The price of imperfect competition for a spanning network

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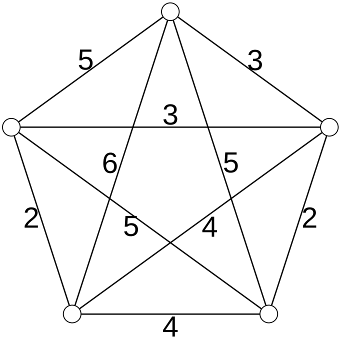
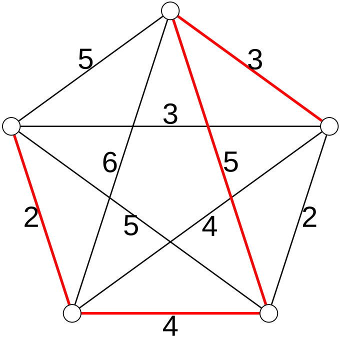
Network procurement problem



- A buyer needs to procure a network spanning a set of nodes.
- Sellers (or contractors) can provide (construct) the different edges. Basic model: there is a different seller for each edge.

Network procurement problem



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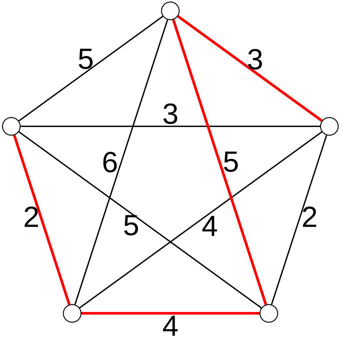
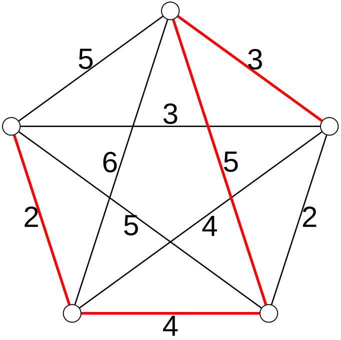


- First price auction (complete info).
- VCG Pivotal mechanism (incomplete info).

Network procurement problem



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- VCG Pivotal mechanism (incomplete info).

What network is built and how much does the buyer pay for it?

Bertrand competition in the procurement of a network

Imperfect competition à la Bertrand usually implies:

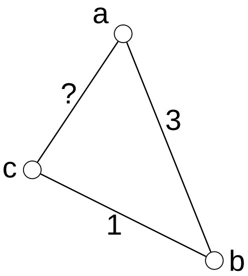
- cheapest technology is used
- overcharge to the buyer(s)

Here too.

Our contribution: to give a universal upper bound of the overhead under a natural type of imperfection.

Δr inequality

- the cost matrix c is triangular if for all edges e, f, g, forming a triangle, $c_e \leq c_f + c_g$



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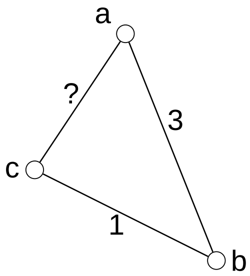
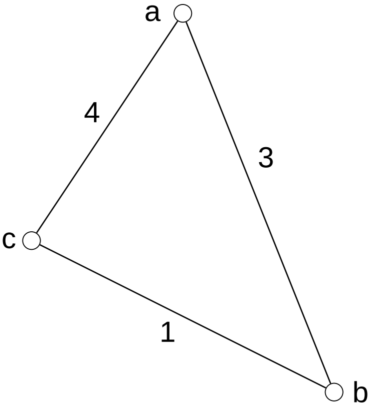


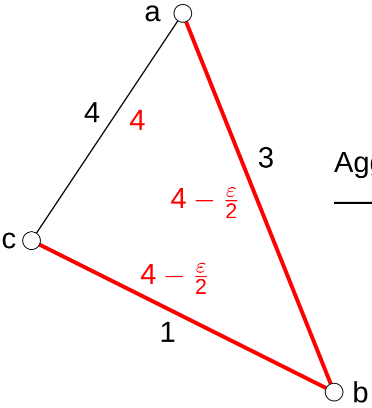
- Natural form of substitutability: (i) metric space with costs proportional to length of shortest path; (ii)...
- Benchmark: Generalize to metric space with cost a function of length, with bounded average; missing edges.

Three-node case (Δr inequality)



Efficient cost 4

Three-node case (Δr inequality): ε-equilibrium



$$\frac{\text{Aggregate payment}}{\text{Efficient cost}} = \frac{8-\varepsilon}{4} \leq 2$$

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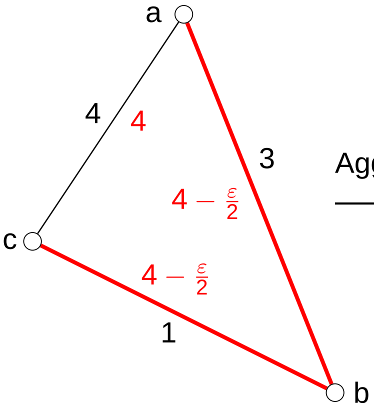

$$\frac{\text{Aggregate payment}}{\text{Efficient cost}} = \frac{8 - \varepsilon}{4} \leq 2$$

- General bound for this ratio?

Summary

Model

- A buyer procures a spanning tree to connect a given set of nodes
- No seller has monopoly power.

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Punch line: in any non-cooperative limit equilibrium:

- a minimal cost spanning tree is built
- If (i) costs satisfy the *triangle inequality*, and (ii) each seller bids to provide *one edge*, then the overhead charge is essentially at most, and up to, 100% of the real cost:

$$\frac{\textit{aggregate payment}}{\textit{efficient cost}} \quad \textit{is essentially at most}, \textit{ and up to}, \ 2$$

Related literature

Price of anarchy: universal worst case bound on welfare loss from decentralized behavior

- negative congestion externalities in network routing: Koutsoupias-Papadimitriou (TACS'99); Roughgarden-Tardos (JACM, 2002)
- positive congestion in network routing: Roughgarden (STOC'01); Anshelevich et al. (STOC'03)
- exploiting a technology with decreasing returns: Johari-Tsitsiklis (MOR, 2004); Juarez (ET, 2008); Moulin (ET, 2008; GEB, 2010)
- assignment of indivisible goods: Guo-Conitzer (EC'08); Moulin (JET, 2009; SCW, 2010); de Clippel et al. (2010)

Related literature

Frugality of a mechanism: evaluation of the surcharge to the buyer in a mechanism with respect to a benchmark cost

- true minimal (efficient) cost: Archer-Tardos (FOCS’01, SODA’02)
- cheapest cost of a spanning tree with no edge in common with the efficient tree: Talwar (STOC’03)
- solution to a linear program that coincides with the most expensive equilibrium in our model: Karlin-Kempe-Tamir (SODA’05)

Selling a spanning tree: (selling bases of a matroid) Chen-Karlin (SODA’07), Bikhchandani et.al. (2010)

Model

- V the set of nodes, E the set of edges
- each edge is owned by a different agent
- Γ the set of spanning trees
- actual cost of e is c_e; cost matrix $c \in \mathbb{R}_+^E$
- minimal cost of a spanning tree $\lambda(c)$
- $\Gamma(c)$ the set of minimal cost spanning trees

Bertrand game

- e posts price p_e to supply his edge
- buyer purchases a spanning tree $\gamma(p) \in \Gamma(p)$
- profit $u_e = (p_e - c_e) \cdot 1_{\{e \in \gamma(p)\}}$
- non-cooperative equilibrium concept: limit (Nash) equilibrium (Fudenberg-Levine, 1986)

limit of ε*-equilibria as* $\varepsilon \to 0$

Preliminary result

Proposition

In each limit equilibrium of the Bertrand game, the buyer purchases a minimal cost spanning tree –for the real costs– though he typically pays more than $\lambda(c)$.

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In each limit equilibrium of the Bertrand game, the buyer purchases a minimal cost spanning tree –for the real costs– though he typically pays more than $\lambda(c)$.

- Holds under general ownership structure.
- Greedy algorithm (Kruskal).
- The cost of an inefficient tree can be reduced by replacing a single edge.
- If the cost of an edge in an efficient tree is reduced, the edge belongs to each efficient tree at the new costs.

Price of Imperfect Competition

$$PIC(c) \equiv \frac{\textit{maximal aggregate price payed in equilibrium}}{\lambda(c)}$$

Computing the PIC

Define

- optimal edges at c:

$$E(c) \equiv \{e \in E : \textit{some } \gamma \in \Gamma(c) \textit{ contains } e\}$$

- For $e \in E(c)$,

$$\mu_e(c) \equiv \max\{b \geq 0 : e \in E(c_{-e}, b)\}$$

Computing the PIC

Lemma

For any $\gamma, \gamma' \in \Gamma(c)$*,*

$$\sum_{e \in \gamma} \mu_e(c) = \sum_{f \in \gamma'} \mu_f(c),$$

so this sum can be written as $\mu(c)$*.*

Computing the PIC

Proposition

When each seller is the exclusive bidder for exactly one edge:

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- *For each c and each $\gamma \in \Gamma(c)$, the prices*

$$p_e = \mu_e(c) \text{ if } e \in \gamma; \; p_e = c_e \text{ if } e \notin \gamma$$

is a limit equilibrium where the buyer pays $\mu(c)$.

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When each seller is the exclusive bidder for exactly one edge:

- *In each limit equilibrium of the Bertrand game, the buyer pays at least* $\lambda(c)$ *and at most* $\mu(c)$.
- *For each* c *and each* $\gamma \in \Gamma(c)$*, the prices*

  $$p_e = \mu_e(c) \text{ if } e \in \gamma; \; p_e = c_e \text{ if } e \notin \gamma$$

  is a limit equilibrium where the buyer pays $\mu(c)$.
- *Then,*

  $$PIC(c) = \frac{\mu(c)}{\lambda(c)}$$

▸ Example

Pivotal interpretation of PIC

Pivotal mechanism:

- e's action: p_e
- buyer purchases a cheapest spanning tree $\gamma(p) \in \Gamma(p)$
- the buyer pays seller e

$$\lambda(p_{-e}, +\infty) - \lambda(p) + p_e \cdot 1_{\{e \in \gamma(p)\}}$$

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Proposition

When each seller is the exclusive bidder for exactly one edge: For each c the buyer pays $\mu(c)$ in the dominant strategy truthful equilibrium of the pivotal mechanism.

Main result: bounding the PIC (Δr model)

- the cost matrix c is triangular.
- $T(V)$ the set of triangular (metric) costs
- $n \equiv |V|$ the number of nodes

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- the cost matrix c is triangular.
- $T(V)$ the set of triangular (metric) costs
- $n \equiv |V|$ the number of nodes

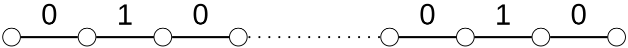
Theorem

If each seller is the exclusive bidder for exactly one edge

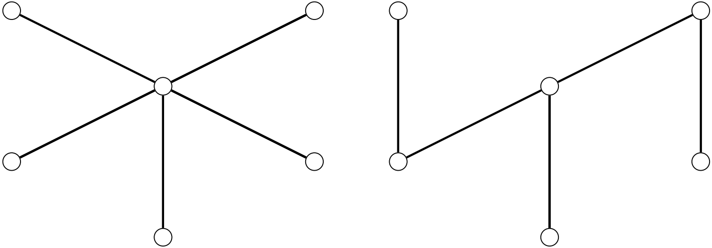
$$\max_{c \in T(V)} PIC(c) = \begin{cases} 2 & \text{if } n \text{ is odd,} \\ 2\frac{n-1}{n-2} & \text{if } n \text{ is even.} \end{cases}$$

The bound is reached

- fix an arbitrary spanning tree $\gamma \in \Gamma$ and choose c: $c_e \equiv 1$ if $e \in \gamma$; $c_e \equiv 2$ otherwise; then $PIC(c) = 2$.
- for n even $PIC(c) = 2\frac{n-1}{n-2}$ is achieved by the line tree



Could the bound be reached (n even)?



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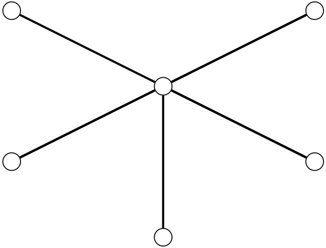
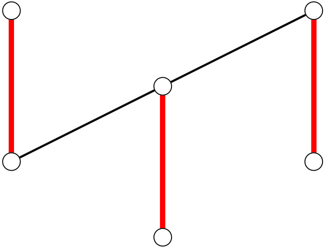




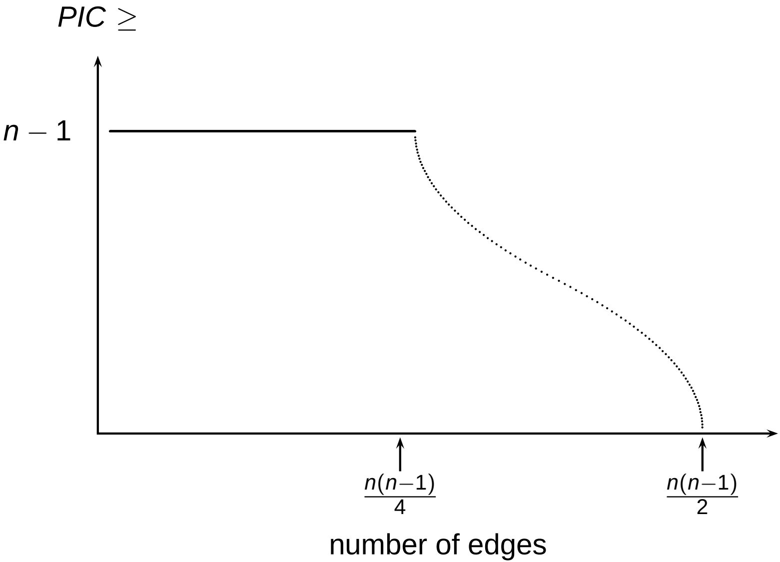
Lemma (n even)

There is c $\in T(V)$ *such that* $\gamma \in \Gamma(c)$ *and* $PIC(c) = 2\frac{n-1}{n-2}$ *if and only if* γ *contains a perfect matching.*

Beyond Δr inequality

1. Less substitutability.
    - Relaxed Δr inequality: for all edges e, f, g, forming a triangle, $c_e \leq \rho(c_f + c_g)$ with $\rho \geq 1$.
    - Missing edges.
2. More substitutability.
    - Ultrametric inequality.

Missing edges (n nodes)



Bounding the PIC (Ultrametric model)

- The cost matrix c is ultrametric if for all edges e, f, g, forming a triangle, $c_e \leq \max\{c_f, c_g\}$
- Stronger substitutability condition

Bounding the PIC (Ultrametric model)

- Example: set of node for which the connection cost is determined by the compatibility of attributes that are ordered in decreasing levels of complexity.
    - Cost to connect nodes in North America (3 countries).
    - An edge has the largest cost if its end nodes are in different countries, the second largest cost if they are in the same country but in different states or provinces, ...

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- write $U(V)$ for the set of ultrametric costs

Theorem

If each seller is the exclusive bidder for exactly one edge

$$\max_{c \in U(V)} PIC(c) = \max_{c \in T(V)} PIC(c)$$

▸ See additional results

Expected overhead

Although the worst cases coincide, in a simple probabilistic model the expected overhead for the metric costs is at least twice that for ultrametric costs.

▸ Go to Figures

The role of the exclusive ownership assumption

- if each seller owns a set of mutually disconnected edges, both theorems still hold
- fix k, $2 \leq k \leq n - 1$, assume that no seller owns a cut, and that the edges owned by any seller are contained in a k-clique. Then, we conjecture that the *PIC* is bounded by approximately k for n large

Thank you for your attention

Computing the PIC

Define

- substitutes of e in $\gamma \ni e$:

$$\Delta(e,\gamma) \equiv \{f \in E \setminus \{e\} : \gamma - e + f \in \Gamma\}$$

- recall that the optimal edges at c are:

$$E(c) \equiv \{e \in E : \textit{some } \gamma \in \Gamma(c) \textit{ contains } e\}$$

Lemma

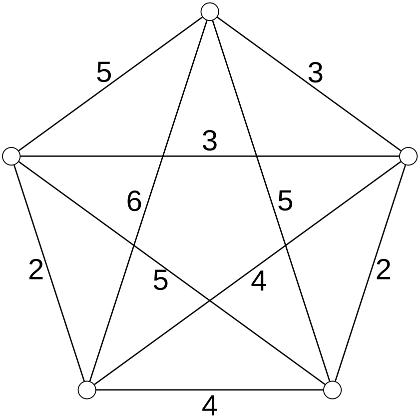
Fix c and $e \in E(c)$, then

$$\mu_e(c) \equiv \max\{b \geq 0 : e \in E(c_{-e}, b)\} = \min\{c_f : f \in \Delta(e,\gamma)\}$$

▸ Return to presentation

Calculating $\mu_e(c)$

Network costs c



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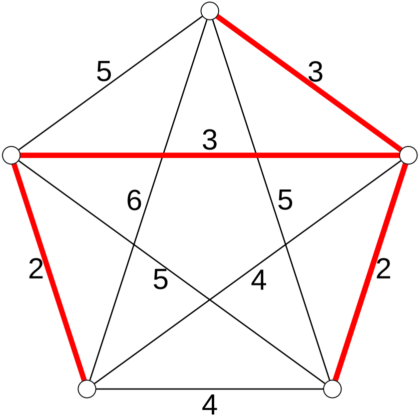
Calculating $\mu_e(c)$

Minimal cost spanning tree and $\lambda(c) = 10$



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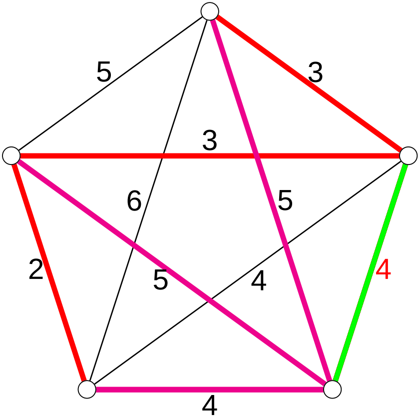
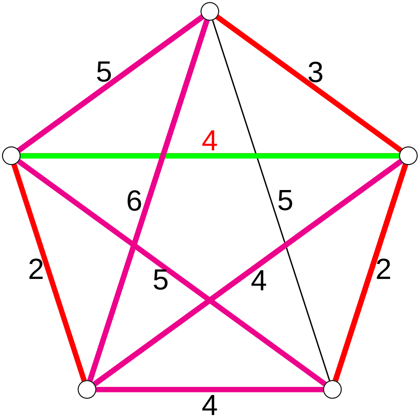


Calculating $\mu_e(c)$



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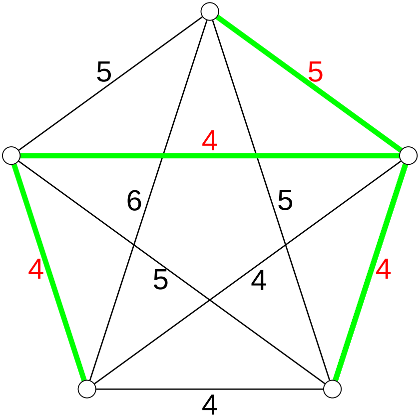
Calculating $\mu(c)$ and $PIC(c)$

$\mu(c) = 17$ and $PIC(c) = \frac{17}{10} = 1.7$



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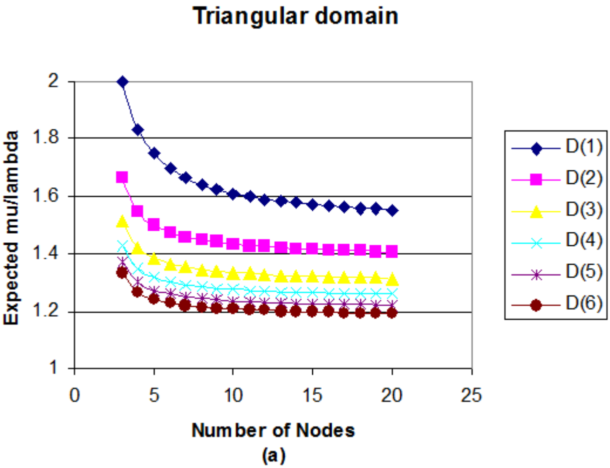
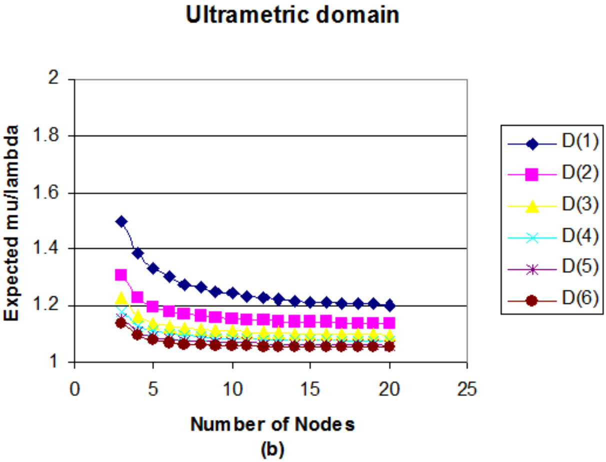
Expected PIC Δr domain



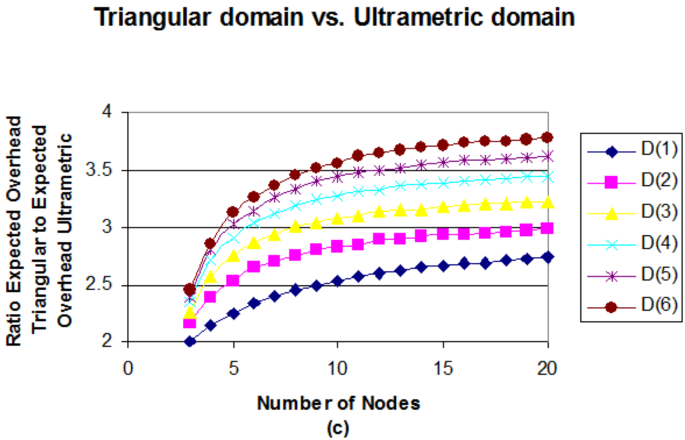
Return to Expected overhead

Expected PIC ultrametric domain



Return to Expected overhead

Expected overhead Δr domain vs. ultrametric domain



▸ Return to Expected overhead

Bounding the PIC (Ultrametric model): additional results

Theorem

- *If n is odd, the spanning tree $\gamma \in \Gamma$ admits a cost c such that $\gamma \in \gamma(c)$ and $\frac{\mu(c)}{\lambda(c)} = 2$ if and only if γ has at least one leaf edge of which the inner end point is of degree two.*
- *If n is even, the spanning tree $\gamma \in \Gamma$ admits a cost c such that $\gamma \in \gamma(c)$ and $\frac{\mu(c)}{\lambda(c)} = 2\frac{n-1}{n-2}$ if and only if the edges of γ contain a perfect matching of V.*

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