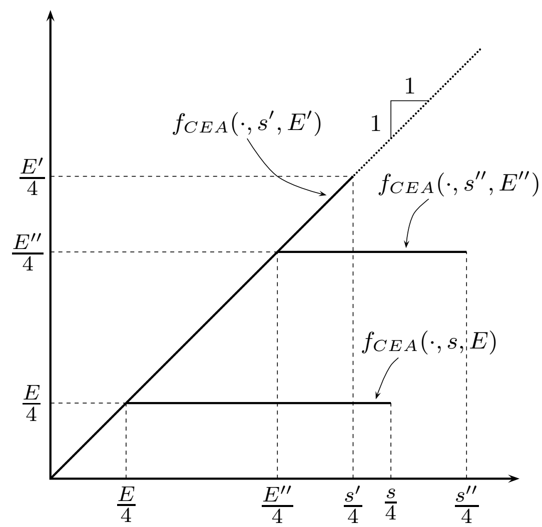

Figure 3: Constrained equal awards initial minimal award function (Example 2).

same interval, $m_{f_{CEL}}(\cdot_r, s, E)$ is composed of two pieces: a first segment of slope zero and a second segment of slope one. Formally, for each $t \in \{0\} \cup \mathbb{N}$ and each $(r, s, E) \in Y_3$ such that $r \in [a_t s, a_{t+1} s]$,

$$m_{f_{CEA}}(r, s, E) = \begin{cases} a_t E + (r - a_t s) & \text{if } r \in [a_t s, a_t s + \frac{E}{4^{t+1}}], \\ a_{t+1} E & \text{if } r \in [a_t s + \frac{E}{4^{t+1}}, a_{t+1} s], \end{cases} \tag{3}$$

and

$$m_{f_{CEL}}(r, s, E) = \begin{cases} a_t E & \text{if } r \in [a_t s, a_{t+1} s - \frac{E}{4^{t+1}}], \\ a_t E + r - (a_{t+1} s - \frac{E}{4^{t+1}}) & \text{if } r \in [a_{t+1} s - \frac{E}{4^{t+1}}, a_{t+1} s]. \end{cases} \tag{4}$$

One can easily see that for each $f \in \mathcal{F}$, $f_{CEL} \leq f \leq f_{CEA}$. Thus, a corollary of Lemma 4 is that $S^{f_{CEA}}$ and $S^{f_{CEL}}$ are Lorenz extremes in the family of *continuous* rules that satisfy the *core axioms*. We omit the straightforward proof.

Corollary 2. Assume $n = 3$. Then, $S^{f_{CEA}}$ and $S^{f_{CEL}}$ are respectively the unique Lorenz maximal and minimal rules in the family of *continuous* rules that satisfy the *core axioms.*

There is a connection between the lattice structure of the family of *continuous* rules that satisfy the *core axioms* and its duality structure. One can easily see that $S^{f_{CEA}}$ and $S^{f_{CEL}}$ are dual rules. This is not a coincidence, however. It is well known that Lorenz domination is reversed by the duality operator among rules satisfying *OPG* and *OPL* (Bosmans and Lauwers, 2007; Thomson, 2007). Consider a family of rules that satisfy *OPG* and *OPL* and is closed under the duality operator. If there is a Lorenz maximal rule in the family, then its dual is the Lorenz minimal rule. Symmetrically, if there is a Lorenz minimal rule in the family, then its dual is the Lorenz maximal rule.

5 Discussion and concluding remarks

In this paper we characterized the family of *continuous* rules that satisfy the *core axioms* when there are three agents. We provided a geometric construction that spans the whole family. In particular, we showed that there are infinitely many rules that satisfy the axioms. The most prominent members of the family are its Lorenz extremes (they exist) and the proportional rule.

One may be interested in rules that satisfy further normative requirements. In what follows, we consider this question for some central additional axioms in the literature (see Thomson, 2008 for a discussion of the axioms).

First, we remark that all these rules satisfy "anonymity." Now, consider the requirement that awards be monotonically increasing with respect to the endowment, i.e., **resource monotonicity**. Let m be a minimal award function. One can easily see that S^m satisfies *resource monotonicity* if and only if (i) m is monotone in its third argument, and (ii) for each $(r, s, E) \in Y_3$ and each E' such that $E < E' \leq s$, $\frac{m(r,s,E') - m(r,s,E)}{E' - E} \leq \frac{1}{2}$.

Another basic monotonicity axiom of rules is the requirement that the award to an agent is not reduced by an increase in her claim, i.e., **own-claim monotonicity**. In contrast to *resource monotonicity*, this axiom imposes less straightforward joint monotonicity restrictions on the first and second arguments of a minimal award function. We omit the details.

When a rule coincides with its dual, it is said to be **self-dual** (Aumann and Maschler, 1985). The *self-dual* rules are the ones whose minimal award functions are symmetric with respect to the straight line of slope $\frac{1}{2}$. There is an infinity of these rules in the family. The proportional rule is among them. Moreover, since the family is closed under the convexity and duality operators, one can associate to each of its members the *self-dual* rule obtained as the average of the rule and its dual.

Next we consider two independence properties. A rule S satisfies **composition down** if for each $(c, E) \in \mathcal{C}$ and each $E' < E$, we have $S(c, E') = S(S(c, E), E')$ (Moulin, 2000). It satisfies **composition up** if for each

$(c, E) \in \mathcal{C}$ and each $E' > E$ such that $\bar{c} \geq E'$, we have $S(c, E') = S(c, E) + S(c - S(c, E), E' - E)$ (Young, 1988). The proportional rule satisfies these two properties. The following proposition states that so do $S^{f_{CEA}}$ and $S^{f_{CEL}}$.

Proposition 3. $S^{f_{CEA}}$ and $S^{f_{CEL}}$ satisfy *composition down* and *composition up.*

The proof is in the Appendix.

In contrast to the case when there are more than three agents, our *core axioms* do not single out the proportional rule when there are three agents. Indeed, we showed that then, infinitely many rules satisfy the *core axioms* and *continuity* (Theorem 1). This case is even more special. If there are three agents, the proportional rule is not the only *continuous* rule satisfying the *core axioms*, *composition down*, and *composition up.*[18]

Finally, the *core axioms* are incompatible with the so called axioms of "claims truncation invariance" and "minimal rights first."[19] (`See Additional material for the referees`)

Appendix

Proof of Theorem 1. ($\Rightarrow$) Let S be a rule that satisfies the properties in the theorem. By Lemma 1, there is a continuous function $m : Y_3 \to \mathbb{R}_+$ satisfying P1, P2, and P3 such that for each $(c, E) \in \mathcal{C}$ each $i \in N$,

$$S_i(c, E) = \begin{cases} m(c_i, \overline{c}, E) & \text{if } c_i = \min c < \max c, \\ E - 2m(\frac{\overline{c} - c_i}{2}, \overline{c}, E) & \text{if } c_i = \max c. \end{cases}$$

Suppose that $i \in N$ is such that $\min c < c_i < \max c$. Since $\sum_{j \in N} S_j(c, E) = E$, then $S_i(c, E) = E - m(\min c, \overline{c}, E) - [E - 2m(\frac{\overline{c} - \max c}{2}, \overline{c}, E)]$.

($\Leftarrow$) Let $m : Y_3 \to \mathbb{R}_+$ be a continuous function that satisfies P1, P2, and P3. Let S^m be defined by (2). We prove that S^m is a well-defined *continuous* rule that satisfies the *core axioms*. Let $(c, E) \in \mathcal{C}$.

• **Proving that $\mathbf{0 \leq S^m(c, E) \leq c}$.** Let $i \in N$. We prove that $0 \leq S_i^m(c, E) \leq c_i$. There are three cases.

Case 1: $c_i = \min c < \max c$. Since $m(\cdot, \bar{c}, E)$ is non-decreasing and $m(0, \bar{c}, E) = 0$, then $0 \leq m(c_i, \bar{c}, E)$. Since m satisfies P2, $m(c_i, \bar{c}, E) - m(0, \bar{c}, E) \leq c_i - 0$. Thus, $m(c_i, \bar{c}, E) \leq c_i$.

[18] The proportional rule is the only *self-dual* rule that satisfies *composition down* or its dual, *composition up* (Young, 1988). We thank two referees for suggesting our investigation of the implications of these additional requirements.

[19] The proof is available from the authors upon request.

Case 2: $\min c < c_i < \max c$. Then, $\bar{c} = \min c + c_i + \max c$. Thus,

$$S_i^m(c,E) = m\left(\frac{\min c + c_i}{2}, \bar{c}, E\right) + \left[m\left(\frac{\min c + c_i}{2}, \bar{c}, E\right) - m\left(\min c, \bar{c}, E\right)\right].$$

Since $m(\cdot, \bar{c}, E)$ is non-decreasing and $m(0, \bar{c}, E) = 0$, then $0 \leq S_i^m(c,E)$. Since m satisfies P2, $S_i^m(c,E) \leq \frac{\min c + c_i}{2} + \frac{\min c + c_i}{2} - \min c = c_i$.

Case 3: $c_i = \max c$. Then, $\frac{\bar{c}-c_i}{2} \leq \frac{\bar{c}}{3}$. Since $m(\cdot, \bar{c}, E)$ is non-decreasing, $m(\frac{\bar{c}-c_i}{2}, \bar{c}, E) \leq m(\frac{\bar{c}}{3}, \bar{c}, E) = \frac{E}{3}$. Thus, $S_i^m(c,E) = E - 2m\left(\frac{\bar{c}-c_i}{2}, \bar{c}, E\right) \geq E - \frac{2}{3}E \geq 0$. Now,

$$\begin{aligned} S_i^m(c,E) &= \tfrac{E}{3} + 2\left[\tfrac{E}{3} - m\left(\tfrac{\bar{c}-c_i}{2}, \bar{c}, E\right)\right] \\ &= \tfrac{E}{3} + 2\left[m\left(\tfrac{\bar{c}}{3}, \bar{c}, E\right) - m\left(\tfrac{\bar{c}-c_i}{2}, \bar{c}, E\right)\right]. \end{aligned}$$

Recall that $E \leq \bar{c}$. Since m satisfies P2, then $S_i^m(c,E) \leq \frac{\bar{c}}{3} + 2\left[\frac{\bar{c}}{3} - \frac{\bar{c}-c_i}{2}\right] = c_i$.

- **Proving that $\sum_{i\in N} S_i^m(c,E) = E$.** There are four cases.

Case 1: There are $\{i,j\} \subset N$ such that $c_i = c_j = \min c < \max c$. Then, $c_i = c_j = \frac{\bar{c} - \max c}{2}$. Thus, $\sum_{k\in N} S_k^m(c,E) = m(\min c, \bar{c}, E) + m(\min c, \bar{c}, E) + [E - 2m(\min c, \bar{c}, E)] = E$.

Case 2: There are $\{i,j\} \subset N$ such that $c_i = c_j = \max c > \min c$. Then, $\max c \in \left[\frac{\bar{c}}{3}, \frac{\bar{c}}{2}\right]$. Thus, $\frac{\bar{c}-\max c}{2} \in \left[\frac{\bar{c}}{4}, \frac{\bar{c}}{3}\right]$. Since $4\frac{\bar{c}-\max c}{2} - \bar{c} = \bar{c} - 2\max c = \min c$ and m satisfies P3, then $m\left(\frac{\bar{c}-\max c}{2}, \bar{c}, E\right) = \frac{E}{4} + \frac{1}{4}m(\min c, \bar{c}, E)$.[20] Thus,

$$\sum_{k\in N} S_k^m(c,E) = 2\left[E - 2m\left(\frac{\bar{c} - \max c}{2}, \bar{c}, E\right)\right] + m(\min c, \bar{c}, E) = E.$$

Case 3: For each $i \in N$, $c_i = \bar{c}/3$. Since $m\left(\bar{c}/3, \bar{c}, E\right) = E/3$, then

$$\sum_{k\in N} S_k^m(c,E) = 3\left[E - 2m\left(\bar{c}/3, \bar{c}, E\right)\right] = E.$$

Case 4: There is $i \in N$, such that $\min c < c_i < \max c$. Then,

$$\sum_{k\in N} S_k^m(c,E) = \left\{\begin{array}{l} m(\min c, \bar{c}, E) + \\ E - m(\min c, c, E) - [E - 2m(\frac{\bar{c}-\max c}{2}, \bar{c}, E)] + \\ E - 2m(\frac{\bar{c}-\max c}{2}, \bar{c}, E) \end{array}\right\} = E.$$

[20] The function, $r' \in [s/4, s/3] \mapsto 4r' - s$, is a bijection from $[s/4, s/3]$ to $[0, s/3]$. Thus, the statement in P3 is equivalent to: for each $(r,s,E) \in Y_3$ such that $r \in [s/4, s/3]$, $m(r,s,E) = \frac{E}{4} + \frac{1}{4}m(4r - s, s, E)$.

• **Proving that *continuity* of S.** Let $N \equiv \{1,2,3\}$. Let $\{(c^k, E^k)\}_{k\in\mathbb{N}}$ be a convergent sequence of problems. Let $(c, E) \equiv \lim_{k\to\infty}(c^k, E^k)$. We prove that $S^m(c^k, E^k) \underset{k\to\infty}{\longrightarrow} S^m(c,E)$. Assume w.l.o.g. that $c_1 \le c_2 \le c_3$. Note that m is continuous. There are four cases.

Case 1: $c_1 < c_2 = c_3$. Since $c^k \underset{k\to\infty}{\longrightarrow} c$, then there is $K \in \mathbb{N}$ such that for each $k \ge K$, $c_1^k < c_2^k$ and $c_1^k < c_3^k$, and thus

$$S_2^m(c^k,E^k) = \begin{cases} 2m(\frac{\bar{c}^k - \max c^k}{2}, \bar{c}^k, E^k) - m(\min c^k, \bar{c}^k, E^k) & \text{if } \min c^k < c_2^k < \max c^k \\ E^k - 2m(\frac{\bar{c}^k - \max c^k}{2}, \bar{c}^k, E^k) & \text{if } c_2^k = \max c^k, \end{cases}$$

and $S_1^m(c^k,E^k) = m(\min c^k, \bar{c}^k, E^k)$. Thus, $S_1^m(c^k,E^k) \underset{k\to\infty}{\longrightarrow} S_1^m(c,E)$. Now, since $\frac{\bar{c}-\max c}{2} \in [\frac{\bar{c}}{4}, \frac{\bar{c}}{3}]$ and $\min c = 4\frac{\bar{c}-\max c}{2} - \bar{c}$, then by P3,

$$2m\left(\frac{\bar{c}-\max c}{2}, \bar{c}, E\right) - m(\min c, \bar{c}, E) = E - 2m\left(\frac{\bar{c}-\max c}{2}, \bar{c}, E\right).$$

Thus, $S_2^m(c^k,E^k) \underset{k\to\infty}{\longrightarrow} S_2^m(c,E)$. Since $S_3^m(c^k,E^k) = E^k - S_1^m(c^k,E^k) - S_2^m(c^k,E^k)$, then $S_3^m(c^k,E^k) \underset{k\to\infty}{\longrightarrow} S_3^m(c,E)$.

Case 2: $c_1 = c_2 < c_3$. Since $c^k \underset{k\to\infty}{\longrightarrow} c$, then there is $K \in \mathbb{N}$ such that for each $k \ge K$, $c_1^k < c_3^k$ and $c_2^k < c_3^k$, and thus

$$S_1^m(c^k,E^k) = \begin{cases} m(\min c^k, \bar{c}^k, E^k) & \text{if } c_1^k = \min c^k < \max c^k \\ 2m(\frac{\bar{c}^k - \max c^k}{2}, \bar{c}^k, E^k) - m(\min c^k, \bar{c}^k, E^k) & \text{if } \min c^k < c_1^k < \max c^k, \end{cases}$$

and $S_3^m(c^k,E^k) = E^k - 2m(\frac{\bar{c}^k-\max c^k}{2}, \bar{c}^k, E^k)$. Thus, $S_3^m(c^k,E^k) \underset{k\to\infty}{\longrightarrow} E - 2m(\frac{\bar{c}-\max c}{2}, \bar{c}, E) = S_3^m(c,E)$. Now, since $\frac{\bar{c}^k-\max c^k}{2} \underset{k\to\infty}{\longrightarrow} \min c$, then

$$S_1^m(c^k,E^k) \underset{k\to\infty}{\longrightarrow} m(\min c, \bar{c}, E) = S_1^m(c,E).$$

Since $S_2^m(c^k,E^k) = E^k - S_1^m(c^k,E^k) - S_3^m(c^k,E^k)$, then $S_2^m(c^k,E^k) \underset{k\to\infty}{\longrightarrow} S_2^m(c,E)$.

Case 3: $c_1 = c_2 = c_3$. Then, for each $k \in \mathbb{N}$,

$$S_1^m(c^k,E^k) = \begin{cases} m(\min c^k, \bar{c}^k, E^k) & \text{if } c_1^k = \min c^k < \max c^k \\ 2m(\frac{\bar{c}^k - \max c^k}{2}, \bar{c}^k, E^k) - m(\min c^k, \bar{c}^k, E^k) & \text{if } \min c^k < c_1^k < \max c^k \\ E^k - 2m\left(\frac{\bar{c}^k - \max c^k}{2}, \bar{c}^k, E^k\right) & \text{if } c_1^k = \max c^k. \end{cases}$$

Since $\min c^k \underset{k\to\infty}{\longrightarrow} \frac{\bar{c}}{3}$ and $\max c^k \underset{k\to\infty}{\longrightarrow} \frac{\bar{c}}{3}$, then $m(\min c^k, \bar{c}^k, E^k) \underset{k\to\infty}{\longrightarrow} \frac{E}{3}$, and $m\left(\frac{\bar{c}^k-\max c^k}{2}, \bar{c}^k, E^k\right) \underset{k\to\infty}{\longrightarrow} \frac{E}{3}$. Thus, $S_1^m(c^k,E^k) \underset{k\to\infty}{\longrightarrow} \frac{E}{3} = S_1^m(c,E)$. A symmetric argument shows that $S_2^m(c^k, E^k) \underset{k\to\infty}{\longrightarrow} S_2^m(c, E)$ and $S_3^m(c^k, E^k) \underset{k\to\infty}{\longrightarrow} S_3^m(c, E)$.

Case 4: $c_1 < c_2 < c_3$. Since $c^k \underset{k\to\infty}{\longrightarrow} c$, then there is $K \in \mathbb{N}$ such that for each $k \geq K$, $c_1^k < c_2^k < c_3^k$ and thus,

$$S_1^m(c^k, E^k) = m(c_1^k, \bar{c}^k, E^k) \quad \text{and} \quad S_3^m(c^k, E^k) = E^k - 2m\left(\frac{\bar{c}^k - c_3^k}{2}, \bar{c}^k, E^k\right).$$

Thus, $S_1^m(c^k, E^k) \underset{k\to\infty}{\longrightarrow} m(c_1, \bar{c}, E) = S_1^m(c, E)$ and $S_3^m(c^k, E^k) \underset{k\to\infty}{\longrightarrow} E - 2m(\frac{\bar{c}-c_3}{2}, \bar{c}, E) = S_3^m(c, E)$. Since $S_2^m(c^k, E^k) = E^k - S_1^m(c^k, E^k) - S_3^m(c^k, E^k)$, then $S_2^m(c^k, E^k) \underset{k\to\infty}{\longrightarrow} S_2^m(c, E)$.

• **Proving that *S* satisfies *OPG* and *OPL*.** Let $(c, E) \in \mathcal{C}$. Assume $\{1, 2\} \subset N$ are such that $c_1 \leq c_2$. We prove that $S_1^m(c, E) \leq S_2^m(c, E)$ and $c_1 - S_1^m(c, E) \leq c_2 - S_2^m(c, E)$. There are five cases.

Case 1: $c_1 = c_2 = \min c$ and $\min c \neq \max c$. Then, $S_1^m(c, E) = S_2^m(c, E)$ and $c_1 - S_1^m(c, E) = c_2 - S_2^m(c, E)$.

Case 2: $c_1 = \min c$ and $\min c < c_2 < \max c$. Since m is non-decreasing, $m(c_1, \overline{c}, E) \leq m(\frac{c_1+c_2}{2}, \overline{c}, E)$. Thus,

$$m(c_1, \overline{c}, E) \leq E - m(c_1, \overline{c}, E) - \left[E - 2m\left(\frac{c_1 + c_2}{2}, \overline{c}, E\right)\right],$$

i.e., $S_1^m(c, E) \leq S_2^m(c, E)$ (note that $\frac{\overline{c}-\max c}{2} = \frac{c_1+c_2}{2}$). By P2 (take $r = c_1$ and $r' = \frac{c_1+c_2}{2}$), $m(\frac{c_1+c_2}{2}, \overline{c}, E) - m(c_1, \overline{c}, E) \leq \frac{c_1+c_2}{2} - c_1$. Thus,

$$c_1 - m(c_1, \overline{c}, E) \leq c_2 - 2m\left(\frac{c_1 + c_2}{2}, \overline{c}, E\right) + m(c_1, \overline{c}, E),$$

i.e., $c_1 - S_1^m(c, E) \leq c_2 - S_2^m(c, E)$.

Case 3: $\min c < c_1 < \max c$ and $c_2 = \max c$. There are two subcases.

Case 3 (a): If $c_2 \leq \frac{\overline{c}}{2}$. Then, $\frac{\overline{c}-c_2}{2} \in [\frac{\overline{c}}{4}, \frac{\overline{c}}{3}]$. By P3, $m(\frac{\overline{c}-c_2}{2}, \overline{c}, E) = \frac{E}{4} + \frac{1}{4}m(\overline{c}-2c_2, \overline{c}, E)$. Since m is non-decreasing, $m(\overline{c}-2c_2, \overline{c}, E) = m(\min c + (c_1 - c_2), \overline{c}, E) \leq m(\min c, \overline{c}, E)$. Thus, $m(\frac{\overline{c}-c_2}{2}, \overline{c}, E) \leq \frac{E}{4} + \frac{1}{4}m(\min c, \overline{c}, E)$, i.e., $S_1^m(c, E) \leq S_2^m(c, E)$. By P2 (take $r = \min c + (c_1 - c_2)$ and $r' = \min c$), $m(\min c, \overline{c}, E) - m(\min c + (c_1 - c_2), \overline{c}, E) \leq c_2 - c_1$. Thus, $c_1 - m(\overline{c} - 2c_2, \overline{c}, E) \leq c_2 - m(\min c, \overline{c}, E)$. Hence $c_1 - 4m(\frac{\overline{c}-c_2}{2}, \overline{c}, E) + E \leq c_2 - m(\min c, \overline{c}, E)$, i.e., $c_1 - S_1^m(c, E) \leq c_2 - S_2^m(c, E)$.

Case 3 (b): If $c_2 > \frac{\overline{c}}{2}$. Then, $\frac{\overline{c}-c_2}{2} \in [0, \frac{\overline{c}}{4}[$. Since $m(\frac{\overline{c}}{4}, \overline{c}, E) = \frac{E}{4}$ and m is non-decreasing, $m(\frac{\overline{c}-c_2}{2}, \overline{c}, E) \leq \frac{E}{4}$. Since $m(\min c, \overline{c}, E) \geq 0$, we have $m(\frac{\bar{c}-c_2}{2}, \bar{c}, E) \leq \frac{E}{4} + \frac{1}{4}m(\min c, \bar{c}, E)$, i.e., $S_1^m(c,E) \leq S_2^m(c,E)$. By P2 (take $r = \frac{\overline{c}-c_2}{2}$ and $r' = \frac{\overline{c}}{4}$), $m(\frac{\overline{c}}{4}, \overline{c}, E) - m(\frac{\overline{c}-c_2}{2}, \overline{c}, E) \leq \frac{\overline{c}}{4} - \frac{\overline{c}-c_2}{2}$. Since $\min c - m(\min c, \overline{c}, E) \geq 0$, we have

$$m\left(\frac{\overline{c}}{4}, \overline{c}, E\right) - m\left(\frac{\overline{c}-c_2}{2}, \overline{c}, E\right) \leq \frac{\overline{c}}{4} - \frac{\overline{c}-c_2}{2} + \frac{\min c - m(\min c, \overline{c}, E)}{4},$$

i.e., $c_1 - S_1^m(c,E) \leq c_2 - S_2^m(c,E)$ (note that $m(\frac{\overline{c}}{4}, \overline{c}, E) = \frac{E}{4}$ and $\frac{\overline{c}}{4} - \frac{\overline{c}-c_2}{2} + \frac{\min c}{4} = \frac{c_2 - c_1}{4}$).

Case 4: $c_1 = \min c$ and $c_2 = \max c$. It follows from Cases 2 and 3.

Case 5: $c_1 = c_2 = \max c$. Then, $S_1^m(c,E) = S_2^m(c,E)$ and $c_1 - S_1^m(c,E) = c_2 - S_2^m(c,E)$.

• **Proving that *S* satisfies *IPG*.** Let $s \in \mathbb{R}_+$ and $\{(c,E),(c',E)\} \subset \mathcal{C}(s)$ be such that $c' \succeq_L c$. We prove that $S^m(c',E) \succeq_L S^m(c,E)$. It is enough to show that $S_{[1]}^m(c',E) \geq S_{[1]}^m(c,E)$ and $S_{[3]}^m(c',E) \leq S_{[3]}^m(c,E)$. Note that $\overline{c} = \overline{c}' = s$. There are three cases.

Case 1: $\min c < \bar{c}/3$ and $\min c' < \bar{c}'/3$. Since S^m satisfies *OPG*,

$$S_{[1]}^m(c',E) = m(\min c', \overline{c}, E) \quad \text{and} \quad S_{[1]}^m(c,E) = m(\min c, \overline{c}, E).$$

Since $c' \succeq_L c$, $\min c' \geq \min c$. Since m is non-decreasing, $m(\min c', \overline{c}, E) \geq m(\min c, \overline{c}, E)$. Thus, $S_{[1]}^m(c',E) \geq S_{[1]}^m(c,E)$. Similarly, by *OPG*,

$$S_{[3]}^m(c',E) = E - 2m\left(\frac{\overline{c}-\max c'}{2}, \overline{c}, E\right) \quad \text{and} \quad S_{[3]}^m(c,E) = E - 2m\left(\frac{\overline{c}-\max c}{2}, \overline{c}, E\right).$$

Since $c' \succeq_L c$, $\max c' \leq \max c$. Since m is non-decreasing, $m(\frac{\overline{c}-\max c'}{2}, \overline{c}, E) \geq m(\frac{\overline{c}-\max c}{2}, \overline{c}, E)$. Thus, $S_{[3]}^m(c',E) \leq S_{[3]}^m(c,E)$.

Case 2: $\min c < \bar{c}/3$ and $\min c' = \bar{c}'/3$. Since S^m satisfies *OPG*, then $S_{[1]}^m(c',E) = E/3$ and $S_{[1]}^m(c,E) = m(\min c, \overline{c}, E)$. Since m is non-decreasing, $m(\min c, \overline{c}, E) \leq m(\bar{c}/3, \overline{c}, E) = E/3$. Thus, $S_{[1]}^m(c',E) \geq S_{[1]}^m(c,E)$. Similarly, by *OPG*,

$$S_{[3]}^m(c',E) = \frac{E}{3} \quad \text{and} \quad S_{[3]}^m(c,E) = E - 2m\left(\frac{\overline{c}-\max c}{2}, \overline{c}, E\right).$$

Since m is non-decreasing, $m(\frac{\overline{c}-\max c}{2}, \overline{c}, E) \leq m(\frac{\overline{c}}{3}, \overline{c}, E) = \frac{E}{3}$. Thus, $S_{[3]}^m(c',E) \leq S_{[3]}^m(c,E)$.

Case 3: $\min c = \bar{c}/3$. Since $c' \succeq_L c$, $\min c' = \bar{c}'/3$. Then,

$$S_{[1]}^m(c',E) = S_{[1]}^m(c,E) = S_{[3]}^m(c',E) = S_{[3]}^m(c,E) = E/3.$$

Thus, $S^m_{[1]}(c', E) \geq S^m_{[1]}(c, E)$ and $S^m_{[3]}(c', E) \leq S^m_{[3]}(c, E)$.

• **Proving that S satisfies *IPL*.** Let $s \in \mathbb{R}_+$ and $\{(c, E), (c', E)\} \subset \mathcal{C}(s)$ be such that $c' \succeq_L c$. We prove that $c' - S^m(c', E) \succeq_L c - S^m(c, E)$. It is enough to show that $[c' - S^m(c', E)]_{[1]} \geq [c - S^m(c, E)]_{[1]}$ and $[c' - S^m(c', E)]_{[3]} \leq [c - S^m(c, E)]_{[3]}$. Note that $\overline{c} = \overline{c}' = s$. There are three cases.

Case 1: $\min c < \overline{c}/3$ and $\min c' < \overline{c}'/3$. Since S^m satisfies *OPL*,

$$[c' - S^m(c', E)]_{[1]} = \min c' - m(\min c', \overline{c}, E)$$

and

$$[c - S^m(c, E)]_{[1]} = \min c - m(\min c, \overline{c}, E).$$

Since $c' \succeq_L c$, $\min c' \geq \min c$. By P2 (take $r = \min c$ and $r' = \min c'$), $m(\min c', \overline{c}, E) - m(\min c, \overline{c}, E) \leq \min c' - \min c$. Thus, $[c' - S^m(c', E)]_{[1]} \geq [c - S^m(c, E)]_{[1]}$. Similarly, by *OPL*,

$$[c' - S^m(c', E)]_{[3]} = \max c' - E + 2m\left(\frac{\overline{c} - \max c'}{2}, \overline{c}, E\right) \quad \text{and}$$

$$[c - S^m(c, E)]_{[3]} = \max c - E + 2m\left(\frac{\overline{c} - \max c}{2}, \overline{c}, E\right).$$

Since $c' \succeq_L c$, $\max c' \leq \max c$. By P2 (take $r = \frac{\overline{c} - \max c}{2}$ and $r' = \frac{\overline{c} - \max c'}{2}$), $m(\frac{\overline{c} - \max c'}{2}, \overline{c}, E) - m(\frac{\overline{c} - \max c}{2}, \overline{c}, E) \leq \frac{\max c - \max c'}{2}$. Thus, $[c' - S^m(c', E)]_{[3]} \leq [c - S^m(c, E)]_{[3]}$.

Case 2: $\min c < \overline{c}/3$ and $\min c' = \overline{c}'/3$. Since S^m satisfies *OPL*,

$$[c' - S^m(c', E)]_{[1]} = \overline{c}'/3 - E/3 \quad \text{and} \quad [c - S^m(c, E)]_{[1]} = \min c - m(\min c, \overline{c}, E).$$

By P2 (take $r = \min c$ and $r' = \overline{c}/3$), $m(\overline{c}/3, \overline{c}, E) - m(\min c, \overline{c}, E) \leq \overline{c}/3 - \min c$, i.e., $E/3 - m(\min c, \overline{c}, E) \leq \overline{c}'/3 - \min c$. Thus, $[c' - S^m(c', E)]_{[1]} \geq [c - S^m(c, E)]_{[1]}$. Similarly, by *OPL*,

$$[c' - S^m(c', E)]_{[3]} = \frac{\overline{c}'}{3} - \frac{E}{3},$$

and

$$[c - S^m(c, E)]_{[3]} = \max c - E + 2m\left(\frac{\overline{c} - \max c}{2}, \overline{c}, E\right).$$

By P2 (take $r = \frac{\overline{c} - \max c}{2}$ and $r' = \frac{\overline{c}}{3}$),

$$m\left(\frac{\overline{c}}{3}, \overline{c}, E\right) - m\left(\frac{\overline{c} - \max c}{2}, \overline{c}, E\right) \leq \frac{\overline{c}}{3} - \frac{\overline{c} - \max c}{2},$$

i.e., $\frac{E}{3} - m\left(\frac{\bar{c}-\max c}{2}, \bar{c}, E\right) \leq \frac{\bar{c}'}{3} - \frac{\bar{c}'-\max c}{2}$. Thus,

$$[c' - S^m(c', E)]_{[3]} \leq [c - S^m(c, E)]_{[3]}.$$

Case 3: $\min c = \bar{c}/3$. Since $c' \succeq_L c$, $\min c' = \bar{c}'/3$. Then,

$$\begin{aligned}[c' - S^m(c', E)]_{[1]} &= [c - S^m(c, E)]_{[1]} = [c' - S^m(c', E)]_{[3]} \\ &= [c - S^m(c, E)]_{[3]} = \bar{c}/3 - E/3.\end{aligned}$$

Thus, $[c' - S^m(c', E)]_{[1]} \geq [c - S^m(c, E)]_{[1]}$ and $[c' - S^m(c', E)]_{[3]} \leq [c - S^m(c, E)]_{[3]}$. □

Proof of Proposition 1. We first prove that m_f is well-defined. It is enough to show that for each $(r, s, E) \in Y_3$ such that $r \in]0, s/3[$, there is a unique $t \in \{0\} \cup \mathbb{N}$ such that $r \in]a_t s, a_{t+1} s]$; moreover, for such a t, $4^t(r - a_t s) \in]0, s/4]$. Since $\{a_t\}_{t\in\{0\}\cup\mathbb{N}}$ is an increasing sequence such that $a_0 = 0$ and $a_t \underset{t\to\infty}{\longrightarrow} 1/3$, then for each $r \in]0, s/3[$, there is a unique $t \in \{0\}\cup\mathbb{N}$ such that $r \in]a_t s, a_{t+1} s]$. If $t = 0$, then $r = 4^0(r - a_0 s) \in]a_0 s, a_1 s] =]0, s/4]$. Let $T \in \mathbb{N}$. Suppose that for each $t \leq T - 1$, if $r \in]a_t s, a_{t+1} s]$ then $4^t(r - a_t s) \in]0, s/4]$. We prove that for each $r \in]a_T s, a_{T+1} s]$, $4^T(r - a_T s) \in]0, s/4]$. Let $r \in]a_T s, a_{T+1} s]$. Then, $4r - s \in]4a_T s - s, 4a_{T+1} s - s] =]a_{T-1} s, a_T s]$. By the induction hypothesis, $4^{T-1}(4r - s - a_{T-1} s) = 4^T(r - a_T s) \in]0, s/4]$.

We now prove the proposition.

($\Rightarrow$) Let $m : Y_3 \to \mathbb{R}_+$ be a continuous function that satisfies P1, P2, and P3. We prove that there is $f \in \mathcal{F}$ such that $m = m_f$. Recall that since m satisfies P3, then for each $(s, E) \in \mathbb{R}^2_+$ such that $s \geq E$ and each $r \in [0, s/3]$,

$$m\left(\frac{s + r}{4}, s, E\right) = \frac{E}{4} + \frac{1}{4} m(r, s, E). \tag{5}$$

We complete the proof in two steps.

Step 1: Identifying a candidate initial minimal award function. Let $f : X \to \mathbb{R}_+$ be the restriction of m to X, i.e., $f \equiv m|_X$. We prove that $f \in \mathcal{F}$. Since m is continuous, so is f. Let $(s, E) \in \mathbb{R}^2_+$ be such that $s \geq E$.

• **Proving that f satisfies p1.** By property P1, $f(\cdot, s, E) : [0, s/4] \to \mathbb{R}_+$ is a non-decreasing function such that $f(0, s, E) = 0$.

• **Proving that f satisfies p2.** By property P2, for each $\{r, r'\} \subset [0, s/4]$ such that $r' \geq r$, $f(r', s, E) - f(r, s, E) \leq r' - r$.

• **Proving that f satisfies a1.** Taking $r = 0$ in (5), $m\left(\frac{s}{4}, s, E\right) = \frac{E}{4} + \frac{1}{4} m(0, s, E) = \frac{E}{4}$. Thus, $f \in \mathcal{F}$.

Step 2: Proving that $m = m_f$. Let $z \equiv (r, s, E) \in Y_3$. We prove that $m(z) = m_f(z)$. There are three cases.

Case 1: $r = 0$. By P1, $m(0, s, E) = 0$. Thus, $m(z) = m_f(z)$.

Case 2: $r = s/3$. By P1, $m(s/3, s, E) = E/3$. Thus, $m(z) = m_f(z)$.

Case 3: $r \in]0, s/3[$. We assert that for each $t \in \{0\} \cup \mathbb{N}$, if $r \in]a_t s, a_{t+1} s]$, then $m(z) = m_f(z)$. We prove this assertion by induction on t. Since $f = m|_X$, then the assertion is true for $t = 0$. Suppose now that the assertion is true for each $t < T$. Let $r \in]a_T s, a_{T+1} s]$. Since $r \in]s/4, s/3[$ then $4r - s \in]0, s/3[$. Thus, by (5), $m(z) = \frac{E}{4} + \frac{1}{4} m(4r - s, s, E)$. By the induction hypothesis (recall that $4r - s \in]a_{T-1} s, a_T s]$),

$$m(z) = \frac{E}{4} + \frac{1}{4}\left[a_{T-1}E + \frac{1}{4^{T-1}} f(4^{T-1}(4r - s - a_{T-1}s), s, E)\right].$$

Thus, $m(z) = a_T E + \frac{1}{4^T} f(4^T(r - a_T s), s, E)$. Thus, $m(z) = m_f(z)$.

Uniqueness. If $m' : Y_3 \to \mathbb{R}_+$ is a continuous function that satisfies P1, P2, P3, and coincides with m on X, then $m'|_X = f$. The same argument as above shows that $m' = m_f$. Thus, $m = m'$.

($\Leftarrow$) Let $f \in \mathcal{F}$. We prove that m_f is continuous, satisfies P1, P2, P3, and has values in $\mathbb{R}_+$.

• **Proving that m_f is continuous.** Let $\{z^k \equiv (r^k, s^k, E^k)\}_{k \in \mathbb{N}}$ be a convergent sequence in Y_3. Let $z = (r, s, E) \equiv \lim_{k\to\infty} z^k$. We prove that $m_f(z^k) \underset{k\to\infty}{\longrightarrow} m_f(z)$. There are four cases.

Case 1: $r = 0$. Then there is $K \in \mathbb{N}$ such that for each $k \geq K$, $m_f(z^k) = f(z^k)$. Since $f \in \mathcal{F}$, $z^k \underset{k\to\infty}{\longrightarrow} z$, then $m_f(z^k) \underset{k\to\infty}{\longrightarrow} f(z) = m_f(z)$.

Case 2: There is $t > 0$ such that $r = a_t s$. For each $k \in \mathbb{N}$, let $r_+^k \equiv \max\{a_t s^k, r^k\}$ and $r_-^k \equiv \min\{a_t s^k, r^k\}$. Observe that $4^t(r_+^k - a_t s^k) \underset{k\to\infty}{\longrightarrow} 0$ and $4^{t-1}(r_-^k - a_{t-1}s^k) \underset{k\to\infty}{\longrightarrow} s/4$. Since $s^k \underset{k\to\infty}{\longrightarrow} s$ and $r^k \underset{k\to\infty}{\longrightarrow} a_t s$, then there is $K \in \mathbb{N}$ such that for each $k \geq K$, $r_k \in]a_{t-1}s^k, a_{t+1}s^k[$, and thus,

$$m_f(r^k, s^k, E^k) = \begin{cases} a_t E^k + \frac{1}{4^t} f\left(4^t(r_+^k - a_t s^k), s^k, E^k\right) & \text{if } r^k = r_+^k \\ a_{t-1}E^k + \frac{1}{4^{t-1}} f\left(4^{t-1}(r_-^k - a_{t-1}s^k), s^k, E^k\right) & \text{otherwise.} \end{cases}$$

Since $f \in \mathcal{F}$ then

$$f\left(4^t(r_+^k - a_t s^k), s^k, E^k\right) \underset{k\to\infty}{\longrightarrow} 0$$

and

$$f\left(4^{t-1}(r_-^k - a_{t-1}s^k), s^k, E^k\right) \underset{k\to\infty}{\longrightarrow} E/4.$$

Thus, $m_f(z^k) \underset{k\to\infty}{\longrightarrow} a_t E = m_f(a_t s, s, E) = m_f(z)$.

Case 3: There is $t \in \mathbb{N}$ such that $r \in]a_t s, a_{t+1}s[$. Since $s^k \underset{k\to\infty}{\longrightarrow} s$ and $r^k \underset{k\to\infty}{\longrightarrow} a_t s$, then there is $K \in \mathbb{N}$ such that for each $k \geq K$, $r_k \in]a_t s^k, a_{t+1}s^k[$ and thus,

$$m_f(z^k) = a_t E^k + \frac{1}{4^t} f\left(4^t(r^k - a_t s^k), s^k, E^k\right).$$

Since $f \in \mathcal{F}$, then $m_f(z^k) \underset{k\to\infty}{\longrightarrow} a_t E + \frac{1}{4^t} f\left(4^t(r - a_t s), s, E\right) = m_f(z)$.

Case 4: $r = s/3$. For each $k \in \mathbb{N}$, let $t(k) \in \mathbb{N}$ be such that $r^k \in]a_{t(k)}s^k, a_{t(k)+1}s^k]$. Since $s^k \underset{k\to\infty}{\longrightarrow} s$ and $r^k \underset{k\to\infty}{\longrightarrow} s/3$, then $t(k) \underset{k\to\infty}{\longrightarrow} \infty$. Now, since f is continuous, then $f(\cdot, s, E)$ is bounded. Thus, $m_f(z^k) \underset{k\to\infty}{\longrightarrow} \lim_{k\to\infty} a_{t(k)} E^k = E/3 = m_f(z)$.

• **Proving that $\boldsymbol{m_f}$ satisfies P1.** Let $(s, E) \in \mathbb{R}^2_+$ be such that $s \geq E$. Let $\{r, r'\} \subset [0, s/3]$ be such that $r' \geq r$. We prove that $m_f(r', s, E) \geq m_f(r, s, E)$. There are four cases.

Case 1: $r = 0$. Since $f \geq 0$, then $m_f(r', s, E) \geq 0 = m_f(r, s, E)$.

Case 2: There is $t \in \{0\} \cup \mathbb{N}$ such that $\{r, r'\} \subset]a_t s, a_{t+1}s]$. Since $r' \geq r$, then $4^t(r' - a_t s) \geq 4^t(r - a_t s)$. Since $f \in \mathcal{F}$, then $f(\cdot, s, E) : [0, s/4] \to [0, E/4]$ is non-decreasing. Thus,

$$\begin{aligned} m_f(r', s, E) - m_f(r, s, E) &= \tfrac{1}{4^t}\left[f\left(4^t(r' - a_t s), s, E\right) - f\left(4^t(r - a_t s), s, E\right)\right] \\ &\geq 0. \end{aligned}$$

Case 3: There are $\{t, t'\} \subset \{0\} \cup \mathbb{N}$ such that $t \neq t'$, $r \in]a_t s, a_{t+1}s]$ and $r' \in]a_{t'}s, a_{t'+1}s]$. Since $r' \geq r$, then $t' \geq t + 1$. Now, since $f \in \mathcal{F}$, then $0 \leq f(\cdot, s, E) \leq E/4$. Thus,

$$m_f(r, s, E) \leq a_t E + E/4^{t+1} = a_{t+1}E \leq a_{t'}E.$$

Consequently, $m_f(r, s, E) \leq a_{t'}E + f(4^{t'}(r - a_{t'}s), s, E)/4^{t'} = m_f(r', s, E)$.

Case 4: $r > 0$ and $r' = s/3$. Let $t \in \{0\} \cup \mathbb{N}$ be such that $r \in]a_t s, a_{t+1}s]$. Since $f \in \mathcal{F}$, then $0 \leq f(\cdot, s, E) \leq E/4$. Thus, $m_f(r, s, E) \leq a_{t+1}E \leq E/3 = m_f(r', s, E)$.

Finally, from the definition of m_f, $m(0, s, E) = 0$ and $m(s/3, s, E) = E/3$.

• **Proving that $\boldsymbol{m_f}$ satisfies P2.** Let $(s, E) \in \mathbb{R}^2_+$ be such that $s \geq E$. Let $\{r, r'\} \subset [0, s/3]$ be such that $r' \geq r$. We prove that $m_f(r', s, E) - m_f(r, s, E) \leq r' - r$. There are four cases.

Case 1: There is $t \in \{0\} \cup \mathbb{N}$ such that $\{r, r'\} \subset]a_t s, a_{t+1} s]$. Thus,

$$m_f(r', s, E) - m_f(r, s, E) = \frac{1}{4^t}\left[f\left(4^t(r' - a_t s), s, E\right) - f\left(4^t(r - a_t s), s, E\right)\right].$$

Since $f \in \mathcal{F}$, then $f\left(4^t(r' - a_t s), s, E\right) - f\left(4^t(r - a_t s), s, E\right) \leq 4^t(r' - r)$. Thus, $m_f(r', s, E) - m_f(r, s, E) \leq r' - r$.

Case 2: There are $\{t, t'\} \subset \{0\} \cup \mathbb{N}$ such that $r = a_t s$ and $r' = a_{t'} s$. Since $4^t(r - a_t s) = 0$ and $4^{t'}(r' - a_{t'} s) = 0$, then $m_f(r', s, E) - m_f(r, s, E) = a_{t'} E - a_t E \leq a_{t'} s - a_t s = r' - r$.

Case 3: There are $\{t, t'\} \subset \{0\} \cup \mathbb{N}$ such that $t \neq t'$, $r \in]a_t s, a_{t+1} s]$ and $r' \in]a_{t'} s, a_{t'+1} s]$. By Cases 1 and 2,

$$\begin{aligned} m_f(r', s, E) - m_f(r, s, E) &= \left\{ \begin{array}{l} m_f(r', s, E) - m_f(a_{t'} s, s, E) + \\ m_f(a_{t'} s, s, E) - m_f(a_t s, s, E) + \\ m_f(a_t s, s, E) - m_f(r, s, E) \end{array} \right\} \\ &\leq r' - a_{t'} + a_{t'} - a_t + a_t - r. \end{aligned}$$

Thus, $m_f(r', s, E) - m_f(r, s, E) \leq r' - r$.

Case 4: $r = 0$ or $r' = \frac{s}{3}$. Let $\{r^k\}_{k \in \mathbb{N}}$ be a sequence in $]0, \frac{r'+r}{2}[$ such that $r^k \underset{k\to\infty}{\longrightarrow} r$, and let $\{v^k\}_{k \in \mathbb{N}}$ be a sequence in $]\frac{r'+r}{2}, \frac{s}{3}[$ such that $v^k \underset{k\to\infty}{\longrightarrow} r'$. By Cases 1 to 3, for each $k \in \mathbb{N}$, $m_f(v^k, s, E) - m_f(r^k, s, E) \leq v^k - r^k$. Since m_f is continuous, then $m_f(r', s, E) - m_f(r, s, E) \leq r' - r$.

• **Proving that m_f satisfies P3.** Let $(s, E) \in \mathbb{R}^2_+$ be such that $s \geq E$. We prove that for each $r \in [0, s/3]$,

$$m_f\left(\frac{s + r}{4}, s, E\right) = \frac{E}{4} + \frac{1}{4} m_f(r, s, E).$$

The function, $r' \in [s/4, s/3] \mapsto 4r' - s$, is a bijection from $[s/4, s/3]$ to $[0, s/3]$. Thus, the statement above is equivalent to: for each $r \in [s/4, s/3]$, $m_f(r, s, E) = \frac{E}{4} + \frac{1}{4} m_f(4r - s, s, E)$. We prove this last statement. Let $r \in [s/4, s/3]$. There are three cases.

Case 1: $r = s/4$. Since $f \in \mathcal{F}$, then $f(r, s, E) = E/4$. Since $m_f(4r - s, s, E) = 0$, then $m_f(r, s, E) = \frac{E}{4} + \frac{1}{4} m_f(4r - s, s, E)$.

Case 2: $r \in]s/4, s/3[$. Let $t \in \mathbb{N}$ be such that $r \in]a_t s, a_{t+1} s]$. Thus, $(4r - s) \in]a_{t-1} s, a_t s]$. Consequently,

$$m_f(4r - s, s, E) = a_{t-1} E + \frac{1}{4^{t-1}} f\left(4^{t-1}[(4r - s) - a_{t-1} s], s, E\right).$$

Since $m_f(r,s,E) = a_t E + \frac{1}{4^t} f\left(4^t(r - a_t s), s, E\right)$, then the direct calculation shows that $m_f(r,s,E) = \frac{E}{4} + \frac{1}{4} m_f(4r - s, s, E)$.

Case 3: $r = s/3$. Since $m_f(r,s,E) = E/3$ and $4r - s = r$, then

$$m_f(r,s,E) = \frac{E}{4} + \frac{1}{4}\left(\frac{E}{3}\right) = \frac{E}{4} + \frac{1}{4} m_f(4r - s, s, E).$$

□

Proof of Lemma 3. Let be S be a rule that satisfies *IPG*. We claim that S^d satisfies *IPL*. Let $(c,E) \in \mathcal{C}$ and $c' \in \mathbb{R}^N_+$ such that $\bar{c} = \sum_{i \in N} c'_i$ and $c \succeq_L c'$. By *IPG*, $S(c, \bar{c} - E) \succeq_L S(c', \bar{c} - E)$. Thus, $c - (c - S(c, \bar{c} - E)) \succeq_L c' - (c' - S(c', \bar{c} - E))$. Thus, $c - S^d(c,E) \succeq_L c' - S^d(c', E)$. A similar argument shows that if S satisfies *IPL*, then S^d satisfies *IPG*. □

Proof of Lemma 4. Let $\{f, f'\} \subseteq \mathcal{F}$. Let $g \equiv \max\{f, f'\}$. We first prove that S^g is an upper bound of $\{S^f, S^{f'}\}$. One can easily see that $g \in \mathcal{F}$. Let $(c,E) \in \mathcal{C}$. We assert that $S^g(c,E) \succeq_L S^f(c,E)$. Since $f \leq g$, then $m_f \leq m_g$. There are two cases.

Case 1: $\min c = \max c$. Then $S^g(c,E) = S^f(c,E)$ and $S^g(c,E) \succeq_L S^f(c,E)$.

Case 2: $\min c < \max c$. Then

$$S^f_{[1]}(c,E) = m_f(\min c, \bar{c}, E) \leq m_g(\min c, \bar{c}, E) = S^g_{[1]}(c,E),$$

and

$$S^f_{[3]}(c,E) = E - 2m_f\left(\tfrac{\bar{c} - \max c}{2}, \bar{c}, E\right) \geq E - 2m_g\left(\tfrac{\bar{c} - \max c}{2}, \bar{c}, E\right) = S^g_{[3]}(c,E).$$

Thus, $S^g(c,E) \succeq_L S^f(c,E)$.

Now, we prove that S^g is the least upper bound of $\{S^f, S^{f'}\}$. Let $h \in \mathcal{F}$ be such that S^h is an upper bound of $\{S^f, S^{f'}\}$. We assert that $h \geq g$. Suppose by contradiction that there is $(r,s,E) \in X$ such that $g(r,s,E) > h(r,s,E)$. Suppose w.l.o.g. that $g(r,s,E) = f(r,s,E)$. Then,

$$S^f_{[1]}(r,s,E) = f(r,s,E) = g(r,s,E) > h(r,s,E) = S^h_{[1]}(r,s,E).$$

But, since $S^h \succeq_L S^f$, $S^h_{[1]}(r,s,E) \geq S^f_{[1]}(r,s,E)$. This is a contradiction. Since $\max\{h,g\} = h$, then by the argument above, S^h is an upper bound of $\{S^h, S^g\}$. Thus, $S^h \succeq_L S^g$.

A symmetric argument proves that $S^{\min\{f,f'\}}$ is the greatest lower bound of $\{S^f, S^{f'}\}$ under $\succeq_L$. □

Proof of Proposition 3. To simplify we write A and L instead of $S^{f_{CEA}}$ and $S^{f_{CEL}}$, respectively. Likewise, we write m_A and m_L instead of $m_{f_{CEA}}$ and $m_{f_{CEL}}$, respectively. First, we prove that L satisfies *composition down*. Let $(c, E) \in \mathcal{C}$ and $E' < E$. We claim that $L(c, E') = L(L(c, E), E')$. First, we prove that $L_{[1]}(c, E') = L_{[1]}(L(c, E), E')$, which, by *OPG*, is equivalent to

$$m_L(\min c, \bar{c}, E') = m_L(m_L(\min c, \bar{c}, E), E, E'). \tag{6}$$

Let $t \in \{0\} \cup \mathbb{N}$ be such that $\min c \in [a_t\bar{c}, a_{t+1}\bar{c}]$. By (4), $m_L(\min c, \bar{c}, E) \in [a_t E, a_{t+1}E]$. One can prove that:[21] (i) If $\min c \in [a_{t+1}\bar{c} - \frac{E'}{4^{t+1}}, a_{t+1}\bar{c}]$, then $m_L(m_L(\min c, \bar{c}, E), E, E') = a_t E' + \min c - a_{t+1}\bar{c} + \frac{E'}{4^{t+1}}$; and (ii) If $\min c \in [a_t\bar{c}, a_{t+1}\bar{c} - \frac{E'}{4^{t+1}}]$, then $m_L(m_L(\min c, \bar{c}, E), E, E') = a_t E'$. In both cases, $m_L(m_L(\min c, \bar{c}, E), E, E') = m_L(\min c, \bar{c}, E')$.

We now prove that $L_{[3]}(c, E') = L_{[3]}(L(c, E), E')$. Recall that $L_{[3]}(c, E) = E - 2m_L\left(\frac{\bar{c} - \max c}{2}, \bar{c}, E\right)$. By *OPG*, $L_{[3]}(L(c, E), E') = E' - 2m_L\left(\frac{E - L_{[3]}(c,E)}{2}, E, E'\right)$. Thus, $L_{[3]}(L(c, E), E') = E' - 2m_L\left(m_L\left(\frac{\bar{c} - \max c}{2}, \bar{c}, E\right), E, E'\right)$. By (6),

$$L_{[3]}(L(c, E), E') = E' - 2m_L\left(\frac{\bar{c} - \max c}{2}, \bar{c}, E'\right) = L_{[3]}(c, E').$$

Feasibility implies that $L_{[2]}(c, E') = L_{[2]}(L(c, E), E')$. Thus, by *OPG*, $L(c, E') = L(L(c, E), E')$.

Since *composition down* and *composition up* are dual, then A satisfies *composition up*.

A similar argument shows that L satisfies *composition up*. Thus, A satisfies *composition down*. □

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