Reflecting inequality of claims in gains and losses

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Abstract

We consider the problem of dividing a resource among a group of agents who have conflicting claims on it. We follow the axiomatic approach and investigate the class of rules satisfying claims-inequality and claims-order preservation in gains and losses. We show that these axioms single out the proportional rule when there are more than three agents. This result confirms the central role of this rule and furthers our understanding of it in claims problems.

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1 Introduction

How should a resource be divided among a group of agents when they have conflicting claims on it? A typical situation is when a firm goes bankrupt and its liquidation value has to be divided among its creditors. An alternative problem that fits this mathematical structure is that of collecting an amount of a resource from a group of agents whose individual endowments add up to at least the amount required. Here, an illustration is a government taxing incomes in order to implement a public project.

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We refer to each situation of this kind as a claims problem.[1] We require that each agent receive a non-negative amount that is at most as large as her claim, and that the individual amounts received by the agents add up to the amount to distribute, the endowment.

A rule is a function that associates with each problem an awards vector for it. In our study of rules, we follow the axiomatic approach.[2] Our first and central two axioms, "claims-inequality preservation in gains" and "claims-inequality preservation in losses," pertain to the class of "distributional" properties of rules. This class has been the object of much attention in recent literature, e.g., Hougaard and Thorlund-Petersen (2001); Hougaard and Østerdal (2005); Moreno-Ternero and Villar (2006a,b); Ju and Moreno-Ternero (2006, 2008); Bosmans and Lauwers (2007); and Thomson (2007). Our two requirements are that inequalities in claims, under the Lorenz criterion, be reflected in awards and losses, respectively (these axioms were first proposed by Hougaard and Thorlund-Petersen 2001). More precisely, fix the endowment and suppose that two claims vectors, whose coordinates add up to the same amount, are related by Lorenz domination. We require that the awards and losses vectors associated with these claims vectors be also related by Lorenz domination (of course, in the same direction).

We illustrate the idea behind these two axioms for the taxation interpretation of our model. Suppose that a government decides to tax incomes in order to implement a public project. If the main purpose of taxation is to cover the cost of the project or if the project benefits evenly all the agents, then one may want to limit the impact of taxation on income distribution. There are several ways to evaluate skewedness of income distributions, among which the Lorenz criterion occupies a central place. In order to limit the extent to which taxation disturbs income distribution, it is natural then to require that Lorenz comparability be preserved after taxation. The effects of taxation can be evaluated in two ways: by the agents' after-tax income and by the amount of taxes paid by the agents (both of them are the reflection of the pre-tax income distribution). This motivates the requirement that Lorenz comparability be preserved for both distributions, i.e., the distribution of after-tax incomes (gains) and the distribution of taxes (losses).

We study the implications of these two axioms in the presence of two basic additional axioms: "claims-order preservation in gains" and "claims-order preservation in losses" (Aumann and Maschler, 1985). Our main result is that claims-inequality preservation in gains and losses and claims-order preservation in gains and losses single out the proportional rule when there are more than three agents.

Characterizing the class of rules satisfying "continuity," claims-inequality preservation

[1] Alternative names in the literature are "bankruptcy problem," "taxation problem," and "rationing problem."

[2] O'Neill (1982) pioneered the application of the axiomatic approach to claims problems. The problems have been studied extensively in the literature since then, e.g., Hokari and Thomson (2003), Alcalde et al (2005), and Dominguez and Thomson (2006) – see Moulin (2002) and Thomson (2003, 2008) for surveys.

in gains and losses, and claims-order preservation in gains and losses is first addressed by Hougaard and Østerdal (2005). They assert that for more than two agents, the proportional rule is the only rule satisfying these axioms. However, a step in their proof contains a subtle error.[3] In fact, their assertion itself is incorrect when there are three agents. In a companion paper (Kasajima and Velez 2009), we characterize the family of such rules when there are three agents and show that there are infinitely many of them. For more than three agents, it follows from our main theorem in the current paper that Hougaard and Østerdal's assertion is indeed true. However, our proofs show that a significantly more intricate argument is needed to establish it. Moreover, since we do not impose continuity, our characterization is tighter. The complications do not come from our dropping continuity, however. One can easily see that they would remain even if we imposed this axiom.

In the context of claims problems, a growing literature has investigated the implications of precluding certain forms of coalitional manipulation. The basic axiom, "no advantageous transfer," requires that no group of agents receive more in the aggregate by transferring claims among themselves. This axiom singles out the proportional rule for more than two agents (Moulin 1985; Chun 1988; Ju et al 2007; Ju 2007; and Thomson 2008). Our axioms preclude manipulation among coalitions composed of either the agents with the k highest claims, or the agents with the k lowest claims, when the members of a coalition transfer claims among themselves and the order of their claims remains the same. Our proofs reveal that this weaker axiom, together with continuity and "equal treatment of equals," characterize the proportional rule for more than three agents. This result is interesting because in the weaker formulation of no advantageous transfer, manipulating coalitions emerge "endogenously" from the data of the problem and the scope of their manipulation is limited.

In the public finance literature, Moyes (1989, 1994) and Arnold (1990) analyze the class of functions that "transform" an income distribution into another one, assuming that each agent's final income depends only on her initial income. The difference between each agent's initial and final incomes is that agent's tax (or subsidy). However, unlike for our model, the total tax to collect (or the total subsidy to distribute) is not exogenously given. In that context, Moyes (1989) and Arnold (1990) show that if there are at least two agents, affine transformations are the only transformations that preserve Lorenz-inequality of income distributions. Moyes (1989, 1994) further studies the class of functions that preserve inequality of income distributions for a variety of indicators of inequality (see also Ebert 2004 for related results). In our model, which includes the specification of how much is available (in theirs, this would be how much to be collected or distributed), their

[3]Their proof fails in their Step 3. They write "let c''_k be the highest c_k for which there exists a pair (c'_k, c_k) satisfying $e_k(c'_k) = 0$, $e_k(c_k) = 0$, $e_k(\widehat{c}_k) > 0$, for all $c'_k < \widehat{c}_k < c_k$ (p.359 lines 9-10)." However, the variable c''_k in their statement may not exist, and in fact, examples can be constructed in which it does not exist. Thus their proof fails for any number of agents greater than two.

assumption that each agent's final income depends only on her initial income corresponds to assuming that the award to each agent depends only on her claim, the total claim, and the amount available. This assumption itself singles out the proportional rule for more than two agents (Moulin 1988; Ching and Kakkar 2001; Thomson 2008).

The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 presents our characterization of the proportional rule and discusses extensions of our result.

2 The model

Let $\boldsymbol{N} \equiv \{1, \ldots, n\}$ be a set of agents. A **claims problem for $\boldsymbol{N}$** is a pair (c, E) where $c = (c_i)_{i \in N} \in \mathbb{R}^N_+$ and $E \in \mathbb{R}_+$ are such that $\sum_{i \in N} c_i \geq E$. The list c is interpreted as a vector of **claims** agents hold and the number E as the amount available of an infinitely divisible resource, the **endowment**; this amount is not sufficient to honor all claims. The **set of claims problems** is denoted $\boldsymbol{\mathcal{C}}$. An **awards vector for $\boldsymbol{(c, E)} \in \boldsymbol{\mathcal{C}}$** is a vector $x \in \mathbb{R}^N_+$, such that $x \leq c$ and $\sum_{i \in N} x_i = E$.[4] A **rule**, denoted generically S, is a function that associates with each problem an awards vector for it.

For each $c \in \mathbb{R}^N_+$, let $\boldsymbol{\bar{c}} \equiv \sum_{i \in N} c_i$, $\mathbf{min}\, \boldsymbol{c} \equiv \min_{i \in N} c_i$, and $\mathbf{max}\, \boldsymbol{c} \equiv \max_{i \in N} c_i$. For each $s \in \mathbb{R}_+$, let $\boldsymbol{\mathcal{C}(s)} \equiv \{(c, E) \in \mathcal{C} : \bar{c} = s\}$.

A rule S is **continuous** if for each convergent sequence of problems $\{(c_k, E_k)\}_{k \in \mathbb{N}}$ the sequence $\{S(c_k, E_k)\}_{k \in \mathbb{N}}$ converges to $S(\lim_{k \to \infty}(c_k, E_k))$. A rule S is **claims-order preserving in gains (OPG)**, if for each $(c, E) \in \mathcal{C}$ and each $\{i, j\} \subseteq N$ such that $c_i \leq c_j$, $S_i(c, E) \leq S_j(c, E)$; it is **claims-order preserving in losses (OPL)** if for each $(c, E) \in \mathcal{C}$ and each $\{i, j\} \subseteq N$ such that $c_i \leq c_j$, $c_i - S_i(c, E) \leq c_j - S_j(c, E)$.[5] Let us remark that if a rule S is *OPG* or *OPL*, then it satisfies **equal treatment of equals (ETE)**, i.e., for each $(c, E) \in \mathcal{C}$ and each $\{i, j\} \subseteq N$ such that $c_i = c_j$, $S_i(c, E) = S_j(c, E)$.

For each $y \in \mathbb{R}^N_+$ and each $t \in \{1, \ldots, n\}$, let $y_{[t]}$ be the t-order statistic of y, i.e., the vector $(y_{[t]})_{t=1}^n$ is a permutation of y such that $y_{[1]} \leq y_{[2]} \leq \cdots \leq y_{[n]}$. Let $\{y, y'\} \subseteq \mathbb{R}^N_+$ be such that $\sum_{i \in N} y_i = \sum_{i \in N} y'_i$. Then, **$\boldsymbol{y}$ Lorenz dominates $\boldsymbol{y'}$**, denoted $y \succeq_L y'$, if for each $k \in \{1, \ldots, n\}$, $\sum_{t=1}^k y_{[t]} \geq \sum_{t=1}^k y'_{[t]}$.[6] A rule S is **claims-inequality preserving in gains (IPG)** if for each $s \in \mathbb{R}_+$ and each $\{(c, E), (c', E)\} \subset \mathcal{C}(s)$ such that $c \succeq_L c'$, $S(c, E) \succeq_L S(c', E)$; it is **claims-inequality preserving in losses (IPL)** if for each $s \in \mathbb{R}_+$ and each $\{(c, E), (c', E)\} \subset \mathcal{C}(s)$ such that $c \succeq_L c'$, $c - S(c, E) \succeq_L c' - S(c', E)$.[7]

We refer to *OPG*, *OPL*, *IPG*, and *IPL* as the **core axioms**.

[4] Vector inequalities: $x \geq y$ allows x and y to be equal; $x \gneq y$ does not; $x > y$ means that each coordinate of x is greater than the corresponding coordinate of y.

[5] *OPG* and *OPL* are introduced by Aumann and Maschler (1985).

[6] See Dalton (1920) and Marshall and Olkin (1979) for more information regarding the Lorenz criterion.

[7] *IPG* and *IPL* are introduced by Hougaard and Thorlund-Petersen (2001).

The **proportional rule**, $\boldsymbol{P}$, divides the endowment proportionally to claims, i.e., for each $(c, E) \in \mathcal{C}$, $P(c, E) \equiv \lambda c$, where $\lambda \in \mathbb{R}_+$ is chosen such that $\sum_{i \in N} P_i(c, E) = E$.

3 Characterization of the proportional rule

In this section, we characterize the proportional rule as the unique *continuous* rule satisfying the *core axioms* when there are more than three agents. In fact, our theorem states that even without requiring *continuity*, the *core axioms* single out this rule.

Lemma 1 states a critical structural property satisfied by any rule satisfying the *core axioms*: the award to a minimal claimant is a function of her claim, the aggregate claim, and the endowment. We call this function the "minimal award function" (associated with the rule). A symmetric statement is also true for the award of a maximal claimant, i.e., her award is a function of her claim, the aggregate claim, and the endowment. The lemma also lists four key properties satisfied by minimal award functions.[8]

Let $\boldsymbol{Y_n} \equiv \{(r, s, E) \in \mathbb{R}^3_+ : r \leq \frac{s}{n} \text{ and } s \geq E\}$.

Lemma 1. Assume $n \geq 2$. If a rule S satisfies the *core axioms*, then there is $m : Y_n \to \mathbb{R}_+$ such that for each $(c, E) \in \mathcal{C}$ and each $i \in \arg\min_{k \in N} c_k$, $S_i(c, E) = m(\min c, \bar{c}, E)$, and for each $j \in \arg\max_{k \in N} c_k$, $S_j(c, E) = E - (n-1)\, m\Big(\frac{\bar{c} - \max c}{n-1}, \bar{c}, E\Big)$. Moreover, m satisfies the following properties:

- **P1**: For each $(s, E) \in \mathbb{R}^2_+$ such that $s \geq E$, $m(\cdot, s, E) : [0, \frac{s}{n}] \to \mathbb{R}_+$ is a non-decreasing function such that $m(0, s, E) = 0$ and $m\left(\frac{s}{n}, s, E\right) = \frac{E}{n}$.

- **P2**: For each $(s, E) \in \mathbb{R}^2_+$ such that $s \geq E$, and each $\{r, r'\} \subset [0, \frac{s}{n}]$ such that $r' \geq r$, $m(r', s, E) - m(r, s, E) \leq r' - r$.

- **P3**: For each $(r, s, E) \in Y_n$,

$$m\Big(\frac{s + (n-2)r}{2(n-1)}, s, E\Big) = \frac{E}{2(n-1)} + \frac{n-2}{2(n-1)} m\left(r, s, E\right). \tag{1}$$

- **P4**: m is continuous in its first argument; moreover, it is continuous whenever S is.

Proof. Let S be a rule satisfying the *core axioms*. We proceed in four steps. We omit the proof of Steps 1 and 2 since they are similar to arguments in Hougaard and Østerdal (2005).

[8] We state this lemma for $n \geq 2$ since these properties are crucial to the characterization of the family of rules satisfying the *core axioms* for the two-agent and three-agent cases. See Kasajima and Velez (2009) for the three-agent case. The two-agent case is available from the authors upon request.

Step 1: For each $s \in \mathbb{R}_+$, each $\{(c,E),(c',E)\} \subset \mathcal{C}(s)$ such that $\min c = \min c'$, each $i \in \arg\min_{k\in N} c_k$, and each $j \in \arg\min_{k\in N} c'_k$, $S_i(c,E) = S_j(c',E)$.

Step 2: For each $s \in \mathbb{R}_+$, each $\{(c,E),(c',E)\} \subset \mathcal{C}(s)$ such that $\max c = \max c'$, each $i \in \arg\max_{k\in N} c_k$, and each $j \in \arg\max_{k\in N} c'_k$, $S_i(c,E) = S_j(c',E)$.

Step 3: Determining the awards to minimal and maximal claimants.

• Minimal claimants: Let $s \in \mathbb{R}_+$, $r \in [0, \frac{s}{n}]$, and $i \in N$. Let $c(r,s,i) \in \mathcal{C}(s)$ be the vector defined by: $c_i(r,s,i) \equiv r$ and for each $j \in N \setminus \{i\}$, $c_j(r,s,i) \equiv \frac{s-r}{n-1}$. Let $m : Y_n \to \mathbb{R}_+$ be the function defined as follows: for each $(r,s,E) \in Y_n$, $m(r,s,E) \equiv S_i(c(r,s,i),E)$ for some $i \in N$. By Step 1 and since $\min c(r,s,i) = r$, then m is well-defined, and for each $(c,E) \in \mathcal{C}$ and each $i \in \arg\min_{k\in N} c_k$, $S_i(c,E) = m(\min c, \bar{c}, E)$.

• Maximal claimants: Let $s \in \mathbb{R}_+$, $r \in [\frac{s}{n}, s]$, and $i \in N$. Let $c'(r,s,i) \in \mathcal{C}(s)$ be the vector defined by: $c'_i(r,s,i) \equiv r$ and for each $j \in N \setminus \{i\}$, $c_j(r,s,i) \equiv \frac{s-r}{n-1}$. Since $\min c'(r,s,i) = \frac{s-r}{n-1}$, then for each $E \in \mathbb{R}_+$ such that $E \leq s$, and each $j \in N \setminus \{i\}$, $S_j(c'(r,s,i),E) = m\left(\frac{s-r}{n-1}, s, E\right)$. Thus, $S_i(c'(r,s,i),E) = E - (n-1)m\left(\frac{s-r}{n-1}, s, E\right)$. By Step 2 and since $\max c'(r,s,i) = r$, then for each $(c,E) \in \mathcal{C}$ and each $i \in \arg\max_{k\in N} c_k$,

$$S_i(c,E) = E - (n-1)\, m\!\left(\frac{\bar{c} - \max c}{n-1}, \bar{c}, E\right).$$

Step 4: Properties of m.

• **Proving P1.** Let $(s,E) \in \mathbb{R}^2_+$ be such that $s \geq E$. Let $\{r,r'\} \subset [0, \frac{s}{n}]$ be such that $r' \geq r$. Let $i \in N$. Let $c(r,s,i) \in \mathcal{C}(s)$ be the vector defined by: $c_i(r,s,i) \equiv r$ and for each $j \in N \setminus \{i\}$, $c_j(r,s,i) \equiv \frac{s-r}{n-1}$. Since $c(r',s,i) \succeq_L c(r,s,i)$ and S satisfies *IPG*, then $S(c(r',s,i),E) \succeq_L S(c(r,s,i),E)$. Thus, $m(r',s,E) = S_i(c(r',s,i),E) \geq S_i(c(r,s,i),E) = m(r,s,E)$. Therefore, the function $m(\cdot,s,E)$ is non-decreasing. Since $S_i(c(0,s,i),E) = 0$, then $m(0,s,E) = 0$. Note that $c\left(\frac{s}{n}, s, i\right) = \left(\frac{s}{n}, \cdots, \frac{s}{n}\right)$. Since S satisfies *ETE*, then $S_i\left(c\left(\frac{s}{n}, s, i\right), E\right) = \frac{E}{n}$. Thus, $m\left(\frac{s}{n}, s, E\right) = \frac{E}{n}$.

• **Proving P2.** Let $(s,E) \in \mathbb{R}^2_+$ be such that $s \geq E$. Let $\{r,r'\} \subset [0, \frac{s}{n}]$ be such that $r' \geq r$. Let $i \in N$. Since $c(r',s,i) \succeq_L c(r,s,i)$ and S satisfies *IPL*, then

$$c(r',s,i) - S(c(r',s,i),E) \succeq_L c(r,s,i) - S(c(r,s,i),E).$$

Thus, $r' - m(r',s,E) = r' - S_i(c(r',s,i),E) \geq r - S_i(c(r,s,i),E) = r - m(r,s,E)$. Therefore, $m(r',s,E) - m(r,s,E) \leq r' - r$.

• **Proving P3.** Clearly, P1 implies P3 when $n = 2$. Assume now that $n \geq 3$. Let $(r,s,E) \in Y_n$. We prove (1). Note that $r \in [0, \frac{s}{n}]$. Let $c^* \equiv (r, \ldots, r, \frac{s-(n-2)r}{2}, \frac{s-(n-2)r}{2}) \in \mathcal{C}(s)$. Since $\min c^* = r$, then by Step 3, $S_{n-2}(c^*,E) = m(r,s,E)$. Now, since $\max c^* =$

$\frac{s-(n-2)r}{2}$, then by Step 3,

$$S_{n-2}(c^*,E) = E-(n-3)m\,(r,s,E) - 2\left[E-(n-1)\,m\!\left(\frac{s-\frac{s-(n-2)r}{2}}{n-1},s,E\right)\right].$$

We obtain (1) by equating $S_{n-2}(c^*,E)$ in the two expressions above.

• **Proving P4.** Clearly, if S is continuous, so is m. One easily sees that (P1) and (P2) imply $m(\cdot,s,E)$ is a contraction and thus continuous. □

We are now ready to state our theorem, namely, when there are more than three agents, the *core axioms* single out the proportional rule. This result confirms the central role of this rule and furthers our understanding of it in claims problems.

Theorem 1. Assume $n > 3$. A rule satisfies the *core axioms* if and only if it is the proportional rule.

First, we sketch the proof. Consider a rule satisfying the *core axioms*. From Lemma 1 we know that there is a minimal award function associated with the rule, say m, that satisfies P1 to P4. Our proof consists on showing that m coincides with the minimal award function associated with the proportional rule. Let $c_1 \leq \cdots \leq c_n$ and $c'_1 \leq \cdots \leq c'_n$ be two claims vectors with the same aggregate claim, $\bar{c} \geq E$. Let us omit the second and third arguments of m, with the understanding that they are the fixed values $\bar{c}$ and E. A crucial step of our proof is showing that if c_1 and c'_1 are "small and close to each other," then there is $\alpha \in (0,1)$ such that $m(\alpha c_1 + (1-\alpha)c'_1) = \alpha m(c_1) + (1-\alpha)m(c'_1)$. Since m is continuous in its first argument, this turns out to be enough to guarantee that it is linear in its first argument on $\left[0, \frac{2\bar{c}}{3(n-1)}\right]$. Using P3, we extend this result to $[0, \frac{\bar{c}}{n}]$. Since m satisfies P1, then it interpolates the proportional value when $c_1 = \frac{\bar{c}}{n}$. Thus, it coincides with the proportional minimal award function.

Let us remark that our proof relates only problems with equal aggregate claims and equal endowments. Since the *core axioms* imply continuity in the first argument of m, then even without imposing any additional notion of continuity, we obtain the proportional rule.

Proof. It is easy to see that the proportional rule satisfies the properties listed in the theorem. Conversely, let S be a rule satisfying these properties. We prove that $S = P$. Let $m : Y_n \to \mathbb{R}_+$ be the function whose existence is stated in Lemma 1. Recall that for each $(c,E) \in \mathcal{C}$, each $i \in \arg\min_{k\in N} c_k$, and each $j \in \arg\max_{k\in N} c_k$,

$$S_i(c,E) = m(\min c, \bar{c}, E) \text{ and } S_j(c,E) = E-(n-1)\,m\!\left(\frac{\bar{c}-\max c}{n-1}, \bar{c}, E\right). \tag{2}$$

By Lemma 1, m satisfies property P3, i.e, for each $(r,s,E)\in Y_n$,

$$m\left(\frac{s+(n-2)r}{2(n-1)},s,E\right)=\frac{E}{2(n-1)}+\frac{n-2}{2(n-1)}m\left(r,s,E\right). \tag{3}$$

We complete the proof in six steps. Step 1 is similar to an argument in Hougaard and Østerdal (2005).

Step 1: Let $(s,E)\in\mathbb{R}^2_+$ be such that $s\geq E$, $\{(c,E),(c',E)\}\subset\mathcal{C}(s)$ such that $c_1\leq\cdots\leq c_n$ and $c'_1\leq\cdots\leq c'_n$, and $1\leq t\leq n$.

(a) If for each $i\leq t$, $c_i=c'_i$, then for each $i\leq t$, $S_i(c,E)=S_i(c',E)$.

(b) If for each $i\geq t$, $c_i=c'_i$, then for each $i\geq t$, $S_i(c,E)=S_i(c',E)$.

First, we prove (a). Let s, E, $\{(c,E),(c',E)\}$, and t be as in the statement above. We prove the assertion by induction on t. If $t=1$, then by (2), if $c_1=c'_1$, then $S_1(c,E)=S_1(c',E)$. Let $t\in\{2,\ldots,n\}$. Suppose that if for each $i\leq t-1$, $c_i=c'_i$, then for each $i\leq t-1$, $S_i(c,E)=S_i(c',E)$. Assume that for each $i\leq t$, $c_i=c'_i$. By the induction hypothesis, we already know that for each $i\leq t-1$, $S_i(c,E)=S_i(c',E)$. Thus, it is enough to show that $S_t(c,E)=S_t(c',E)$. Let $c^*\in\mathcal{C}(s)$ be the vector defined as follows: for each $i\leq t$, $c^*_i\equiv c_i$, and for each $j>t$, $c^*_j\equiv\frac{s-\sum_{l\leq t}c_l}{n-t}$. Observe that $c^*\succeq_L c$. Then by *OPG* and *IPG*, $S_1(c,E)+\cdots+S_t(c,E)\leq S_1(c^*,E)+\cdots+S_t(c^*,E)$. Now, by *OPL* and *IPL*,

$$[c_1-S_1(c,E)]+\cdots+[c_t-S_t(c,E)]\leq[c^*_1-S_1(c^*,E)]+\cdots+[c^*_t-S_t(c^*,E)].$$

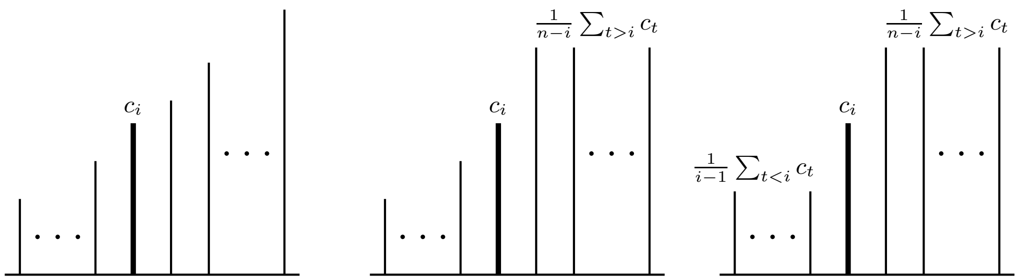
Altogether, $S_1(c,E)+\cdots+S_t(c,E)=S_1(c^*,E)+\cdots+S_t(c^*,E)$. A symmetric argument shows that $S_1(c',E)+\cdots+S_t(c',E)=S_1(c^*,E)+\cdots+S_t(c^*,E)$. Since for each $i\leq t-1$, $S_i(c,E)=S_i(c',E)$, then $S_t(c,E)=S_t(c',E)$.

We omit the proof of (b) since it is parallel to that of (a).

Step 2: Determining the award to an intermediate claimant (Figure 1). Let $c\in\mathbb{R}^N_+$ and suppose that $c_1\leq\cdots\leq c_n$. Let $i\in N$ be such that $1<i<n$. Let $c^*(c,i)\in\mathcal{C}(\bar{c})$ be the vector defined as follows: for each $j<i$, $c^*_j(c,i)\equiv\frac{1}{i-1}\sum_{t<i}c_t$, $c^*_i(c,i)\equiv c_i$, and for each $j>i$, $c^*_j(c,i)\equiv\frac{1}{n-i}\sum_{t>i}c_t$. By Step 1, $S_i(c,E)=S_i(c^*(c,i),E)$. By (2),

$$\begin{aligned}S_i(c,E)=\;&E-(i-1)m\left(\tfrac{1}{i-1}\textstyle\sum_{t<i}c_t,\bar{c},E\right)\\&-(n-i)\left[E-(n-1)m\left(\frac{\bar{c}-\frac{1}{n-i}\sum_{t>i}c_t}{n-1},\bar{c},E\right)\right].\end{aligned}$$

Let $(s,E)\in\mathbb{R}^2_+$ be such that $s>0$ and $s\geq E$. Since s and E are fixed throughout Steps 3 to 6, we omit the second and third arguments of m.



(a) Initial claims vector c (b) Intermediate claims vector (c) Claims vector $c^*(c,i)$

Figure 1: Illustrating Step 2. Agent i's award is the same for claims vectors in (a), (b), and (c).

Step 3: For each $\boldsymbol{r \in \left[0, \frac{s}{n}\right[}$, and each $\boldsymbol{y \in \left[r, \frac{2s+(n-3)r}{3(n-1)}\right]}$,

$$\boldsymbol{m\left(\frac{n-3}{2(n-2)}r + \frac{n-1}{2(n-2)}y\right) = \frac{n-3}{2(n-2)}m(r) + \frac{n-1}{2(n-2)}m(y).} \tag{4}$$

Let $r \in [0, \frac{s}{n}[$. Since $r < \frac{s}{n}$, then $r < \frac{2s+(n-3)r}{3(n-1)}$. Let $y \in \left]r, \frac{2s+(n-3)r}{3(n-1)}\right]$ and $z \equiv \frac{(n-1)y-(n-3)r}{2}$. Since $y > r$, then $z > r$. Let $c^* \equiv (r, \ldots, r, z, z, s-(n-3)r-2z) \in \mathcal{C}(s)$. Since $c^*_n = s-(n-1)y$ and $y \in \left]r, \frac{2s+(n-3)r}{3(n-1)}\right]$, then $c^*_n \geq z$. Thus, $c^*_1 \leq \cdots \leq c^*_n$. By Step 2,

$$S_{n-2}(c^*, E) = E - (n-3)m(r) - 2\left[E - (n-1)m\left(\frac{s - \frac{z+s-(n-3)r-2z}{2}}{n-1}\right)\right]. \tag{5}$$

Since $\sum_{i\in N} S_i(c^*, E) = E$, $\min c^* = r$ and $\max c^* = c^*_n$, then by (2),

$$\begin{aligned} S_{n-2}(c^*, E) = \;& E - (n-3)m(r) - S_{n-1}(c^*, E) \\ & - \left[E - (n-1)m\left(\tfrac{s-(s-(n-3)r-2z)}{n-1}\right)\right]. \end{aligned} \tag{6}$$

Since $c^*_{n-2} = c^*_{n-1}$ and S satisfies ETE, then $S_{n-2}(c^*, E) = S_{n-1}(c^*, E)$. Replacing (5) in (6) yields

$$m(r) = \frac{4(n-1)}{n-3}m\left(\frac{s+(n-3)r+z}{2(n-1)}\right) - \frac{n-1}{n-3}m\left(\frac{(n-3)r+2z}{n-1}\right) - \frac{2}{n-3}E. \tag{7}$$

Since $y \leq \frac{2s+(n-3)r}{3(n-1)}$ and $r \leq \frac{s}{n}$, then $y \leq \frac{s}{n}$. Thus,

$$\frac{(n-3)r+z}{n-2} = \frac{(n-3)r+(n-1)y}{2(n-2)} \leq \frac{s}{n}.$$

Thus, by (3),

$$m\left(\frac{s+(n-3)r+z}{2(n-1)}\right)=\frac{E}{2(n-1)}+\frac{n-2}{2(n-1)}m\left(\frac{(n-3)r+z}{n-2}\right). \tag{8}$$

Since $y=\frac{(n-3)r+2z}{n-1}$, then replacing (8) in (7) and rearranging terms yields

$$m\left(\frac{(n-3)r+z}{n-2}\right)=\frac{n-3}{2(n-2)}m(r)+\frac{n-1}{2(n-2)}m(y).$$

Now, since $\frac{(n-3)r+z}{n-2}=\frac{n-3}{2(n-2)}r+\frac{n-1}{2(n-2)}y$, then

$$m\left(\frac{n-3}{2(n-2)}r+\frac{n-1}{2(n-2)}y\right)=\frac{n-3}{2(n-2)}m(r)+\frac{n-1}{2(n-2)}m(y),$$

as asserted.[9]

Step 4: For each $r\in\left]0,\frac{2s}{3(n-1)}\right[$, $m(r)=\frac{3(n-1)}{2s}m\left(\frac{2s}{3(n-1)}\right)r$. Let $L(r)\equiv\{r'\in[0,r]:m(r')=\frac{3(n-1)}{2s}m\left(\frac{2s}{3(n-1)}\right)r'\}$. Since $0\in L(r)$, then $L(r)\neq\emptyset$. Let $U(r)\equiv\{r'\in[r,\frac{2s}{3(n-1)}]:m(r')=\frac{3(n-1)}{2s}m\left(\frac{2s}{3(n-1)}\right)r'\}$. Since $\frac{2s}{3(n-1)}\in U(r)$, then $U(r)\neq\emptyset$. We prove that $\sup L(r)=\inf U(r)$. Suppose by contradiction that $\sup L(r)<\inf U(r)$. By (P4) –Lemma 1–, $m(\cdot,s,E)$ is continuous. Then, $\sup L(r)\in L(r)$ and $\inf U(r)\in U(r)$. Now, since $\inf U(r)\leq\frac{2s}{3(n-1)}$ and $n>3$, then $\inf U(r)\leq\frac{2s+(n-3)\sup L(r)}{3(n-1)}$. By (4),

$$\begin{aligned}m\left(\tfrac{n-3}{2(n-2)}\sup L(r)+\tfrac{n-1}{2(n-2)}\inf U(s)\right)= {}&\tfrac{n-3}{2(n-2)}m(\sup L(r))\\&+\tfrac{n-1}{2(n-2)}m(\inf U(s)).\end{aligned}$$

Let $v\equiv\frac{n-3}{2(n-2)}\sup L(r)+\frac{n-1}{2(n-2)}\inf U(s)$. Since, $\sup L(r)\in L(r)$ and $\inf U(r)\in U(r)$ then,

$$m(v)=\frac{3(n-1)}{2s}m\left(\frac{2s}{3(n-1)}\right)v.$$

Now, since $v\in\left[0,\frac{2s}{3(n-1)}\right]$, then $v\in L(r)\cup U(r)$. This is a contradiction, because $\sup L(r)<v<\inf U(r)$.

Step 5: For each $r\in[0,\frac{s}{n}]$, $m(r)=\frac{r}{s}E$. Let $r\in[0,\frac{s}{n}]$. Since $m(0)=0$, then by (3), $m\left(\frac{s}{2(n-1)}\right)=\frac{E}{s}\frac{s}{2(n-1)}$. Now, since $\frac{s}{2(n-1)}\leq\frac{2s}{3(n-1)}$, then $m\left(\frac{s}{2(n-1)}\right)=$

[9] Let us remark that this equation is trivially satisfied when $n=3$. Kasajima and Velez (2009) show that the properties stated in Lemma 1 are the only restrictions induced on minimal award rules by the *core axioms* in the three-agent case. Moreover, they prove that there is a continuum of minimal award rules satisfying these properties and therefore the class of rules that satisfies these axioms in the three-agent case is infinite.

$\frac{3(n-1)}{2s} m \left(\frac{2s}{3(n-1)}\right) \frac{s}{2(n-1)}$. Thus, $m\left(\frac{2s}{3(n-1)}\right) = \frac{2}{3(n-1)}E$. By Step 4, for each $r \in [0, \frac{2s}{3(n-1)}]$, $m(r) = \frac{r}{s}E$. Now, let $\{b_t\}_{t\in\mathbb{N}}$ be the sequence defined as follows: for each $t \in \mathbb{N}$,

$$b_t \equiv \frac{1}{n}\left[1 - \left(\frac{n-2}{2(n-1)}\right)^t\right].$$

We prove that for each $t \in \mathbb{N}$ and each $r \in [0, b_t s]$, $m(r) = \frac{r}{s}E$. We prove this by induction on t. Since $b_1 s = \frac{s}{2(n-1)}$, then for each $r \in [0, b_1 s]$, $m(r) = \frac{r}{s}E$. Let $t \geq 1$ and suppose that for each $r \in [0, b_t s]$, $m(r) = \frac{r}{s}E$. We prove that for each $r \in [0, b_{t+1}s]$, $m(r) = \frac{r}{s}E$. Let $r \in [b_t s, b_{t+1}s]$. Since $b_1 s = \frac{s}{2(n-1)} \leq r \leq \frac{s}{n}$, then $0 \leq \frac{2(n-1)r-s}{n-2} \leq \frac{s}{n}$. Thus, by (3),

$$m(r) = \frac{E}{2(n-1)} + \frac{n-2}{2(n-1)} m\left(\frac{2(n-1)r-s}{n-2}\right).$$

Since $r \leq b_{t+1}s$, then $\frac{2(n-1)r-s}{n-2} \leq b_t s$. By the induction hypothesis, $m\left(\frac{2(n-1)r-s}{n-2}\right) = \frac{2(n-1)r-s}{s(n-2)}E$. Replacing $m\left(\frac{2(n-1)r-s}{n-2}\right)$ in the expression above yields that $m(r) = \frac{r}{s}E$. Now, since $b_t s \underset{t\to\infty}{\longrightarrow} \frac{s}{n}$, then for each $r \in [0, \frac{s}{n}[$, $m(r) = \frac{r}{s}E$. Since S satisfies ETE, then $m(\frac{s}{n}) = \frac{E}{n}$. Thus, for each $r \in [0, \frac{s}{n}]$, $m(r) = \frac{r}{s}E$.

Step 6: For each $(c, E) \in \mathcal{C}$ such that $\bar{c} > 0$, and each $i \in N$, $S_i(c, E) = \frac{c_i}{\bar{c}}E$.

Assume w.l.o.g. that $c_1 \leq \cdots \leq c_n$. By (2) and Step 5, $S_1(c, E) = \frac{c_1}{\bar{c}}E$ and $S_n(c, E) = \frac{c_n}{\bar{c}}E$. Let $i \in N$ be such that $1 < i < n$. By Steps 2 and 5, $S_i(c, E) = \frac{c_i}{\bar{c}}E$. □

We close this section by discussing a way of strengthening our result.[10]

Theorem 1 holds even on domains of problems smaller than $\mathcal{C}$, i.e., the domains of claims problems where the total claim and the endowment are fixed. More precisely, for each $(s, E) \in \mathbb{R}^2_+$ such that $E \leq s$, let $\boldsymbol{\mathcal{C}(s, E)} \equiv \{(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+ : \bar{c} = s\}$. Now take an arbitrary $(s, E) \in \mathbb{R}^2_+$ such that $E \leq s$. Our proof reveals that the only rule defined on $(s, E) \in \mathbb{R}^2_+$ that satisfies the *core axioms* is the proportional rule. Thus, our characterization can be strengthened by reducing the domain of problems in which the axioms apply.

References

Alcalde J, Marco M, Silva JA (2005) Bankruptcy games and the Ibn Ezra's proposal. Econ Theory 26:103–114

[10]See Ju et al (2007) for related characterizations on "rich" domains.

Arnold B (1990) The Lorenz order and the effects of taxation policies. Bull Econ Res 42:249–264

Aumann R, Maschler M (1985) Game theoretic analysis of a bankruptcy problem from the talmud. J Econ Theory 36:195–213

Bosmans K, Lauwers L (2007) Lorenz comparisons of nine rules for the adjudication of conflicting claims, Mimeo

Ching S, Kakkar V (2001) A market approach to the bankruptcy problem, Mimeo

Chun Y (1988) The proportional solution for rights problems. Math Soc Sci 15:231–246

Dalton H (1920) The measurement of the inequality of incomes. Econ J 30:348–361

Dominguez D, Thomson W (2006) A new solution to the problem of adjudicating conflicting claims. Econ Theory 28:283–307

Ebert U (2004) Coherent inequality views: linear invariant measures reconsidered. Math Soc Sci 47:1–20

Hokari T, Thomson W (2003) Claims problems and weighted generalizations of the Talmud rule. Econ Theory 21:241–261

Hougaard JL, Østerdal LP (2005) Inequality preserving rationing. Econ Lett 87:355–360

Hougaard JL, Thorlund-Petersen L (2001) Bankruptcy rules, inequality, and uncertainty, Mimeo, Copenhagen Business School

Ju BG (2007) Coalitional manipulation on networks, Mimeo

Ju BG, Moreno-Ternero J (2006) Progressivity, inequality reduction and merging-proofness in taxation, Mimeo

Ju BG, Moreno-Ternero J (2008) On the equivalence between progressive taxation and inequality reduction. Soc Choice Welf 30:561–569

Ju BG, Miyagawa E, Sakai T (2007) Non-manipulable division rules in claims problems and generalizations. J Econ Theory 132:1–26

Kasajima Y, Velez R (2009) Non-proportional inequality preservation in gains and losses, Mimeo

Marshall A, Olkin I (1979) Inequalities: theory of majorization and its applications. Academic Press, New York

Moreno-Ternero J, Villar A (2006a) On the relative equitability of a family of taxation rules. J Public Econ Theory 8:283–291

Moreno-Ternero J, Villar A (2006b) The TAL-family of rules for bankruptcy problems. Soc Choice Welf 27:231–249

Moulin H (1985) Egalitarianism and utilitarianism in quasi-linear bargaining. Econometrica 53:49–67

Moulin H (1988) Axioms of cooperative decision making. Cambridge University Press, New York

Moulin H (2002) Axiomatic cost and surplus sharing. Handbook of Social Choice and Welfare 1 (K Arrow, A Sen, and K Suzumura, eds) pp 289–357

Moyes P (1989) Some classes of functions that preserve the inequality and welfare orderings of income distributions. J Econ Theory 49:347–359

Moyes P (1994) Inequality reducing and inequality preserving transformations of incomes: symmetric and individualistic transformations. J Econ Theory 63:271–298

O'Neill B (1982) A problem of rights arbitration from the talmud. Math Soc Sci 2:345–371

Thomson W (2003) Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. Math Soc Sci 45:249–297

Thomson W (2007) Lorenz rankings of rules for the adjudication of conflicting claims, Mimeo, University of Rochester

Thomson W (2008) How to divide when there isn't enough, Book Manuscript