

# Let them cheat!<sup>☆</sup>

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## Abstract

We consider the problem of fairly allocating a bundle of infinitely divisible goods among a group of agents with “classical” preferences. We propose to measure an agent’s “sacrifice” at an allocation by the size of the set of feasible bundles that the agent prefers to her consumption. As a solution, we select the allocations at which sacrifices are equal across agents and this common sacrifice is minimal. We then turn to the manipulability of this solution. In the tradition of Hurwicz (1972), we identify, under some mild assumptions on preferences, the equilibrium allocations of the manipulation game associated with this solution when all commodities are normal: for each preference profile, each equal-division constrained Walrasian allocation is an equilibrium allocation; conversely, each equilibrium allocation is equal-division constrained Walrasian. Furthermore, we show that if normality of goods is dropped, then equilibrium allocations may not be equal-division constrained Walrasian.

*Keywords:* equal-sacrifice solution, manipulation game, equal-division Walrasian solution.

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## 1. Introduction

We consider the problem of fairly allocating a bundle of privately appropriable and infinitely divisible goods among a group of agents having equal rights on these goods. To make the objective of fairness operational, we propose to measure the sacrifice required of an agent at an allocation by the size of the set of feasible bundles that she prefers to her assignment and to select the allocations at which sacrifices are equal across agents and this common sacrifice is minimal. We refer to the resulting solution as the “equal-sacrifice” solution.

First, we prove that this solution is well-defined under general assumptions on preferences, and that under some very mild additional monotonicity assumptions, equal-sacrifice allocations are also efficient.

Crawford (1980) advocates, for the two-agent case, the rule that selects, from the one of agent 1’s indifference curves that divides the Edgeworth box into two regions of equal volumes, the allocation preferred by agent 2. This proposal suffers from treating the two agents asymmetrically, and it is not easily generalized to more than two agents. Nevertheless, basing the choice of an allocation on the size of upper contour sets is a natural assumption that we have retained. We have adapted it to handle arbitrary populations, and done so in a manner that delivers a symmetric treatment of agents. Our solution can also be thought of as a member of the following family. Specify for each agent a function that represents her preferences, a “welfare index” for her. Then, select the allocations at which these welfare indices take a common value, and this common value is maximal among all feasible allocations.

We then turn to the question of manipulability. To each rule can be associated a manipulation “game form” as follows: the strategy space of each agent is the space of preferences satisfying the properties that her relation is known to satisfy; the outcome function is the rule itself. If in this game, it is a dominant strategy for each agent to announce her true preferences, we say that the rule is “strategy-proof”. It is well-known that on the domain on which we are operating, strategy-proofness is very restrictive. Indeed, no selection from the correspondence that associates with each economy its set of efficient allocations at which each agent finds her consumption at least as desirable as her endowment, is strategy-proof (Hurwicz, 1972, Serizawa, 2002). This conclusion also applies to all selections from the Pareto solution

satisfying “equal treatment of equals” (Serizawa, 2002).<sup>1</sup> It follows from this latter result that the equal-sacrifice solution is not strategy-proof. It is of course easy to construct examples directly establishing this fact, and we will provide some.

The study of a solution should not stop with the observation that it is not strategy-proof, however. A violation of this property simply means that there are preference profiles such that, if all agents but possibly one tell the truth about their preferences, the last agent may benefit from not doing so. However, what should really concern us is not so much that agents may not be truthful, but rather that the allocations that the solution would specify for the true profile may not be reached. Thus, a determination of which allocations will be obtained is called for. Only knowing that an agent may benefit by behaving strategically, keeping fixed the announcements of the others, does not suffice for that purpose. Several agents may be in that position, and any agent who is considering misrepresenting her preferences has to entertain the thought that others could do the same, and should take that fact into consideration when selecting her strategy. Consequently, we are led to associating with the solution a manipulation game, identifying its equilibria, and evaluating them in terms of the true preference profile.

Thus, the second objective of this paper is to characterize, for each preference profile, the pure strategy equilibria of the manipulation game associated with the equal-sacrifice solution for that profile. We achieve this under a mild monotonicity condition on preferences and the assumption that all goods are normal. Our main result is the following: the set of equilibrium outcomes of the manipulation game associated with the equal-sacrifice solution coincides with the set of equal-division constrained Walrasian allocations for the true preferences!<sup>2</sup>

It is not uncommon that the equilibria of a manipulation game associated with an allocation rule contain the equal-division constrained Walrasian al-

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<sup>1</sup>Barberà and Jackson (1995) drop efficiency and provide a characterization of the class of strategy-proof rules in the two-agent case, and under some additional property in the case of more than two agents.

<sup>2</sup>A constrained Walrasian allocation is defined as a Walrasian allocation except that maximization of preferences takes place in “truncated” budget sets. Only consumption bundles that are part of a feasible allocation and meet the budget constraint are admissible. This variant of the Walrasian solution, introduced by Hurwicz (1979), only differs from it when some agents consume on the boundary of their consumption sets.

locations (see below). Thus, the striking part of our results is the converse inclusion. No other allocation is reached at equilibrium. Typically, manipulation does not lead to such a relatively happy conclusion (these allocations are efficient on the domain of preferences to which our main characterization pertains; moreover, they have been a focal point in axiomatic studies of fair allocation). For instance, for each preference profile, the equilibrium allocations of a game similarly associated with the Walrasian solution itself are not necessarily constrained Walrasian for that profile. (In the two-agent case, they are all the allocations in the lens-shaped area defined by the true offer curves, as shown by Hurwicz, 1972.)

The following observations should provide some intuition for our result. First, the equal-sacrifice solution is very “sensitive”. In particular, any change in any agent’s indifference curve through her assigned bundle at some profile is likely to bring about a change in the allocation the rule selects.<sup>3</sup> Moreover, if an agent’s announced indifference curve through her assigned bundle is not linear, switching to a preference relation for which her indifference curve through her assigned bundle is flatter than it was originally (this is an “anti-monotonic” transformation of her preferences at that point, where “monotonicity” is understood in the sense of Maskin, 1999), increases the agent’s apparent sacrifice. This calls for her sacrifice to be reduced. It is tempting at this point to conclude that equilibrium has to occur when all announcements are linear. One more step is needed however—and it turns out to be a technically delicate one—because a reduction in an agent’s sacrifice in terms of preferences to which she may switch does not necessarily cause a parallel reduction in terms of her true preferences, which is what really matters to her. What is needed is an understanding of the circumstances under which this implication does hold. This is where the assumption of normality of goods comes into play. Under that assumption, the agent will indeed benefit. We also show that this assumption is necessary. Without it—and we give examples—equilibria exist involving non-linear preferences and whose outcomes are not equal-division constrained Walrasian for the true preferences.

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<sup>3</sup>The sensitivity of our solution explains in part the difference between the conclusions we reach for it and what we know about the Walrasian solution. These issues are discussed by Thomson (1984), who shows the relevance of Maskin-monotonicity to the characterization of the set of equilibria.

## 2. Related literature

The manipulability of allocation rules on various domains has been the object of a number of studies.

In the context of exchange economies, early studies are Sobel (1981), for two agents, and Thomson (1984, 1987, 1988) for quasi-linear preferences. Constrained Walrasian allocations (or equal-income constrained Walrasian allocations) are shown to be equilibrium allocations, but there can be others. More recent papers have dealt with the manipulability of certain solutions to Nash's bargaining problem (Sobel, 2001, Kıbrıs, 2002) and have also derived equal-income constrained Walrasian allocations as equilibrium allocations.<sup>4</sup> This is why earlier we wrote that our conclusion that these allocations are equilibrium allocations of the game associated with the equal-sacrifice solution is not the surprising part. What is remarkable is that here there are no other equilibrium allocations. We noted above that for two agents and for each specification of their preferences, the equilibrium allocations of the game associated with the Walrasian rule are delimited by the true offer curves. This region does contain the true Walrasian allocations but it also contains the endowment allocation, and a continuum of allocations in between. Thus, Pareto efficiency is a possible outcome but it may also be that no gains from trade are achieved at equilibrium.

The context of the allocation of a single infinitely divisible good when agents have single-peaked preferences (Sprumont, 1991) is a rare one in which a strategy-proof rule exists. It is called the uniform rule. Other rules have been proposed as providing fair outcomes for this class of problems. It so happens that for each preference profile, the manipulation game associated with a number of these rules has a unique equilibrium allocation, which is none other than the uniform allocation for that profile (Thomson, 1990, Bochet and Sakai, 2008.) Consequently, not only does manipulation not necessarily cause violations of efficiency, but it leads to a rule that has a number of other desirable properties (Thomson, 2010, provides an overview of these properties.) The similarity between these results and the main result of the current paper is due, in part, to the fact that the uniform rule can be thought of as a counterpart for the single-peaked model of the Walrasian concept when operated from equal division.

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<sup>4</sup>See also Crawford and Varian (1979) for an earlier study of the manipulation of utility functions in the context of bargaining.

The problem of allocating an indivisible good when monetary compensations are feasible is studied by Tadenuma and Thomson (1995).<sup>5</sup> Consider an allocation rule that selects envy-free allocations. Then, the equilibrium correspondence of its associated manipulation game is the entire no-envy solution. This conclusion can also be related to our main result. Indeed, for that model, the no-envy solution coincides with the equal-income Walrasian solution (Svensson, 1983). Nevertheless, no such coincidence takes place in either the classical model or the model with which we are concerned here.

In the context of matching (Roth and Sotomayor, 1990), no selection from the stable solution exists that is strategy-proof. However, for each preference profile, the set of undominated Nash equilibria of the manipulation game associated with either the “man-optimal” rule or the “woman-optimal” rule is the entire set of stable outcomes for that profile (Roth, 1984, Gale and Sotomayor, 1985.) Moreover, the set of Nash equilibria is the entire set of individually rational outcomes for the true preferences (Alcalde, 1996).

The manipulability of solutions has been studied under alternative behavioral assumptions. Crawford (1980) shows that for the rule that he had defined (see above), under maximin behavior, agent 1 would announce the one of her true indifference curves that divides the Edgeworth box into equal areas (volumes). The behavioral assumption under which he addresses the manipulation issue supposes an extreme form of risk aversion. We have found it more natural to assume Nash behavior.

Manipulation of preferences in economies with public goods is studied by Thomson (1979) and manipulation through endowments by Postlewaite (1979) and Thomson (1987). Manipulation of voting procedures is studied by Sanver and Zwicker (2004).

### 3. Model

#### 3.1. The environment

We consider the problem of allocating a fixed social endowment  $\Omega \equiv (\Omega^1, \dots, \Omega^K) \in \mathbb{R}_{++}^K$ , for some  $K \in \mathbb{N}$ , among a group of agents  $N \equiv \{1, \dots, n\}$ . Their preferences are complete and transitive binary relations on  $\mathbb{R}_+^K$ . The generic preference is  $R_0$ . The symmetric and asymmetric parts

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<sup>5</sup>Related results have been obtained by Āzaciš (2008), Beviá (2010), and Velez (2011a) for the allocation of  $n$  objects and an amount of money among  $n$  agents.

of  $R_0$  are  $I_0$  and  $P_0$ , respectively. A preference  $R_0$  is **monotone** if for each pair  $\{b, b'\} \subseteq \mathbb{R}_+^K$  such that  $b' \gg b$ , we have that  $b' P_0 b$ .<sup>6</sup> Let  $\mathcal{U}$  denote the domain of convex, continuous, and monotone preferences. A preference  $R_0 \in \mathcal{U}$  is **strictly monotone in the interior of the consumption space** if for each pair  $\{b, b'\} \subseteq \mathbb{R}_+^K$  such that  $b' \succeq b$ ,  $b P_0 0$  implies  $b' P_0 b$ . The sub-domain of  $\mathcal{U}$  of preferences that are strictly monotone in the interior of the consumption space is  $\mathcal{R}$ .<sup>7</sup>

Let  $B$  be a binary relation on  $\mathbb{R}_+^K$  and  $x \in \mathbb{R}_+^K$ . The **upper section of  $B$  at  $x$**  is the set  $U(B, x) \equiv \{x' \in \mathbb{R}_+^K : x' B x\}$ . The **constrained upper section of  $B$  at  $x$**  is the set  $U^c(B, x) \equiv \{x' \in \mathbb{R}_+^K : x' \leq \Omega, x' B x\}$ .<sup>8</sup>

Let  $R_0 \in \mathcal{U}$  and  $x \in \mathbb{R}_+^K$ . The **set of prices that support  $U(R_0, x)$  at  $x$**  is  $\text{Supp}(R_0, x)$ . Let  $p \in \Delta^{K-1} \equiv \{p \in \mathbb{R}_+^K : \sum_{k=1}^K p_k = 1\}$ . An **income expansion path for  $R_0$  at prices  $p$**  is a function  $V : \mathbb{R}_+ \rightarrow \mathbb{R}_+^K$ , such that for each  $w \in \mathbb{R}_+$ ,  $w = p \cdot V(w)$  and  $p \in \text{Supp}(R_0, V(w))$ ;  $V$  is **quasi-strictly increasing** if its component functions  $\{V_k\}_{k=1}^K$  are strictly increasing up to a possible “flat part” containing the origin. Formally,  $V$  is quasi-strictly increasing if for each  $k \in \{1, \dots, K\}$  and each pair  $\{w, w'\} \subseteq \mathbb{R}_+$  such that  $w < w'$ ,  $0 < V_k(w)$  implies  $V_k(w) < V_k(w')$ .

We consider four additional preference domains.

- A preference  $R_0$  has **quasi-strictly increasing income expansion paths** (i.e., all goods are normal) if for each  $x \in \mathbb{R}_+^K \setminus \{0\}$  and each  $p \in \text{Supp}(R_0, x)$  such that  $p \cdot x > 0$ , there is a quasi-strictly increasing income expansion path for  $R_0$  at prices  $p$  that passes through  $x$ .<sup>9</sup> Let  $\mathcal{U}_{\mathcal{I}}$  and  $\mathcal{R}_{\mathcal{I}}$  be the sub-domains of  $\mathcal{U}$  and  $\mathcal{R}$  of preferences that have quasi-strictly increasing income expansion paths, respectively.
- A preference  $R_0$  is **smooth** if for each  $x \in \mathbb{R}_{++}^K$ , there is a unique price

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<sup>6</sup>We use the following vector inequalities. For each  $L \in \mathbb{N}$  and each pair  $\{x, x'\} \subseteq \mathbb{R}^L$ :  $x' \geq x$  if for each  $l \in \{1, \dots, L\}$ ,  $x'_l \geq x_l$ ;  $x' \succeq x$  if  $x' \geq x$  and  $x' \neq x$ ; and  $x' \gg x$  if for each  $l \in \{1, \dots, L\}$ ,  $x'_l > x_l$ .

<sup>7</sup>A preference  $R^0$  is strictly monotone if for each pair  $\{x, y\} \subseteq \mathbb{R}_+^K$  such that  $x \succeq y$ ,  $x P_0 y$ . Strict monotonicity in the interior of the consumption space is a weaker requirement than strict monotonicity. For instance, Cobb-Douglas preferences are in  $\mathcal{R}$ , but are not strictly monotone (violations of the strict form of the property occur on the boundary.)

<sup>8</sup>Let us remark that under our notation, the indifference and constrained indifference sets of  $R_0$  at  $x$  respectively are  $U(I_0, x)$  and  $U^c(I_0, x)$ .

<sup>9</sup>Observe that if  $p \in \Delta^{K-1}$  is such that  $p \gg 0$  and  $R_0 \in \mathcal{R}$ , then each income expansion path relative to prices  $p$  that passes through  $x$  starts at  $0 \in \mathbb{R}_+^K$ .

in  $\text{Supp}(R_0, x)$ . Let  $\mathcal{U}_{\mathcal{S}}$  and  $\mathcal{R}_{\mathcal{S}}$  be the domains of smooth preferences in  $\mathcal{U}$  and  $\mathcal{R}$ , respectively.

- A preference  $R_0$  is **homothetic** if for each  $x \in \mathbb{R}_+^K \setminus \{0\}$  and each  $p \in \text{Supp}(R_0, x)$ , the ray passing through  $x$  is an income expansion path for  $R_0$  at prices  $p$ .<sup>10</sup> Let  $\mathcal{U}_{\mathcal{H}}$  and  $\mathcal{R}_{\mathcal{H}}$  be the domains of homothetic preferences in  $\mathcal{U}$  and  $\mathcal{R}$ , respectively.
- A preference  $R_0$  is **linear**, if there is  $p \in \Delta^{K-1}$  such that for each pair  $\{x, x'\} \subseteq \mathbb{R}_+^K$ ,  $x R_0 x'$  if and only if  $p \cdot x \geq p \cdot x'$ . For each  $p \in \mathbb{R}_+^K$ , the linear preference associated with  $p$  is  $L^p$  and its associated indifference relation  $I(L^p)$ . Let  $\mathcal{U}_{\mathcal{L}}$  and  $\mathcal{R}_{\mathcal{L}}$  be the domains of linear preferences in  $\mathcal{U}$  and  $\mathcal{R}$ , respectively.

One can easily see that the following inclusion relations hold among the preference sub-domains of  $\mathcal{U}$ :  $\mathcal{U}_{\mathcal{L}} \subsetneq \mathcal{U}_{\mathcal{H}} \subsetneq \mathcal{U}_{\mathcal{I}} \subsetneq \mathcal{U}$  and  $\mathcal{U}_{\mathcal{L}} \subsetneq \mathcal{U}_{\mathcal{S}} \subsetneq \mathcal{U}$ . Parallel statements also hold for the sub-domains of  $\mathcal{R}$ , i.e.,  $\mathcal{R}_{\mathcal{L}} \subsetneq \mathcal{R}_{\mathcal{H}} \subsetneq \mathcal{R}_{\mathcal{I}} \subsetneq \mathcal{R}$  and  $\mathcal{R}_{\mathcal{L}} \subsetneq \mathcal{R}_{\mathcal{S}} \subsetneq \mathcal{R}$ .

Agent  $i$ 's generic preference is  $R_i \in \mathcal{U}$ , and the generic preference profile is  $R \equiv (R_i)_{i \in N} \in \mathcal{U}^N$ . For each  $R \in \mathcal{U}^N$ , each  $i \in N$ , and each  $R'_i \in \mathcal{U}$ , the profile  $(R_{-i}, R'_i) \in \mathcal{U}^N$  is obtained from  $R$  by replacing  $R_i$  by  $R'_i$ .

The **set of feasible allocations** is  $Z \equiv \{z \equiv (z_i)_{i \in N} \in \mathbb{R}_+^{K \times N} : \sum_{i \in N} z_i = \Omega\}$ .<sup>11</sup> Agent  $i$ 's allotment at  $z \in Z$  is  $z_i \equiv (z_i^k)_{k=1}^K \in \mathbb{R}_+^K$ . A **solution** associates with each preference profile a non-empty subset of  $Z$ . The generic solution is  $F$ . A **selector**  $f$  from a solution  $F$  is a function that associates with each  $R \in \mathcal{R}^N$  an element of  $F(R)$ . We write  $f \in F$ .

A solution  $F$  is **essentially single-valued** if for each  $R \in \mathcal{R}^N$ , each pair  $\{z, z'\} \subseteq F(R)$ , and each  $i \in N$ ,  $z I_i z'$ .

### 3.2. Manipulation of a solution

When a solution  $F$  recommends for a particular economy a set of allocations, as opposed to a singleton, one has to ask the question: how will agents manipulate  $F$ ? Consider some profile of announced preferences. Suppose that a given  $F$ -optimal allocation, say  $z$ , is chosen for that profile. If an

<sup>10</sup>Since indifference sets with linear pieces are allowed for preferences in  $\mathcal{R}$ , then for a given price vector, income expansion paths are not necessarily unique.

<sup>11</sup>All our results generalize to economies with free disposal, i.e., in which the set of feasible allocations is  $\{z \equiv (z_i)_{i \in N} \in \mathbb{R}_+^{K \times N} : \sum_{i \in N} z_i \leq \Omega\}$ .

agent unilaterally deviates, and the set of  $F$ -optimal allocations for the new profile is not a singleton, then there could be some allocations in the set at which she is better off, some at which she is worse off, and some at which her welfare is unaffected.<sup>12</sup>

Several behavioral assumptions can be formulated to deal with this indeterminacy. For instance, one can suppose that the agent is optimistic, i.e., given the others' reports, if by reporting different preferences, there is at least one  $F$ -optimal allocation for the new profile that she prefers to  $z$ , then she will not stick with her initial announcement. Alternatively, one can suppose that she will switch only if she prefers all of the  $F$ -optimal allocations for the new profile to  $z$ . Of course, one can think of yet other behavioral assumptions.

We solve this issue in a way that bypasses any speculation about which behavioral assumption is the most appropriate.<sup>13</sup> We complete the allocation process by assuming that each agent is asked to report not only her preferences, but also a bundle, which we interpret as the one she requests. Moreover, we add to the specification of the outcome function a “selector” to break ties between the  $F$ -optimal outcomes for the reported preferences when the  $F$ -optimal set is not a singleton and the profile of reported bundles is not in this set. Now the outcome function is complete: if the profile of reported bundles is an  $F$ -optimal allocation for the profile of reported preferences, then it is the outcome of the allocation process; otherwise, the selector determines this outcome.

Adding a selector allows us to understand the strength of the behavioral assumptions in the strategic analysis of solutions. If the outcomes that obtain in equilibrium are independent of this selector, then these outcomes will result under any of the aforementioned behavioral assumptions. In Section 5 we show that this is the case for the equal-sacrifice solution when all goods are normal.

For a given solution and one of its selectors, we now formally define the game form associated with them for some domain of preferences. Let  $\mathcal{D} \subseteq \mathcal{U}$

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<sup>12</sup>This issue arises even for an *essentially single-valued* solution. For such a solution, agents are seemingly indifferent among the recommended allocations under the reported preferences, but this could not be the case for their true preferences (Thomson, 1979, 1984).

<sup>13</sup>See Velez (2011a) for the application of our approach to the manipulability of solutions for the allocation of indivisible goods with monetary compensation.

be a domain,  $F$  a solution, and  $f \in F$ . The **game form**  $\langle S(\mathcal{D})^N, \mathbf{F}^f \rangle$  is defined as follows: (i) each agent's strategy space is  $S(\mathcal{D}) \equiv \mathcal{D} \times \mathbb{R}_+^K$ ; and (ii) given the strategy profile  $(R, z) \equiv (R_i, z_i)_{i \in N} \in S(\mathcal{D})^N$ , the outcome is

$$\mathbf{F}^f(\mathbf{R}, \mathbf{z}) \equiv \begin{cases} z & \text{if } z \in F(R) \\ f(R) & \text{otherwise.} \end{cases}$$

For each  $R^0 \in \mathcal{U}^N$ , the **game**  $\langle S(\mathcal{D})^N, \mathbf{F}^f, \mathbf{R}^0 \rangle$  is obtained by augmenting the game form  $\langle S(\mathcal{D})^N, \mathbf{F}^f \rangle$  by the preference profile  $R^0$ .

A **Nash equilibrium** of  $\langle S(\mathcal{D})^N, \mathbf{F}^f, \mathbf{R}^0 \rangle$  is a strategy profile  $(R, z) \in S(\mathcal{D})^N$ , such that for each  $i \in N$  and each  $(R'_i, z'_i) \in S(\mathcal{D})$ ,  $F_i^f(R, z) R_i^0 F_i^f(R_{-i}, R'_i, z_{-i}, z'_i)$ . For each game  $\langle S(\mathcal{D})^N, \mathbf{F}^f, \mathbf{R}^0 \rangle$ , the set of **Nash equilibria** is  $\mathcal{N}\langle S(\mathcal{D})^N, \mathbf{F}^f, \mathbf{R}^0 \rangle$  and the set of **Nash equilibrium outcomes** is  $\mathcal{O}\langle S(\mathcal{D})^N, \mathbf{F}^f, \mathbf{R}^0 \rangle$ . If for each pair of selectors of  $F$ ,  $f$  and  $g$ ,  $\mathcal{O}\langle S(\mathcal{D})^N, \mathbf{F}^f, \mathbf{R}^0 \rangle = \mathcal{O}\langle S(\mathcal{D})^N, \mathbf{F}^g, \mathbf{R}^0 \rangle$ , we denote this common set by  $\mathcal{O}\langle S(\mathcal{D})^N, \mathbf{F}, \mathbf{R}^0 \rangle$ .

We also consider the possibility of coalitional manipulations. A **strong Nash equilibrium** of  $\langle S(\mathcal{D})^N, \mathbf{F}^f, \mathbf{R}^0 \rangle$  is a strategy profile  $(R, z) \in S(\mathcal{D})^N$  such that for each  $M \subseteq N$  and each  $(R'_M, z'_M) \in S(\mathcal{D})^M$ , if there is  $i \in M$  such that  $F^f(R_{-M}, R'_M, z_{-M}, z'_M) P_i^0 F^f(R, z)$ , then there is  $j \in M$  such that  $F^f(R, z) P_j^0 F^f(R_{-M}, R'_M, z_{-M}, z'_M)$ . For each game  $\langle S(\mathcal{D})^N, \mathbf{F}^f, \mathbf{R}^0 \rangle$  the set of **strong Nash equilibria** is  $\mathcal{N}^*\langle S(\mathcal{D})^N, \mathbf{F}^f, \mathbf{R}^0 \rangle$  and the set of **strong Nash equilibrium outcomes** is  $\mathcal{O}^*\langle S(\mathcal{D})^N, \mathbf{F}^f, \mathbf{R}^0 \rangle$ . If for any two selections of  $F$ ,  $f$  and  $g$ ,  $\mathcal{O}^*\langle S(\mathcal{D})^N, \mathbf{F}^f, \mathbf{R}^0 \rangle = \mathcal{O}^*\langle S(\mathcal{D})^N, \mathbf{F}^g, \mathbf{R}^0 \rangle$ , then we drop this parameter from our notation and denote this common set by  $\mathcal{O}^*\langle S(\mathcal{D})^N, \mathbf{F}, \mathbf{R}^0 \rangle$ .

### 3.3. Additional notation

For each pair  $\{a, b\} \subseteq \mathbb{R}_+^K$ , let **seg** $\{a, b\}$  be the segment connecting  $a$  and  $b$ . For each list  $\{a, b, \dots, z\} \subseteq \mathbb{R}_+^K$ , let **bro.seg** $\{a, b, \dots, z\}$  be the broken segment connecting these points in that order, and let **con.hull** $\{a, b, \dots, z\}$  be the convex hull of  $\{a, b, \dots, z\}$ . For each pair of vectors  $\{a, b\} \subseteq \mathbb{R}_+^K$  such that  $a \leq b$ , let **rec** $\{a, b\}$  be the rectangle  $\{y \in \mathbb{R}_+^K : a \leq y \leq b\}$ . Finally, for each  $b \in \mathbb{R}_+^2$  and each  $m \in \mathbb{R}_+$ , let **ray** $\{b, m\}$  be the ray emanating from  $b$  with slope  $m$ .

## 4. The equal-sacrifice solution

The **Pareto solution**,  $P$ , and the **weak Pareto solution**,  $P^w$ , are defined as usual: for each  $R \in \mathcal{U}^N$  and each  $z \in Z$ ,  $z \in P(R)$  if and only if

there is no  $z' \in Z$  such that (i) for each  $i \in N$ ,  $z'_i R_i z_i$  and (ii) there is  $j \in N$  such that  $z'_j P_j z_j$ ;  $z \in P^w(R)$  if and only if there is no  $z' \in Z$  such that for each  $i \in N$ ,  $z'_i P_i z_i$ .

Let  $\mu$  be the Lebesgue measure on  $\mathbb{R}^K$ . For each  $R_0 \in \mathcal{U}$  and each  $x \in \mathbb{R}_+^K$ , let  $\mathbf{a}(R_0, x) \equiv \mu(U^c(R_0, x))$ , i.e., the size of the constrained upper contour set of  $R_0$  at  $x$ . Since resources are owned collectively in our model, then when an agent with preferences  $R_0$  consumes  $x$ , she “sacrifices” her option of consuming the bundles in  $U^c(R_0, x)$ . Thus,  $a(R_0, x)$  is a reasonable measure of the sacrifice of  $R_0$  at  $x$ , the measure that assigns equal weights to all bundles.<sup>14</sup>

For each  $R \in \mathcal{U}^N$ , let  $\Psi(R)$  be the set of feasible allocations at which sacrifices are equal across agents, i.e.,  $\Psi(R) \equiv \{z \in Z : \text{for each pair } \{i, j\} \subseteq N, a(R_i, z_i) = a(R_j, z_j)\}$ . The **equal-sacrifice solution**,  $\mathbf{E}$ , associates with each  $R \in \mathcal{U}^N$  the set of allocations at which sacrifices are equal across agents, and this common sacrifice is minimal:

$$E(R) \equiv \{z \in \Psi(R) : \text{for each } i \in N \text{ and each } z' \in \Psi(R), a(R_i, z_i) \leq a(R_i, z'_i)\}.$$

The following theorem states that  $E$  is a well-defined solution.<sup>15</sup> We present the proof in the Appendix.

**Theorem 1.** For each  $R \in \mathcal{U}^N$ ,  $E(R) \neq \emptyset$ .

The following lemma concerns efficiency properties of the equal-sacrifice allocations. First, they are weakly Pareto efficient. Second, the lemma identifies two wide classes of economies in which they are in fact Pareto efficient and establishes welfare bounds for equal-sacrifice allocations.<sup>16</sup> The proof is in the Appendix.

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<sup>14</sup>One can argue in favor of other measures that assign less uniform weights than  $\mu$ . In fact, one can think of equal-sacrifice as a principle and then define the family of equal-sacrifice solutions parameterized by the measure that defines the sacrifice of an agent at an allocation. All of our results extend to this broad family of solutions when  $a(R_0, x) \equiv \tilde{\mu}(U^c(R_0, x))$  for some measure  $\tilde{\mu}$ , on the family of Lebesgue-measurable sets, that is uniformly continuous with respect to  $\mu$  and assigns a positive measure to each open subset of  $\mathbb{R}^K$ .

<sup>15</sup>One can easily see that  $E$  is *essentially single-valued*.

<sup>16</sup>The equal-sacrifice solution may select weakly Pareto efficient allocations that are not Pareto efficient when there are at least three agents and preferences are in  $\mathcal{U} \setminus \mathcal{R}$ . Let  $(R_i)_{i \in N} \in \mathcal{U}^N$  and for each  $i \in N$ , let  $u_i$  be the function defined by: for each  $z_i \in \mathbb{R}_+^K$ ,  $u_i(z_i) \equiv \mu(\text{rec}\{0, \Omega\}) - a(R_i, z_i)$ . It is easily seen that  $E(R)$  contains the allocations

**Lemma 1.** Let  $R \in \mathcal{U}^N$  and  $z \in E(R)$ . Then,

(i-a)  $z \in P^w(R)$  and (i-b) for each  $i \in N$ ,  $z_i P_i 0$ .

(ii) if  $|N| = 2$ , then  $z \in P(R)$ .

(iii) if  $R \in \mathcal{R}^N$ , then (iii-a)  $z \in P(R)$  and (iii-b) for each  $i \in N$ ,  $\Omega P_i z_i$ .

The following lemma is an application of the Second Fundamental Theorem of Welfare Economics. We omit the standard proof (e.g., Mas-Colell et al., 1995).

**Lemma 2.** Let  $R \in \mathcal{U}^N$  and  $z \in E(R)$ . Then there is  $p \in \Delta^{K-1}$  such that for each  $i \in N$ , either  $p \cdot z_i = 0$  or  $p \in \text{Supp}(R_i, z_i)$ . Moreover if  $R \in \mathcal{R}^N$ , then  $p \gg 0$  and for each  $i \in N$ ,  $p \in \text{Supp}(R_i, z_i)$ .

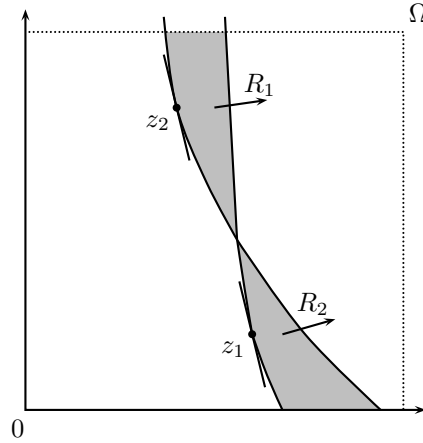


Figure 1: Equal sacrifice allocations.

Figure 1 illustrates an equal sacrifice allocation. The allocation is efficient (Lemma 1). Moreover, the sizes of the agents' upper contour sets at their consumption are equal. This is equivalent to equality of the size of the two shaded areas forming a bow tie in the figure.

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whose image under  $(u_i)_{i \in N}$  is the Kalai-Smorodinsky (K-S) bargaining solution for the comprehensive hull of the problem  $u(Z)$  with disagreement point 0. It is well known that K-S may select weakly Pareto efficient allocations that are not Pareto efficient for more than three agents on the convex domain. An example showing this fact is easily adapted to prove the parallel statement for  $E$ .

We now introduce a solution that is seemingly unrelated with  $E$ . In Section 5, we show that it selects the outcomes that result from the manipulation of  $E$ . The **equal-division constrained Walrasian solution**,  $W_{ed}^c$ , selects the “market-like” outcomes at which each agent’s endowment is an equal share of the aggregate endowment and her budget set is constrained by feasibility.<sup>17</sup> Formally, it associates with each  $R \in \mathcal{U}^N$  the set of allocations

$$W_{ed}^c(R) \equiv \left\{ z \in Z : \begin{array}{l} \text{there is } p \in \Delta^{K-1} \text{ such that for each } i \in N, p \cdot z_i \leq p \cdot \frac{1}{n}\Omega \\ \text{and for each } z'_i \in U^c(P_i, z_i), p \cdot z'_i > p \cdot \frac{1}{n}\Omega \end{array} \right\}.$$

$W_{ed}^c$  is a supersolution of the equal-division Walrasian solution, i.e., the solution that associates with each profile the allocations that can be sustained as a competitive equilibrium from equal endowments. Moreover, it is the smallest solution, in a set inclusion sense, that contains the equal-division Walrasian solution and (i) is monotonic in the sense of Maskin (1999) and (ii) can be implemented in Nash equilibria (Thomson, 1999).<sup>18</sup>

The following lemma concerns efficiency properties of  $W_{ed}^c$ . We omit the standard proof.

**Lemma 3.** Let  $R \in \mathcal{U}^N$  and  $z \in W_{ed}^c(R)$ . Then,

- (i)  $z \in P^w(R)$ .
- (ii) if  $R \in \mathcal{R}^N$ , then (ii-a)  $z \in P(R)$ , (ii-b) if  $p \in \Delta^{K-1}$  supports  $z$  as a member of  $W_{ed}^c(R)$ , then  $p \gg 0$ , (ii-c) for each  $i \in N$  and each  $z'_i \in U^c(R_i, z_i)$ ,  $p \cdot z'_i \geq p \cdot \frac{1}{n}\Omega$ .

## 5. The manipulability of the equal-sacrifice solution

Our main theorem concerns the manipulation of the equal-sacrifice solution. It characterizes both the Nash and strong Nash equilibrium correspondences of the manipulation game associated with this solution when all goods are normal. Let  $e \in E$  and  $R^0 \in \mathcal{R}_{\mathcal{I}}$ . The theorem states that each equal-division constrained Walrasian allocation for  $R^0$  is a strong Nash equilibrium allocation (and thus, a Nash equilibrium allocation) of the game associated

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<sup>17</sup>In general, equal-sacrifice allocations are not equal-division constrained Walrasian. See for instance allocation  $z$  in Figure 1: there is a common supporting price for the upper contour set of each agent at her consumption, but the value of these consumptions at this price is different across agents.

<sup>18</sup>See Section 6.3 for definitions concerning Nash implementation.

with  $\mathcal{R}_{\mathcal{I}}$ ,  $E$ ,  $e$ , and  $R^0$ . Moreover, each Nash equilibrium allocation (and thus, each strong Nash equilibrium allocation) of the game associated with  $\mathcal{R}_{\mathcal{I}}$ ,  $E$ ,  $e$ , and  $R^0$  is an equal-division constrained Walrasian allocation for  $R^0$ .

**Theorem 2.** For each  $R^0 \in \mathcal{R}_{\mathcal{I}}^N$  and each  $e \in E$ ,

$$\mathcal{O}^*\langle S(\mathcal{R}_{\mathcal{I}})^N, E^e, R^0 \rangle = \mathcal{O}\langle S(\mathcal{R}_{\mathcal{I}})^N, E^e, R^0 \rangle = W_{ed}^c(R^0).$$

Since the characterization above is independent of the particular selector that parameterizes the game form, then we can drop it from the notation. We conclude that the outcomes from the manipulation, both individual and coalitional, of  $E$  on  $\mathcal{R}_{\mathcal{I}}$  are exactly the constrained Walrasian outcomes for the true preference profile. Moreover, these outcomes are Pareto efficient for the true preference profile.

**Corollary 1.** For each  $R^0 \in \mathcal{R}_{\mathcal{I}}^N$ ,

$$P(R^0) \supseteq \mathcal{O}^*\langle S(\mathcal{R}_{\mathcal{I}})^N, E, R^0 \rangle = \mathcal{O}\langle S(\mathcal{R}_{\mathcal{I}})^N, E, R^0 \rangle = W_{ed}^c(R^0).$$

Section 6 discusses the tightness of our theorems, the extension of our results when the domain of admissible preferences is  $\mathcal{U}_{\mathcal{I}}$ , and the relevance of our theorems to implementation theory.

The proof of Theorem 2 follows from four lemmas.

Let  $e \in E$ ,  $R^0 \in \mathcal{R}^N$ , and  $\mathcal{D} \subseteq \mathcal{R}$  be such that  $\mathcal{R}_{\mathcal{H}} \subseteq \mathcal{D}$ . Our first lemma identifies action profiles that are not Nash equilibria of  $\langle S(\mathcal{D})^N, E^e, R^0 \rangle$ . It states conditions on an action profile under which at least one agent can benefit by changing her action. Let us provide the intuition in the two-agent case (Figure 2); the intuition extends to the general case. Assume that  $N \equiv \{1, 2\}$ . Let  $(R, z)$  be a strategy profile. The lemma provides conditions guaranteeing that it is not a best response for one agent, say agent 1, to report  $(R_1, z_1)$  if agent 2 reports  $(R_2, z_2)$ . More precisely, these conditions guarantee that there are: (i)  $z' \in Z$  such that agent 1 prefers to  $E^e(R, z)$ , and (ii) preferences  $R'_1$  that agent 1 can report in order to obtain  $z'$ , i.e., such that  $z' = E^e(R'_1, R_2, z)$ .

The critical condition for the existence of  $z'$  and  $R'_1$  satisfying (i) and (ii) above is that for some  $z'$  that agent 1 prefers to  $E^e(R, z)$  and some  $p$

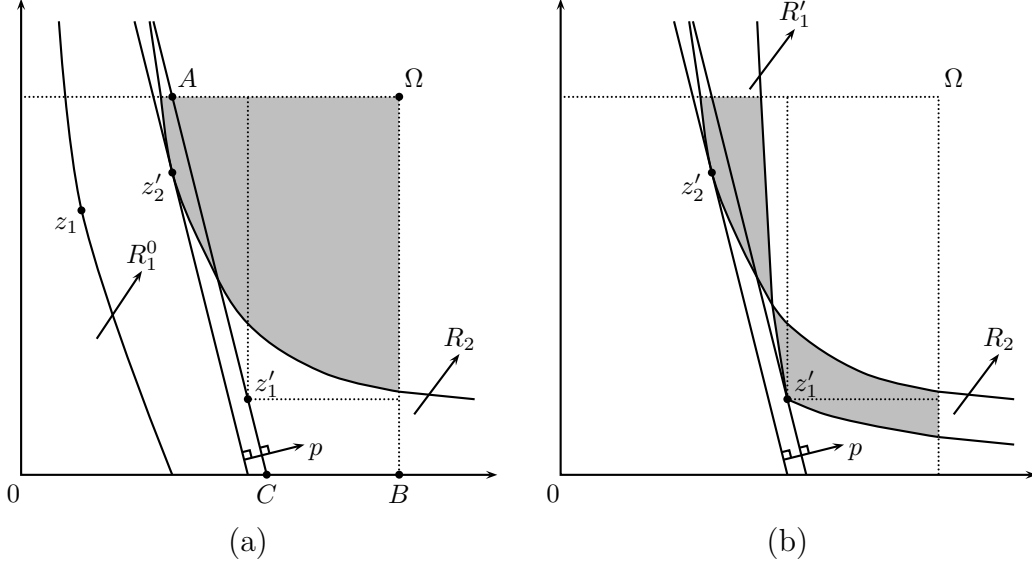


Figure 2: Intuition Lemma 4.

in  $\text{Supp}(R_2, z_2)$ ,  $\mu(\text{rec}\{z_1', \Omega\}) < a(R_2, z_2) < a(L^p, z_1')$ . The intuition of this condition is as follows. Let  $a \equiv a(R_2, z_2)$ . Recall that  $a$  is  $R_2$ 's sacrifice at  $z_2$  (the size of the shaded area in Figure 2 (a)). Thus, if  $z' = E^e(R_1', R_2, z)$ , then the sacrifice of  $R_1'$  at  $z_1'$  must be  $a$ . First, the sacrifice at  $z_1'$  for any  $R_1'$  is greater than the size of  $\text{rec}\{z_1', \Omega\}$ . Thus,  $\mu(\text{rec}\{z_1', \Omega\}) < a$ . Second, by Lemma 2 there is  $p$  in  $\text{Supp}(R_1', z_1') \cap \text{Supp}(R_2, z_2)$ . Thus,  $a \leq a(L^p, z_1')$ . Moreover, a sufficient condition that guarantees  $z' = E^e(R_1', R_2)$ , is  $U(I_1', z_1') \cap U(I(L^p), z_1') = \{z_1'\}$ . For this to be true, the sacrifice at  $z_1'$  for  $L^p$  must be greater than  $a$ , i.e.,  $a < a(L^p, z_1')$ . In Figure 2 (a) this condition amounts to the size of polygon  $A\Omega BC$  be greater than that of the shaded area. It turns out that if these two inequalities are satisfied and if homothetic preferences are possible reports, then there is  $R_1' \in \mathcal{R}_{\mathcal{H}}$  such that  $\{z'\} = E(R_1', R_2)$ . In Figure 2 (b) this amounts to the existence of  $R_1'$  for which the two shaded areas that form a bow-tie have equal size.

**Lemma 4.** Let  $\mathcal{D} \subseteq \mathcal{R}$  be such that  $\mathcal{R}_{\mathcal{H}} \subseteq \mathcal{D}$  and  $e \in E$ . Let  $R^0 \in \mathcal{D}^N$  and  $(R, z) \in S(\mathcal{D})^N$ . If there are  $i \in N$ ,  $z' \in Z$  with  $z'_i \gg 0$ ,  $p \in \Delta^{K-1}$  with  $p \gg 0$ , and  $a \in \mathbb{R}_+$  such that:

- (i) for each  $j \in N \setminus \{i\}$ ,  $p \in \text{Supp}(R_j, z'_j)$ ,
- (ii) for each  $j \in N \setminus \{i\}$ ,  $a(R_j, z'_j) = a$ ,
- (iii)  $z'_i P_i^0 E_i^e(R, z)$ ,

(iv)  $\mu(\text{rec}\{z'_i, \Omega\}) < a < a(L^p, z'_i)$ ,  
then  $(R, z) \notin \mathcal{N}\langle S(\mathcal{D})^N, E^e, R^0 \rangle$ .

*Proof.* We construct a preference  $R'_i \in \mathcal{D}$  such that  $E_i^e(R_{-i}, R'_i, z_{-i}, z'_i) P_i^0 E_i^e(R, z)$ . Let  $\alpha \in [0, 1)$  and  $A$  be the set obtained by translating the boundary of  $\mathbb{R}_+^K$  so that the origin is moved to  $z'_i$ , i.e.,  $A \equiv \{x_i \in \mathbb{R}_+^K : z'_i \leq x_i\} \setminus \{x_i \in \mathbb{R}_+^K : z'_i \ll x_i\}$ . For each ray  $r$  in the direction of a strictly positive vector, the  **$\alpha$ -convex combination of  $A$  and  $U(I(L^p), z'_i)$  through  $r$**  is the point in  $\mathbb{R}_+^K$  obtained as a convex combination of  $r \cap A$  with weight  $\alpha$  and  $r \cap U(I(L^p), z'_i)$  with weight  $1 - \alpha$ . Let  $A_\alpha$  be the set obtained by taking  $\alpha$ -convex combinations of  $A$  and  $U(I(L^p), z'_i)$  through all the strictly positive rays. Let  $R_i(\alpha)$  be the preference whose indifference sets in the interior of  $\mathbb{R}_+^K$  are obtained as homothetic images of the closure of  $A_\alpha$  (the indifference set of  $R_i(\alpha)$  through zero is the complement of the interior indifference sets). Since  $z'_i \gg 0$ , then  $R_i(\alpha)$  is homothetic and such that  $U(I_i(\alpha), z'_i) = A_\alpha$ . The function  $\alpha \mapsto a(R_i(\alpha), z'_i)$  is continuous and as  $\alpha \rightarrow 1$ ,  $a(R_i(\alpha), z'_i) \rightarrow \mu(\text{rec}\{z'_i, \Omega\})$ . By hypothesis (iv),  $\mu(\text{rec}\{z'_i, \Omega\}) < a < a(L^p, z'_i) = a(R_i(0), z'_i)$ . Thus, by the Intermediate Value Theorem, there is  $\beta \in (0, 1)$  such that  $a(R_i(\beta), z'_i) = a$ . Let  $R'_i \equiv R_i(\beta)$ .<sup>19</sup> Since  $\mathcal{R}_{\mathcal{H}} \subseteq \mathcal{D}$ , then  $R'_i \in \mathcal{D}$ .

Since  $U(R'_i, z'_i) \subseteq U(L^p, z'_i)$ , then  $p \in \text{Supp}(R'_i, z'_i)$ . Thus, by hypothesis (i),  $z' \in P(R_{-i}, R'_i)$ . Since  $a(R'_i, z'_i) = a$ , then by hypothesis (ii),  $z' \in E(R_{-i}, R'_i)$ . By construction, the upper contour set of  $R'_i$  at  $z'_i$  intersects the hyperplane with normal  $p$  passing through  $z'_i$  only at  $z'_i$ , i.e.,  $\{z''_i : z''_i R'_i z'_i, p \cdot z''_i = p \cdot z'_i\} = \{z'_i\}$ . Let  $z'' \in E(R_{-i}, R'_i)$ . We prove that  $z''_i = z'_i$ . Since for each  $j \in N \setminus \{i\}$ ,  $z''_j I_j z'_j$ , then for each  $j \in N \setminus \{i\}$ ,  $p \cdot z''_j \geq p \cdot z'_j$  and thus,  $p \cdot z''_i \leq p \cdot z'_i$ . Since  $p \in \text{Supp}(R'_i, z'_i)$  and  $z''_i I'_i z'_i$ , then  $p \cdot z''_i \leq p \cdot z'_i$ . Thus,  $p \cdot z''_i = p \cdot z'_i$ . Thus,  $z''_i = z'_i$ . By hypothesis (iii),  $E_i^e(R_{-i}, R'_i, z_{-i}, z'_i) = z'_i P_i^0 E_i^e(R, z)$ . Thus,  $(R, z) \notin \mathcal{N}\langle S(\mathcal{D})^N, E^e, R^0 \rangle$ .  $\square$

The next lemma states that if a domain  $\mathcal{D} \subseteq \mathcal{R}_{\mathcal{I}}$  is such that  $\mathcal{R}_{\mathcal{H}} \subseteq \mathcal{D}$ , then, for each  $e \in E$ , in each equilibrium of  $\langle S(\mathcal{D})^N, E^e, R^0 \rangle$ , agents report parallel linear indifference sets through their consumption bundles within the “feasible box,” i.e., the set  $\text{rec}\{0, \Omega\}$ .

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<sup>19</sup>  $R'_i$  is strictly monotone in the interior of the consumption space, but it is not strictly monotone. It is easy to construct a preference  $R''_i$  with the same properties as  $R'_i$  but strictly monotone.

**Lemma 5.** Let  $\mathcal{D} \subseteq \mathcal{R}_{\mathcal{I}}$  be such that  $\mathcal{R}_{\mathcal{H}} \subseteq \mathcal{D}$ ,  $e \in E$ , and  $R^0 \in \mathcal{R}^N$ . Let  $(R, z^*) \in \mathcal{N}\langle S(\mathcal{D})^N, E^e, R^0 \rangle$  and  $z \equiv E^e(R, z^*)$ . Then, there is  $p \in \Delta^{K-1}$  such that  $p \gg 0$  and for each  $i \in N$ ,  $U^c(I_i, z_i) = U^c(I(L^p), z_i)$ .

*Proof.* Let  $R$ ,  $z^*$ , and  $z$  be as in the statement of the lemma. By Lemma 2, for each  $i \in N$ ,  $z_i \succeq 0$  and there is  $p \in \Delta^{K-1}$  such that  $p \gg 0$  and for each  $i \in N$ ,  $p \in \text{Supp}(R_i, z_i)$ . We claim that for each  $i \in N$ ,  $U^c(I_i, z_i) = U^c(I(L^p), z_i)$ . Suppose by contradiction that there is  $i \in N$  for whom  $U^c(I_i, z_i) \neq U^c(I(L^p), z_i)$ . By Lemma 1,  $\Omega P_i z_i P_i 0$ . Since preferences are strictly monotone in the interior of the consumption space, then  $\mu(\text{rec}\{z_i, \Omega\}) < a(R_i, z_i)$ . Since  $p \in \text{Supp}(R_i, z_i)$ , then  $a(R_i, z_i) \leq a(L^p, z_i)$ . Since preferences are continuous and  $U^c(I_i, z_i) \neq U^c(I(L^p), z_i)$ , then  $a(R_i, z_i) < a(L^p, z_i)$ .<sup>20</sup> Let  $\delta \equiv a(L^p, z_i) - a(R_i, z_i) > 0$ .

For each  $j \in N \setminus \{i\}$ , let  $V_j$  be a quasi-strictly increasing income expansion path for  $R_j$  at prices  $p$  that passes through  $z_j$ .<sup>21</sup> Since preferences are continuous, then for each  $j \in N \setminus \{i\}$ , the function  $w \in \mathbb{R}_+ \mapsto a(R_j, V_j(w))$  is continuous.<sup>22</sup> Let  $\eta \in \mathbb{R}_+$  be such that  $\eta < \frac{\delta}{2}$ . Since  $\delta < \mu(\text{rec}\{0, \Omega\}) - a(R_i, z_i)$ , then by the Intermediate Value Theorem, there are  $(w_j^\eta)_{j \in N \setminus \{i\}} \ll (p \cdot z_j)_{j \in N \setminus \{i\}}$  such that for each  $j \in N \setminus \{i\}$ ,  $a(R_j, V_j(w_j^\eta)) = a(R_j, z_j) + \eta$ . Let  $z_i^\eta \equiv \Omega - \sum_{j \in N \setminus \{i\}} V_j(w_j^\eta)$ . Since each  $V_j$  is quasi-strictly increasing, then for each  $\eta > 0$ ,  $z_i^\eta \succeq z_i$  and  $z_i^\eta \gg 0$ . Since preferences are strictly monotone in the interior of the consumption space, then  $z_i^\eta P_i^0 z_i$ . Observe that as  $\eta \rightarrow 0$ ,  $z_i^\eta \rightarrow z_i$ . Let  $\nu > 0$  be such that  $a(L^p, z_i) - a(L^p, z_i^\nu) < \frac{\delta}{2}$ . Let  $z_i' \equiv z_i^\nu$ , and for each  $j \in N \setminus \{i\}$ , let  $z_j' \equiv V_j(w_j^\nu)$ . Clearly,  $z' \in Z$ . Let  $a \equiv a(R_i, z_i) + \nu$ . By construction, for each  $j \in N \setminus \{i\}$ ,  $a(R_j, z_j') = a$ .

Agent  $i$  and the objects  $z' \in Z$ ,  $p \in \mathbb{R}_{++}^K$ , and  $a \in \mathbb{R}_{++}$  satisfy the first three properties in Lemma 4. We claim that they also satisfy the fourth property. By construction of  $z'$  and  $a$ ,  $a(L^p, z_i) - \frac{\delta}{2} < a(L^p, z_i')$ . Recall that  $\delta \equiv a(L^p, z_i) - a(R_i, z_i)$ . Thus,  $a(R_i, z_i) + \frac{\delta}{2} < a(L^p, z_i')$ . Since  $\nu < \frac{\delta}{2}$ , then  $a \equiv a(R_i, z_i) + \nu < a(L^p, z_i')$ . Finally, since  $z_i' \geq z_i$ , then  $\mu(\text{rec}\{z_i', \Omega\}) \leq \mu(\text{rec}\{z_i, \Omega\}) < a(R_i, z_i) < a$ . Thus, by Lemma 4,  $(R, z^*) \notin$

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<sup>20</sup>One could also argue that strict monotonicity in the interior of the consumption space implies that  $a(R_i, z_i) < a(L^p, z_i)$ . We thank a reviewer for bringing this remark to our attention.

<sup>21</sup>These income expansion paths exist because  $R \in \mathcal{R}_{\mathcal{I}}^N$ .

<sup>22</sup>Since preferences are continuous, then the function  $x_j \in \mathbb{R}_+^K \mapsto a(R_j, x_j)$  is continuous; since  $V_j$  is quasi-strictly increasing, then it is continuous. Thus,  $a(R_i, V_j(\cdot))$  is the composition of two continuous functions.

$\mathcal{N}\langle S(\mathcal{D})^N, E^e, R^0 \rangle$ . This is a contradiction.  $\square$

The following lemma states that in an equal-sacrifice allocation in which each agent has linear and parallel constrained indifference sets through their consumptions, each agent has equal “expenditure” at the prices that support her preferences.

**Lemma 6.** Let  $R \in \mathcal{U}^N$ ,  $z \in E(R)$ , and  $p \in \Delta^{K-1}$ . If for each  $i \in N$ ,  $U^c(I_i, z_i) = U^c(I(L^p), z_i)$ , then for each  $i \in N$ ,  $p \cdot z_i = p \cdot \frac{1}{n}\Omega$ .

*Proof.* Let  $i \in N$ . Since for each  $j \in N$ ,  $U^c(I_j, z_j) = U^c(I(L^p), z_j)$  and  $z \in E(R)$ , then for each  $j \in N$ ,  $p \cdot z_j = p \cdot z_i$ . Thus,  $p \cdot z_i \leq p \cdot \frac{1}{n}\Omega$ . We claim that  $p \cdot z_i = p \cdot \frac{1}{n}\Omega$ . Suppose by contradiction that  $p \cdot z_i < p \cdot \frac{1}{n}\Omega$ . Thus, for each  $j \in N$ ,  $p \cdot z_j < p \cdot \frac{1}{n}\Omega$ . Thus, for each  $j \in N$ ,  $\frac{1}{n}\Omega P_j z_j$ . Thus,  $z \notin P^w(R)$ . This contradicts Lemma 1.  $\square$

Let  $e \in E$ . The following lemma states that if a domain  $\mathcal{D} \subseteq \mathcal{R}_{\mathcal{I}}$  is such that  $\mathcal{R}_{\mathcal{H}} \subseteq \mathcal{D}$ , then all equilibrium allocations of  $\langle S(\mathcal{D})^N, E^e, R^0 \rangle$  are equal-division constrained Walrasian for  $R^0$ .

**Lemma 7.** Let  $\mathcal{D} \subseteq \mathcal{R}_{\mathcal{I}}$  be such that  $\mathcal{R}_{\mathcal{H}} \subseteq \mathcal{D}$ . Let  $e \in E$ . Then, for each  $R^0 \in \mathcal{R}^N$ ,

$$\mathcal{O}\langle S(\mathcal{D})^N, E^e, R^0 \rangle \subseteq W_{ed}^c(R^0).$$

*Proof.* Let  $(R, z^*) \in \mathcal{N}\langle S(\mathcal{D})^N, E^e, R^0 \rangle$  and  $z \equiv E^e(R, z^*)$ . By Lemma 5, there is  $p \in \Delta^{K-1}$  such that  $p \gg 0$  and for each  $i \in N$ ,  $U^c(I_i, z_i) = U^c(I(L^p), z_i)$ . Since  $z \in E(R)$ , then for each  $\{i, j\} \subseteq N$ ,  $a(L^p, z_i) = a(L^p, z_j)$ . By Lemma 6, for each  $i \in N$ ,  $p \cdot z_i = p \cdot \frac{1}{n}\Omega$ . We claim that for each  $i \in N$  and each  $x_i \in U^c(P_i^0, z_i)$ ,  $p \cdot x_i > p \cdot \frac{1}{n}\Omega$ . Suppose by contradiction that there is  $i \in N$  and  $x_i \in U^c(P_i^0, z_i)$  such that  $p \cdot x_i \leq p \cdot \frac{1}{n}\Omega$ . Since preferences are continuous, then we can suppose without loss of generality that  $x_i \gg 0$ . For each  $\alpha \in [0, 1]$ , let  $z_i^\alpha \equiv \alpha x_i + (1 - \alpha)\Omega$ . Since  $\Omega R_i^0 x_i$  and  $R_i^0$  is convex, then for each  $\alpha \in [0, 1]$ ,  $z_i^\alpha P_i^0 z_i$ . Also, since  $p \cdot x_i \leq p \cdot \frac{1}{n}\Omega$ , then by the Intermediate Value Theorem, there is  $\beta \in (0, 1]$  such that  $0 \ll z_i^\beta \in U^c(P_i^0, z_i)$  and  $p \cdot z_i^\beta = p \cdot \frac{1}{n}\Omega$ . For each  $j \in N \setminus \{i\}$ , let  $z_j^\beta \equiv (\Omega - z_i^\beta)/(n - 1)$ . Thus,  $z^\beta \in Z$ . Moreover, for each  $j \in N \setminus \{i\}$ ,  $p \cdot z_j^\beta = p \cdot \frac{1}{n}\Omega$  and  $p \in \text{Supp}(R_j, z_j^\beta)$ .

For each  $j \in N \setminus \{i\}$ , let  $V_j$  be a quasi-strictly increasing income expansion path for  $R_j$  at prices  $p$  that passes through  $z_j^\beta$ .<sup>23</sup> For each  $j \in N \setminus \{i\}$ , the

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<sup>23</sup>These income expansion paths exist because  $R \in \mathcal{R}_{\mathcal{I}}^N$ .

function  $w \in \mathbb{R}_+ \mapsto a(R_j, V_j(w))$  is continuous.<sup>24</sup> Let  $\eta \in (0, a(L^p, \frac{1}{n}\Omega))$ . By the Intermediate Value Theorem, there are  $(w_j^\eta)_{j \in N \setminus \{i\}} \gg (p \cdot z_j)_{j \in N \setminus \{i\}}$  such that for each  $j \in N \setminus \{i\}$ ,  $a(R_j, V_j(w_j^\eta)) = a(R_j, z_j) - \eta$ . Let  $y_i^\eta \equiv \Omega - \sum_{j \in N \setminus \{i\}} V_j(w_j^\eta) \lesssim z_i^\beta$ . Since each  $V_j$  is quasi-strictly increasing, and thus, continuous, then as  $\eta \rightarrow 0$ ,  $y_i^\eta \rightarrow z_i^\beta$ . Thus, as  $\eta \rightarrow 0$ ,  $\mu(\text{rec}\{y_i^\eta, \Omega\}) \rightarrow \mu(\text{rec}\{z_i^\beta, \Omega\})$ . Moreover, for each  $j \in N \setminus \{i\}$ , as  $\eta \rightarrow 0$ ,  $V_j(w_j^\eta) \rightarrow z_j^\beta$ . Then  $\nu > 0$  can be selected in a neighborhood of zero such that: (i) for each  $j \in N \setminus \{i\}$ ,  $a(R_j, V_j(w_j^\nu)) > \mu(\text{rec}\{y_i^\nu, \Omega\})$ , (ii)  $y_i^\nu \gg 0$ , and (iii)  $y_i^\nu P_i^0 z_i$ . Let  $z'_i \equiv y_i^\nu$  and for each  $j \in N \setminus \{i\}$ , let  $z'_j \equiv V_j(w_j^\nu)$ . Clearly,  $z' \in Z$ . Let  $a \in \mathbb{R}_{++}$  be the common value of  $a(R_j, z'_j) = a(R_j, z_j) - \nu = a(L^p, z_j) - \nu$  for  $j \in N$ . Since  $a < a(L^p, z_i)$  and  $a(L^p, z_i) < a(L^p, z'_i)$ , then  $a < a(L^p, z'_i)$ . Thus,  $\mu(\text{rec}\{z'_i, \Omega\}) < a < a(L^p, z'_i)$ .

Agent  $i$  and the objects  $z' \in Z$ ,  $p \in \Delta^{K-1}$ , and  $a \in \mathbb{R}_{++}$  satisfy the four properties in Lemma 4. Thus,  $(R, z^*) \notin \mathcal{N}\langle S(\mathcal{D})^N, E^e, R^0 \rangle$ . This is a contradiction.

Since for each  $i \in N$ ,  $p \cdot z_i = p \cdot \frac{1}{n}\Omega$  and for each  $x_i \in U^c(P_i^0, z_i)$ ,  $p \cdot x_i > p \cdot \frac{1}{n}\Omega$ , then  $p$  supports  $z$  as a member of  $W_{ed}^c(R^0)$ .  $\square$

The following lemma states that if agents can report linear preferences, then each equal-division constrained Walrasian allocation is a strong Nash equilibrium outcome of the game form associated with  $E$  and any of its selections when true preferences belong to  $\mathcal{R}$ .

**Lemma 8.** Let  $\mathcal{D} \subseteq \mathcal{U}$  be such that  $\mathcal{R}_{\mathcal{L}} \subseteq \mathcal{D}$  and  $e \in E$ . Then, for each  $R^0 \in \mathcal{R}^N$ ,

$$\mathcal{O}^*\langle S(\mathcal{D})^N, E^e, R^0 \rangle \supseteq W_{ed}^c(R^0).$$

*Proof.* Let  $R^0 \in \mathcal{D}^N$  and  $z \in W_{ed}^c(R^0)$ . We show that there is  $R \in \mathcal{R}_{\mathcal{L}}^N$  such that  $(R, z) \in \mathcal{N}^*\langle S(\mathcal{D})^N, E^e, R^0 \rangle$  and  $E^e(R, z) = z$ . Since  $z \in W_{ed}^c(R^0)$ , then by Lemma 3, there is  $p \in \Delta^{K-1}$  such that  $p \gg 0$  and for each  $i \in N$  and each  $z'_i \in U^c(P_i^0, z_i)$ ,  $p \cdot z'_i > p \cdot \frac{1}{n}\Omega$ . Let  $R \equiv (L_i^p)_{i \in N} \in \mathcal{R}_{\mathcal{L}}^N$ .

We claim that for each  $M \subseteq N$  and each  $(R'_i, z'_i)_{i \in M} \in S(\mathcal{D})^M$ , if there is  $i \in M$  such that  $E_i^e(R_{-M}, R'_M, z_{-M}, z'_M) P_i^0 E_i^e(R, z)$ , then there is  $j \in M$  such that  $E_j^e(R, z) P_j^0 E_j^e(R_{-M}, R'_M, z_{-M}, z'_M)$ . By Lemma 3,  $W_{ed}^c(R^0) \subseteq P(R^0)$ . Thus, we can suppose without loss of generality that  $M \subsetneq N$ . Let  $M \subsetneq N$ ,  $R'_M \in \mathcal{D}^M$ , and  $z' \in E(R_{-M}, R'_M)$ . For each pair  $\{j, k\} \subseteq N \setminus M$ ,

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<sup>24</sup>See Footnote 22.

$p \cdot z'_j = p \cdot z'_k$ . Let  $\bar{z}_M \equiv \frac{1}{|M|} \sum_{i \in M} z'_i$ . We claim that  $p \cdot \bar{z}_M \leq p \cdot \frac{1}{n} \Omega$ . Suppose by contradiction that  $p \cdot \bar{z}_M > p \cdot \frac{1}{n} \Omega$ . By Lemma 2, there is  $q \in \Delta^{K-1}$  such that for each  $i \in N \setminus M$ , either  $q \cdot z'_i = 0$  or  $q \in \text{Supp}(R_i, z'_i)$  and for each  $i \in M$ , either  $q \cdot z'_i = 0$  or  $q \in \text{Supp}(R'_i, z'_i)$ . Since  $p \cdot \bar{z}_M > p \cdot \frac{1}{n} \Omega$ , then there is at least an agent in  $N \setminus M$  whose consumption at  $z'$  is worth, at prices  $p$ , less than  $\frac{1}{n} \Omega$ . Let  $j \in N \setminus M$ . Since agents in  $N \setminus M$  have identical preferences, then  $\frac{1}{n} \mu(\text{rec}\{0, \Omega\}) = a(L^p, \frac{1}{n} \Omega) < a(R_j, z'_j)$ . Since  $q \cdot z'_j = 0$  or  $q \in \text{Supp}(R_j, z'_j)$ , then  $q \cdot z'_j < q \cdot \frac{1}{n} \Omega$ . By feasibility,  $\sum_{j \in N} z'_j = \Omega$ . Thus, there is  $i \in M$  such that  $q \cdot z'_i \geq q \cdot \frac{1}{n} \Omega$ . Thus,  $a(R'_i, z'_i) \leq \frac{1}{n} \mu(\text{rec}\{0, \Omega\})$ . This contradicts  $z' \in E(R_{-M}, R'_M)$ . Now, suppose that there is  $i \in M$  such that  $z'_i P_i^0 z_i$ . Then,  $p \cdot z'_i > p \cdot \frac{1}{n} \Omega$ . Since  $p \cdot \bar{z}_M \leq p \cdot \frac{1}{n} \Omega$ , then there is  $j \in M$  such that  $p \cdot z'_j < p \cdot \frac{1}{n} \Omega$ . By Lemma 3 (ii-c), if  $z'_j \in U^c(R_j^0, z_j)$ , then  $p \cdot z'_j \geq p \cdot \frac{1}{n} \Omega$ . Thus,  $z_j P_j^0 z'_j$ .

Thus,  $(R, z) \in \mathcal{N}^* \langle S(\mathcal{D})^N, E^e, R^0 \rangle$ . Since  $z \in E(R)$ , then  $E^e(R, z) = z$ . Thus,  $z \in \mathcal{O}^* \langle S(\mathcal{D})^N, E^e, R^0 \rangle$   $\square$

The proof of Theorem 2 is a straightforward application of Lemmas 7 and 8 in which the domain of admissible preferences is  $\mathcal{D} = \mathcal{R}_{\mathcal{I}}$ .

## 6. Discussion

### 6.1. Tightness of theorems

In this section we discuss the tightness of our theorems. First, we present a lemma that provides conditions that characterize the Nash equilibria in the two-agent case when preferences are not necessarily strictly monotone in the interior of the consumption space. It facilitates the presentation of the examples that follow.

**Lemma 9.** Assume  $N \equiv \{1, 2\}$ . Let  $\mathcal{D} \subseteq \mathcal{U}$ ,  $R^0 \in \mathcal{U}^N$ ,  $R \in \mathcal{D}^N$ , and  $z \in E(R)$ . Suppose that for each  $z'_1 \in U^c(P_1^0, z_1)$ , and each  $p \in \Delta^{K-1}$  supporting  $U^c(R_2, \Omega - z'_1)$  at  $\Omega - z'_1$ ,  $a(L^p, z'_1) < a(R_2, \Omega - z'_1)$ , and that the parallel statement obtained by exchanging the roles of the two agents holds. Then for each  $e \in E$ ,  $(R, z) \in \mathcal{N} \langle S(\mathcal{D})^N, E^e, R^0 \rangle$  and  $z \in \mathcal{O} \langle S(\mathcal{D})^N, E^e, R^0 \rangle$ .

*Proof.* Let  $N$ ,  $\mathcal{D}$ ,  $R^0$ ,  $R$ , and  $z$  be as in the statement of the lemma. We claim that for each  $e \in E$ , each  $i \in N$ , and each  $(R'_i, z'_i) \in S(\mathcal{D})$ ,  $E_i^e(R, z) R_i^0 E_i^e(R_{-i}, R'_i, z_{-i}, z'_i)$ .

Suppose by contradiction that the above statement is false for say,  $e \in E$  and agent 1: there are  $(R'_1, z'_1) \in S(\mathcal{D})$  such that  $E_1^e(R'_1, R_2, z'_1, z_2) P_1^0 E_1^e(R, z)$ .

Thus,  $E_1^e(R'_1, R_2, z'_1, z_2) \neq E_1^e(R, z)$ . We claim that  $E^e(R'_1, R_2, z'_1, z_2) \neq (z'_1, z_2)$ . Suppose by contradiction that  $E^e(R'_1, R_2, z'_1, z_2) = (z'_1, z_2)$ . Since  $z \in Z$  and  $z_1 + z_2 = \Omega$ , then  $z'_1 \leq z_1$ . Thus,  $z_1 = E_1^e(R, z) R_1^0 E_1^e(R'_1, R_2, z'_1, z_2)$ . This is a contradiction.

Since  $E^e(R'_1, R_2, z'_1, z_2) \neq (z'_1, z_2)$ , then  $E^e(R'_1, R_2, z'_1, z_2) = e(R'_1, R_2)$ . Let  $\hat{z} \equiv e(R'_1, R_2)$ . By Lemma 1,  $\hat{z} \in P(R'_1, R_2)$ . Thus, there is  $p \in \Delta^{K-1}$  that supports  $U^c(R_2, \hat{z}_2)$  at  $\hat{z}_2$  and also supports  $U^c(R'_1, \hat{z}_1)$  at  $\hat{z}_1$ . Thus,  $a(R'_1, \hat{z}_1) \leq a(L^p, \hat{z}_1)$ . Moreover, by hypothesis,  $a(L^p, \hat{z}_1) < a(R_2, \hat{z}_2)$ . Thus,  $a(R'_1, \hat{z}_1) < a(R_2, \hat{z}_2)$  and  $\hat{z} \notin E(R'_1, R_2)$ . This is a contradiction.  $\square$

The following example shows that our assumption of normality of goods,  $\mathcal{D} \subseteq \mathcal{R}_{\mathcal{I}}$ , cannot be replaced in Theorem 2 by the assumption  $\mathcal{D} \subseteq \mathcal{R}$ . The constructed equilibrium is neither equal-division constrained Walrasian, nor weakly Pareto efficient for the true preferences. It also involves “non-linear” reports and thus, Lemma 5 is also tight.

**Example 1.** Let  $N \equiv \{1, 2\}$ ,  $\Omega \equiv (1, 1) \in \mathbb{R}_{++}^2$ , and  $\mathcal{D} \subseteq \mathcal{U}$  such that  $\mathcal{R} \setminus \mathcal{R}_{\mathcal{I}} \subseteq \mathcal{D}$ . We specify  $R^0 \in \mathcal{R}^{\{1,2\}}$  and construct  $(R, z) \in (\mathcal{R} \setminus \mathcal{R}_{\mathcal{I}} \times \mathbb{R}_+^2)^{\{1,2\}}$  such that  $(R, z) \in \mathcal{N}\langle S(\mathcal{D})^N, E, R^0 \rangle$ ,  $z \in \mathcal{O}\langle S(\mathcal{D})^N, E, R^0 \rangle$ ,  $U^c(I_1, z_1)$  and  $U^c(I_2, z_2)$  are not linear,  $z \notin W_{ed}^c(R^0)$ , and  $z \notin P^w(R^0)$ .

- Specifying true preferences. Let  $R_1^0 \in \mathcal{R}_{\mathcal{H}}$  be the homothetic preference whose upper contour set at  $(\frac{1}{5}, \frac{4}{5})$  is the intersection of the upper contour sets of  $L^{(8,1)}$  and  $L^{(4,3)}$  (Figure 3 (a)). Let  $R_2^0 \in \mathcal{R}_{\mathcal{H}}$  be the symmetric image of  $R_1^0$  with respect to the  $45^\circ$  line, i.e., for each  $\{x, y\} \subseteq \mathbb{R}_+^2$ ,  $x R_2^0 y$  if and only if  $(x_2, x_1) R_1^0 (y_2, y_1)$ .

- Specifying reported preferences. Let us specify  $R_1$  (Figure 3 (b)). Let  $Q$  be the union of seg  $[(\frac{2}{5}, 0), (\frac{1}{5}, \frac{4}{5})]$  and the half line with slope 4 starting at  $(\frac{1}{5}, \frac{4}{5})$ .<sup>25</sup> The indifference sets of  $R_1$  to the left of  $Q$  are linear with normal  $(2, 1)$ . The indifference sets of  $R_1$  to the right of  $Q$  are linear with normal  $(4, 3)$ .

The indifference sets of  $R_1$  can be alternatively described as follows. For each  $a \in [0, \frac{4}{5}]$ ,  $U(I_1, (0, a)) = U(I(L^{(\frac{4}{5}, \frac{2}{5})}), (0, a))$ . For each  $a \in ]\frac{4}{5}, \frac{6}{5}]$ ,

$$U(I_1, (0, a)) = \text{bro.seg} \left[ (0, a), \left( \frac{4}{5} - \frac{1}{2}a, 2a - \frac{8}{5} \right), \left( \frac{13}{6}a - \frac{4}{3}, 0 \right) \right].$$

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<sup>25</sup>The ray with slope 4 starting at  $(\frac{1}{5}, \frac{4}{5})$  is the set of points  $\{(\frac{1}{5}\lambda, \frac{4}{5}\lambda) \in \mathbb{R}_+^2 : \lambda \geq 1\}$ .

Let  $\hat{R}_1 \in \mathcal{R}_{\mathcal{H}}$  be the preference for which  $U(\hat{I}_1, (0, \frac{6}{5})) = I(R_1, (0, \frac{6}{5}))$ . For each  $a \in ]\frac{6}{5}, +\infty[$ ,  $U(I_1, (0, a)) = U(\hat{I}_1, (0, a))$ .

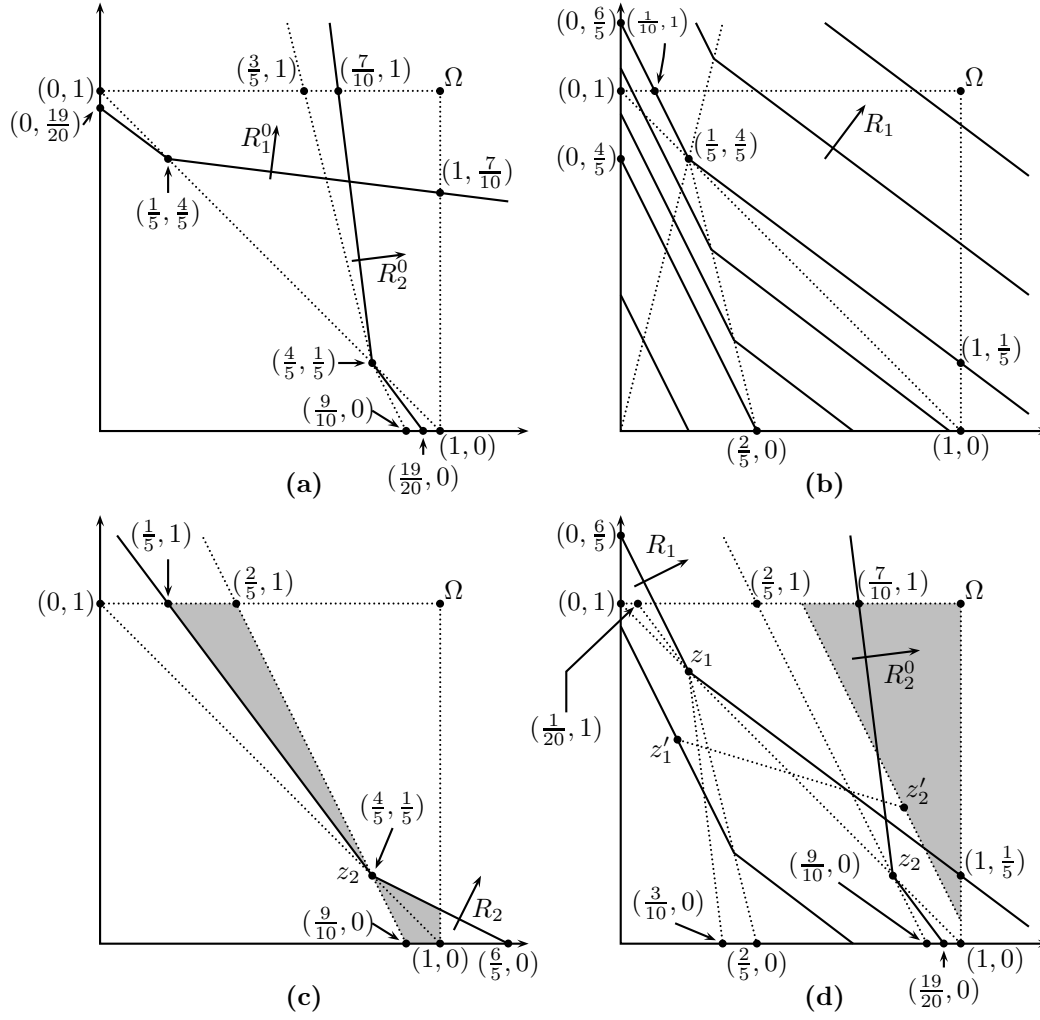


Figure 3: Example 1.

Now,  $R_2$  is the symmetric image of  $R_1$  with respect to the  $45^\circ$  line.

**Claim 1:**  $R \notin \mathcal{U}_{\mathcal{L}}$ . Indeed,  $(1, 1) \in \text{Supp}(R_1, (\frac{1}{5}, \frac{4}{5}))$ , and for each  $0 \leq \Delta \in \mathbb{R}_+^K$  such that  $(\frac{1}{5}, \frac{4}{5}) - \Delta \succeq 0$ ,  $(1, 1) \notin \text{Supp}(R_i, (0, 1) - \Delta)$ . Thus,

no quasi-strictly increasing path of maximizers of  $R_1$  at prices  $(1, 1)$  passes through  $(\frac{1}{5}, \frac{4}{5})$  (Figure 3 (b)).

• Identifying an equilibrium outcome,  $z$ . Let  $z_1 \equiv (\frac{1}{5}, \frac{4}{5})$  and  $z_2 \equiv (\frac{4}{5}, \frac{1}{5})$ .

**Claim 2:**  $(R, z) \in \mathcal{N}\langle S(\mathcal{D})^N, E^c, R^0 \rangle$ . Observe that  $z \in E(R)$ . Thus, by Lemma 9 and the symmetry of our construction, it is enough to show that for each  $R'_2 \in \mathcal{U}$  such that  $e_2(R_1, R'_2) P_2^0 z_2$ , and each  $p \in \Delta^{K-1}$  supporting  $U^c(R_1, e_1(R_1, R'_2))$  at  $e_1(R_1, R'_2)$ , we have  $a(L^p, e_2(R_1, R'_2)) < a(R_1, e_1(R_1, R'_2))$ . Let  $R'_2 \in \mathcal{U}$ ,  $z' \equiv e(R_1, R'_2)$ , and  $A$  the “residual complement” of the constrained upper contour set of  $R_2^0$  at  $z_2$ , i.e.,  $A \equiv \{z'_1 \leq \Omega : \Omega - z'_1 P_2^0, z_2\}$ . It is easy to see that

$$A = \text{con.hull} \left\{ 0, \left( \frac{3}{10}, 0 \right), z_1, \left( \frac{1}{20}, 1 \right), (0, 1) \right\} \setminus \text{bro.seg} \left[ \left( \frac{3}{10}, 0 \right), z_1, \left( \frac{1}{20}, 1 \right) \right].$$

There are three cases.

**Case 1:**  $z'_1 \in A \setminus \text{bro.seg}[(0, 1), 0, (\frac{3}{10}, 0)]$  (Figure 3 (d)). Then, the unique  $p \in \Delta^1$  (up to positive scale multiplication) supporting  $U^c(R_1, z'_1)$  at  $z'_1$  is  $(2, 1)$ . Observe that  $a(L^{(2,1)}, z_2) < a(R_2, z_2)$ , i.e., of the two shaded sets forming a bow-tie in Figure 3 (c), the measure of the upper set is greater than the measure of the lower one. Since  $a(L^{(2,1)}, z'_2) \leq a(L^{(2,1)}, z_2) < a(R_2, z_2) = a(R_1, z_1) \leq a(R_1, z'_1)$ , then  $a(L^{(2,1)}, z'_2) < a(R_1, z'_1)$ .

**Case 2:**  $z'_1 \in \text{seg}[(0, 1), 0]$ . Then, for each  $p \in \Delta^1$  supporting  $U^c(R_1, z'_1)$  at  $z'_1$ ,  $\frac{p_2}{p_1} \leq \frac{1}{2}$ . Since  $z'_2 \in \text{seg}[(1, 0), \Omega]$ , then  $a(L^p, z'_2) \leq a(L^{(2,1)}, z'_1)$  and the argument in Case 1 shows that  $a(L^p, z'_2) < a(R_1, z'_1)$ .

**Case 3:**  $z'_1 \in \text{seg}[0, (\frac{3}{10}, 0)]$ . Then, for each  $p \in \Delta^1$  supporting  $U^c(R_1, z'_1)$  at  $z'_1$ ,  $\frac{p_2}{p_1} \geq \frac{1}{2}$ . Since  $z'_2 \in \text{seg}[(\frac{7}{10}, 1), \Omega]$ , then  $a(L^p, z'_2) < a(R_1, z_1)$ . Now, since  $a(R_1, z_1) < a(R_1, z'_1)$ , then  $a(L^p, z'_2) < a(R_1, z'_1)$ .

**Concluding:** Observe that there is no price vector  $p$  such that for each  $z''_1 \in U^c(P_1^0, z_1)$  and each  $z''_2 \in U^c(P_2^0, z_2)$ ,  $p \cdot z''_1 > p \cdot \frac{\Omega}{2}$  and  $p \cdot z''_2 > p \cdot \frac{\Omega}{2}$ . Thus,  $z \notin W_{ed}^c(R^0)$ . Finally, it is easily seen that  $z \notin P^w(R^0)$ .  $\square$

In light of the example above one can ask whether the assumption of normality of goods in Theorem 2,  $\mathcal{D} \subseteq \mathcal{R}_{\mathcal{I}}$ , could be replaced by the assumption of smoothness of preferences,  $\mathcal{D} \subseteq \mathcal{R}_{\mathcal{S}}$ . The following example shows it cannot. The constructed equilibrium is neither equal-division constrained Walrasian, nor weakly Pareto efficient for the true preferences.

Let  $F$  be a solution. A selector  $f \in F$  is an **equal-division selector**, if for each  $R \in \mathcal{R}^N$  such that  $z_{ed} \equiv \frac{1}{|N|}(\Omega, \dots, \Omega) \in F(R)$ , we have  $f(R) = z_{ed}$ .

**Example 2.** Let  $N \equiv \{1, 2\}$ ,  $\lambda > 4\sqrt{2}$ ,  $\mathcal{D} \subseteq \mathcal{U}_S$  such that  $\mathcal{R}_S \setminus \mathcal{R}_I \subseteq \mathcal{D}$ , and  $\Omega \equiv (\lambda, \lambda) \in \mathbb{R}_{++}^2$ . We specify  $R^0 \in \mathcal{R}_S^N$  and construct  $(R, z) \in (\mathcal{R}_S \setminus \mathcal{R}_I \times \mathbb{R}_+^2)^{\{1, 2\}}$  such that for each equal-division selector  $e \in E$ ,  $(R, z) \in \mathcal{N}\langle S(\mathcal{D})^N, E^e, R^0 \rangle$ ,  $U^c(I_1, z_1)$  and  $U^c(I_2, z_2)$  are linear, and  $z \notin W_{ed}^c(R^0)$ .

- Specifying true preferences (Figure 4 (a)). Let  $R_1^0 \in (\mathcal{R}_H \cap \mathcal{R}_S)^N$  be such that  $U(R_1^0, (\lambda, 0)) \cap \{(x_1, x_2) \in \mathbb{R}_+^K : x_1 + x_2 < \lambda\} \neq \emptyset$  and for each  $x \in U(R_1^0, (\lambda, 0)) \cap \{(x_1, x_2) \in \mathbb{R}_+^K : x_1 + x_2 \leq \lambda\}$ ,  $\|(\lambda, 0) - x\| < \frac{1}{2}$ . Let  $R_2^0 \equiv L^{(1,1)} \in (\mathcal{R}_H \cap \mathcal{R}_S)^N$ .

- Specifying reported preferences (Figure 4 (b)). We construct  $\hat{R} \in \mathcal{R}^N$  and  $z \in Z$ , such that  $(\hat{R}, z) = \mathcal{N}\langle S(\mathcal{D})^N, E^e, R^0 \rangle$  and  $z = \mathcal{O}\langle S(\mathcal{D})^N, E^e, R^0 \rangle$ ; later we smooth out  $\hat{R}$  and construct  $R \in \mathcal{R}_S^N$  such that  $(R, z) = \mathcal{N}\langle S(\mathcal{D})^N, E^e, R^0 \rangle$  and  $z \in \mathcal{O}\langle S(\mathcal{D})^N, E^e, R^0 \rangle$ .

We first specify  $\hat{R}_1$  (Figure 4 (b)).

(i) For each  $a \in [0, \lambda] \subseteq \mathbb{R}_+$ ,  $U(\hat{I}_1, (0, a)) \equiv U(I(L^{(1,1)}), (0, 1))$ . Let  $\alpha \equiv (\frac{\sqrt{2}}{2}, \lambda - \frac{\sqrt{2}}{2})$  and  $\beta \equiv (\sqrt{2}, \lambda - \sqrt{2})$ ; note that  $\|(0, \lambda) - \alpha\| = 1$  and  $\|\alpha - \beta\| = 1$ .

(ii) For each  $a \in ]\lambda, \frac{3}{2}\lambda] \subseteq \mathbb{R}_+$  let  $t(a) \equiv \text{seg}[(0, a), \beta] \cap \text{ray}\{\alpha, 1\}$  and  $U(\hat{I}_1, (0, a)) \equiv \text{bro.seg}[(0, a), t(a), (t_2(a) - \frac{\sqrt{2}}{2}, 0)]$ . Observe that  $\text{seg}[t(a), (t_2(a) - \frac{\sqrt{2}}{2}, 0)]$  has slope  $-1$  and for each  $a \in ]\lambda, \frac{3}{2}\lambda]$ , the line that passes through  $(0, a)$  and whose slope is that of  $\text{seg}[(0, a), t(a)]$ , namely,  $-\frac{a-t_2(a)}{t_1}$ , intersects at  $\beta$  the line that passes through  $(0, \lambda)$  with slope  $-1$ .

Let  $\tilde{R}_1 \in \mathcal{R}_H$  be the homothetic preference for which  $U(\tilde{I}_1, (0, \frac{3}{2}\lambda)) \equiv U(\hat{I}_1, (0, \frac{3}{2}\lambda))$ . For each  $a \in ]\frac{3}{2}\lambda, +\infty[ \subseteq \mathbb{R}_+$ , let  $U(\hat{I}_1, (0, a)) \equiv U(\tilde{I}_1, (0, a))$ . Finally, let  $\hat{R}_2 \equiv \hat{R}_1$ .

**Claim 1:**  $\hat{R}_1 \notin \mathcal{R}_I$ . Indeed,  $(0, \lambda)$  is a maximizer of  $\hat{R}_1$  at prices  $(1, 1)$ , and for each  $0 \preceq \Delta \in \mathbb{R}_+^K$  such that  $\|(0, \lambda) - \Delta\| < \frac{1}{2}$ ,  $(0, \lambda) + \Delta$  is not a maximizer of  $\hat{R}_1$  at prices  $(1, 1)$ . Thus, no quasi-strictly increasing path of maximizers of  $\hat{R}_1$  at prices  $(1, 1)$  passes through  $(0, \lambda)$  (Figure 4 (b)).

Let  $z_1 \equiv (\lambda, 0)$  and  $z_2 \equiv (0, \lambda)$ .

**Claim 2:**  $(\hat{R}, z) \in \mathcal{N}\langle S(\mathcal{R})^N, E^e, R^0 \rangle$ . Note that  $z \in E(\hat{R})$ . Thus, by Lemma 9, it is enough to show that for each  $R'_1 \in \mathcal{R}$  such that  $e_1(R'_1, \hat{R}_2) P_1^0 z_1$ , and each  $p \in \Delta^1$  supporting  $U^c(\hat{R}_2, e_2(R'_1, \hat{R}_2))$  at  $e_2(R'_1, \hat{R}_2)$ ,  $a(L^p, e_1(R'_1, \hat{R}_2)) < a(R_2, e_2(R'_1, \hat{R}_2))$ , and the parallel statement obtained by exchanging the roles of the two agents hold.

We prove the first statement of Claim 2. Let  $R'_1 \in \mathcal{R}$  and  $p \in \Delta^1$  be as specified above. Let  $\tilde{z} \equiv e(R'_1, \hat{R}_2)$ . We prove that  $a(L^p, e_1(R'_1, \hat{R}_2)) <$

$a(R_2, e_2(R'_1, \hat{R}_2))$ . There are five cases.

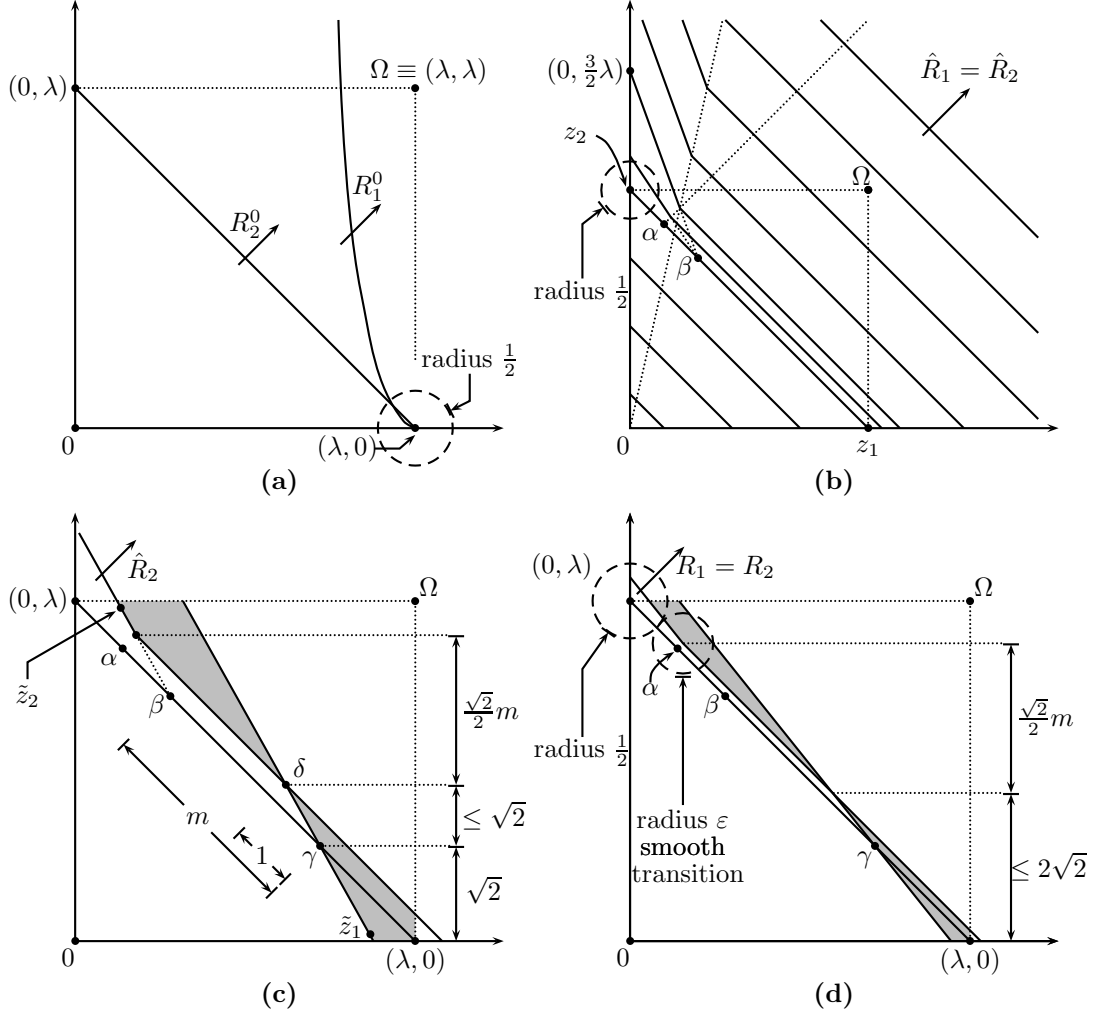


Figure 4: Example 2. In order to help visualize our geometrical argument, we have exaggerated the distance between  $\tilde{z}_1$  and  $(\lambda, 0)$  in Panel (c) with respect to preferences shown in Panel (a).

**Case 1:**  $\tilde{z}_1^1 + \tilde{z}_1^2 > \lambda$  and  $\tilde{z}_1 \ll \Omega$ . Then, the unique  $p$  (up to a positive scale transformation) supporting  $U^c(\hat{R}_2, \tilde{z}_2)$  at  $\tilde{z}_2$  is  $(1, 1)$ . Since  $\tilde{z}_2^1 + \tilde{z}_2^2 < \lambda$ , then  $a(L^{(1,1)}, \tilde{z}_1) < a(L^{(1,1)}, \tilde{z}_2) = a(R_2, \tilde{z}_2)$ .

**Case 2:**  $\tilde{z}_1^1 + \tilde{z}_1^2 > \lambda$  and  $\tilde{z}_1^1 = \lambda$ . The claim is clearly true if  $\tilde{z}_1 = \Omega$ . Suppose now that  $\tilde{z}_1 \neq \Omega$ . Thus,  $\tilde{z}_1^2 < \lambda$  and for each  $p$  supporting  $U^c(\hat{R}_2, \tilde{z}_2)$  at  $\tilde{z}_2$ , we have  $p_1 \leq p_2$ . Thus,  $U^c(L^p, \tilde{z}_1) \subseteq U^c(L^{(1,1)}, \tilde{z}_1)$  and  $a(L^p, \tilde{z}_1) < a(L^{(1,1)}, \tilde{z}_2) = a(R_2, \tilde{z}_2)$ .

**Case 3:**  $\tilde{z}_1^1 + \tilde{z}_1^2 > \lambda$ ,  $\tilde{z}_1^2 = \lambda$ , and  $\tilde{z}_1 \neq \Omega$ . A symmetric argument to the one in Case 2 shows that for each  $p$  supporting  $U^c(\hat{R}_2, \tilde{z}_2)$  at  $\tilde{z}_2$ ,  $a(L^p, \tilde{z}_1) < a(L^{(1,1)}, \tilde{z}_2) = a(R_2, \tilde{z}_2)$ .

**Case 4:**  $\tilde{z}_1^1 + \tilde{z}_1^2 = \lambda$ . Then,  $\tilde{z}_2^1 + \tilde{z}_2^1 = \lambda$ . Moreover, since  $\tilde{z}_1 P_1^0 z_1$ , then  $\tilde{z}_2^2 < \lambda$ . Thus, the unique  $p$  (up to positive scale transformations) supporting  $U^c(\hat{R}_2, \tilde{z}_2)$  at  $\tilde{z}_2$  is  $(1, 1)$ . Since  $z_1 P_1^0 \frac{\Omega}{2}$  and  $e$  is an equal-division selector, then  $(\frac{\Omega}{2}, \frac{\Omega}{2}) \notin E(R'_1, \hat{R}_2)$ , for otherwise  $e(R'_1, \hat{R}_2) = (\frac{\Omega}{2}, \frac{\Omega}{2})$ . Thus,  $p$  supports  $U^c(R'_1, \tilde{z}_1)$  at  $\tilde{z}_1$ ,  $a(R'_1, \tilde{z}_1) < a(L^p, \tilde{z}_1) = a(\hat{R}_2, \tilde{z}_2)$ , and  $\tilde{z} \notin E(R'_1, \hat{R}_2)$ . Thus, this case cannot occur.

**Case 5:**  $\tilde{z}_1^1 + \tilde{z}_1^2 < \lambda$  (Figure 4 (c)). Then,  $\|\tilde{z}_1 - z_1\| < \frac{1}{2}$  and the unique  $p$  (up to positive scale transformations) supporting  $U^c(R_2, \tilde{z}_2)$  at  $\tilde{z}_2$  is  $(p_1, p_2)$  where  $p_1 = \tilde{z}_2^2 - \lambda - \sqrt{2}$  and  $p_2 = \tilde{z}_2^1 - \sqrt{2}$ , i.e., the normal vector to the line that passes through  $\tilde{z}_2$  and  $\beta$  (Figure 4 (b)). Let  $\gamma \equiv U(I(L^p), \tilde{z}_1) \cap \text{seg}[(0, \lambda), (\lambda, 0)]$  and  $\delta = (\delta^1, \delta^2) \equiv U(I(L^p), \tilde{z}_1) \cap U(I_2, \tilde{z}_2)$ . Note that  $\|\gamma - (\lambda, 0)\| = 2$  and  $\delta^2 \leq 2\sqrt{2}$ . We claim that  $a(L^p, \tilde{z}_1) < a(R_2, \tilde{z}_2)$ . This is equivalent to  $\mu(U^c(R_2, \tilde{z}_2) \setminus U^c(L^p, \tilde{z}_1)) > \mu(U^c(L^p, \tilde{z}_1) \setminus U^c(R_2, \tilde{z}_2))$ , i.e., from the measures of the two shaded sets forming a bow-tie in Figure 4 (c), the measure of the upper set is greater than the measure of the lower one. Indeed, let  $m \equiv \sqrt{2}\lambda - 4$ . Then, the upper set is a proper superset of a congruent set to the lower one if  $\frac{\sqrt{2}}{2}m > 2\sqrt{2}$ . Since  $\lambda > 4\sqrt{2}$ , this last inequality holds.

We now prove the second statement of Claim 2, i.e., for each  $R'_2 \in \mathcal{R}$  such that  $e_2(\hat{R}_1, R'_2) P_2^0 z_2$ , and each  $p \in \Delta^1$  supporting  $U^c(R_1, e_1(\hat{R}_1, R'_2))$  at  $e_1(\hat{R}_1, R'_2)$ ,  $a(L^p, e_2(\hat{R}_1, R'_2)) < a(R_1, e_2(\hat{R}_1, R'_2))$ . A symmetric argument to Cases 1, 2, and 3 above proves this statement (there is no counterpart of Cases 4 and 5.)

Now, we smooth out  $\hat{R}$  and construct  $R \in \mathcal{R}_S^N$  such that  $(R, z) = \mathcal{N}\langle S(\mathcal{R}_S)^N, E^e, R^0 \rangle$  and  $z = \mathcal{O}\langle S(\mathcal{R}_S)^N, E^e, R^0 \rangle$ . The only points at which the indifference sets of  $\hat{R}_1$  and  $\hat{R}_2$  have multiple supporting lines are  $\{t(a) : \lambda < a < \frac{3}{2}\lambda\}$  and the half line starting at  $t(\frac{3}{2}\lambda)$  with direction  $t(\frac{3}{2}\lambda)$ . For  $\varepsilon > 0$  small enough, there are smooth preferences, say  $R_1$  and  $R_2$ , whose indifference sets “coincide” with the indifference sets of  $\hat{R}_1$  and  $\hat{R}_2$  outside each open ball with radius  $\varepsilon$  and centered at each of these “kinks” (Fig-

ure 4 (d)). Again, if  $\varepsilon$  is small enough, the same argument that shows that  $(\hat{R}, z) = \mathcal{N}\langle S(\mathcal{R})^N, E^e, R^0 \rangle$  and  $z = \mathcal{O}\langle S(\mathcal{R})^N, E^e, R^0 \rangle$  can be used to show that  $(R, z) = \mathcal{N}\langle S(\mathcal{R}_S)^N, E^e, R^0 \rangle$  and  $z = \mathcal{O}\langle S(\mathcal{R}_S)^N, E^e, R^0 \rangle$ .<sup>26</sup>

**Concluding:** Observe that there is no price vector  $p$  such that for each  $z_1'' \in U^c(P_1^0, z_1)$  and each  $z_2'' \in U^c(P_2^0, z_2)$ ,  $p \cdot z_1'' > p \cdot \frac{\Omega}{2}$  and  $p \cdot z_2'' > p \cdot \frac{\Omega}{2}$ . Thus,  $z \notin W_{ed}^c(R^0)$ .  $\square$

## 6.2. Monotonicity of Preferences

In this section we discuss the manipulation of the equal-sacrifice solution when the admissible domain of preferences is  $\mathcal{U}_{\mathcal{I}}$ .

The following example shows that in Lemma 5, the assumption  $R^0 \in \mathcal{R}_{\mathcal{I}}$  cannot be replaced by  $R^0 \in \mathcal{U}_{\mathcal{I}}$ . That is, when true preferences are not strictly monotone in the interior of the consumption space, there may be equilibria of the manipulation of  $E$  in which an agent does not report a linear indifference set through her equilibrium consumption within the feasible box.

**Example 3.** Let  $N \equiv \{1, 2\}$  and  $\Omega \equiv (1, 1) \in \mathbb{R}_{++}^2$ . We specify  $R^0 \in \mathcal{U}^N$  and construct a profile  $(R, z) \in (\mathcal{U}_{\mathcal{I}} \times \mathbb{R}_+^2)^N$  such that  $R \in (\mathcal{U}_{\mathcal{H}} \setminus \mathcal{R})^N$ ,  $(R, z) \in \mathcal{N}\langle S(\mathcal{U})^N, E, R^0 \rangle$ ,  $z \in \mathcal{O}\langle S(\mathcal{U})^N, E, R^0 \rangle$ , and  $U^c(I_2, z_2)$  is not linear.

- Specifying true preferences (Figure 5 (a)). Let  $q \in \mathbb{R}_{++}^2$  be such that  $q_1 > q_2$  and  $R_1^0 \equiv L^{(q_1+1, q_2)}$ . Let  $R_2^0 \in \mathcal{U}$  be the homothetic preference for which  $U(I_2^0, (0, 1)) \equiv \{(x, 1) \in \mathbb{R}_+^2 : x \in \mathbb{R}_+\}$ .

- Specifying reported preferences. Let  $R_1 \equiv L^q$ . Let  $\alpha \equiv \text{seg}[0, \Omega] \cap U(I_1, (0, 1))$ . Let  $R_2 \in \mathcal{U}$  be the homothetic preference such that  $U(I_2, (1, 0)) \equiv \text{bro.seg}[(0, 1), \alpha, (1, 0)]$ .

- Identifying an equilibrium outcome,  $z$  (Figure 5 (b)). The function  $x \in \mathbb{R}_+ \mapsto a(R_1, (x, 0))$  is continuous and strictly decreasing on  $[\frac{1}{2}, 1]$ ; also, the function  $x \in \mathbb{R}_+ \mapsto a(R_2, \Omega - (x, 0))$  is continuous and strictly increasing on  $[\frac{1}{2}, 1]$ . Moreover,  $a(R_1, (\frac{1}{2}, 0)) > a(R_2, \Omega - (\frac{1}{2}, 0))$  and  $a(R_1, (1, 0)) < a(R_2, \Omega - (1, 0))$ . Thus, by the Intermediate Value Theorem, there is  $a \in [\frac{1}{2}, 1]$  such that  $a(R_1, (a, 0)) = a(R_2, \Omega - (a, 0))$ . In Figure 5 (b), this is equivalent

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<sup>26</sup>Formally, since  $\lambda > 4\sqrt{2}$ , there is  $R_2 \in \mathcal{R}_{\mathcal{S}}$  such that: (i) for each  $a \in [0, \lambda] \subseteq \mathbb{R}_+$ ,  $U(I_2, (0, a)) \equiv U(\hat{I}_2, (0, a))$ ; (ii) there is  $0 < \varepsilon < \frac{\sqrt{2}}{2}m - 2\sqrt{2}$  such that for each  $a \in ]\lambda, \frac{3}{2}\lambda] \subseteq \mathbb{R}_+$ ,  $U(I_2, (0, a)) \cap \{y \in \mathbb{R}_+^K : \|y - t(a)\| < \varepsilon\} \equiv U(\hat{I}_2, (0, a)) \cap \{y \in \mathbb{R}_+^K : \|y - t(a)\| < \varepsilon\}$ ; and (iii) for each  $\tilde{z}_2 \in \mathbb{R}_+^K$  such that  $\|(0, \lambda) - \tilde{z}_2\| < \frac{1}{2}$  and  $\tilde{z}_2^1 + \tilde{z}_2^2 > \lambda$ , the positive normal vector to the the line that passes through  $\tilde{z}_2$  and  $\beta$  supports  $U^c(R_2, \tilde{z}_2)$  at  $\tilde{z}_2$ .

to the equality of the measures of the two shaded areas forming a bow-tie. Let  $z_1 \equiv (a, 0)$  and  $z_2 \equiv \Omega - z_1$ .

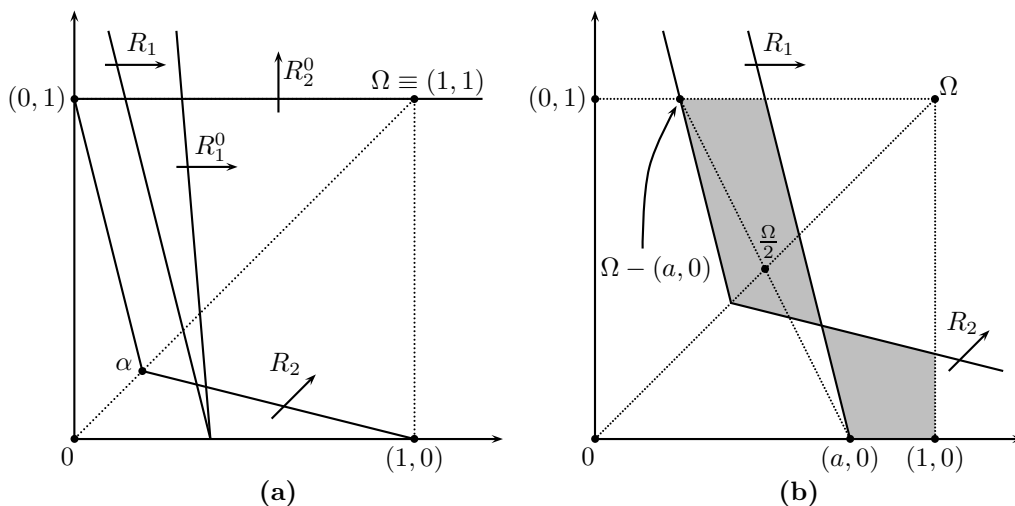


Figure 5: Example 3.

By following an argument similar to that in Examples 1 and 2, one can easily see that for each  $e \in E$ ,  $(R, z) \in \mathcal{N}\langle S(\mathcal{U})^N, E^e, R^0 \rangle$ .

Clearly,  $U^c(I_2, z_2)$  is not linear. Thus, even though an agent reports a “non linear” constrained indifference set at the equilibrium consumption, the outcome is a Constrained Walrasian allocation from equal division: the price vector

$$\frac{1}{(z_2^2 - z_1^2) + (z_1^1 - z_2^1)}(z_2^2 - z_1^2, z_1^1 - z_2^1),$$

supports  $z$  as an element of  $W_{ed}^c(R^0)$ .  $\square$

Example 3 does not close our interest in the study of the manipulation of  $E$  on  $\mathcal{U}_{\mathcal{I}}$ . It only reveals that Lemma 5 does not hold on  $\mathcal{U}_{\mathcal{I}}$ . In a companion paper Velez (2011b) investigates, and it turns out to be an issue of independent interest, the extent to which our results generalize when true preferences and reports are elements of  $\mathcal{U}_{\mathcal{I}}$ . It turns out that our results are robust: (i) for each  $R^0 \in \mathcal{U}^N$  and each  $e \in E$ ,  $W_{ed}^c(R^0) \subseteq$

$\mathcal{O}\langle S(\mathcal{U})^N, E^e, R^0 \rangle \subseteq P^w(R^0)$  and (ii) when  $R^0 \in \mathcal{R}^N$  or there are two commodities (i.e.,  $K = 2$ ), then for each  $e \in E$ ,  $\mathcal{O}\langle S(\mathcal{U}_{\mathcal{I}})^N, E^e, R^0 \rangle = W_{ed}^c(R^0)$  and  $\mathcal{O}^*\langle S(\mathcal{U}_{\mathcal{I}})^N, E^e, R^0 \rangle = W_{ed}^c(R^0) \cap P(R^0)$ . It is an open question to characterize the outcomes from individual, or coalitional, manipulation of  $E$  when there are at least three commodities, admissible preferences are  $\mathcal{U}_{\mathcal{I}}$ , and true preferences belong to  $\mathcal{U}_{\mathcal{I}}^N \setminus \mathcal{R}_{\mathcal{I}}^N$ .

### 6.3. Implementation

Our results have implications for what is called “implementation” – for introductions and surveys, see Corchón (1996), Jackson (2001), and Maskin and Sjöström (2002). A “game form” consists of a profile of strategy spaces, one for each agent, and an outcome function, a function that associates with each preference profile an allocation. Once a preference profile is given, we have a “game.” A game form “implements a solution” if for each preference profile, the set of equilibrium allocations of the resulting game coincides with the set of allocations that the solution would select for this profile.

The study of the manipulation of a solution is relevant for implementation theory. Let  $F$  and  $G$  be two solutions,  $\mathcal{D} \subseteq \mathcal{U}$ , and  $\mathcal{D}' \subseteq \mathcal{U}$ . The pair  $\langle \mathbf{S}(\mathcal{D}')^N, \mathbf{G} \rangle$  implements  $F$  in Nash equilibria on  $\mathcal{D}$  if for each  $R^0 \in \mathcal{D}$ ,  $\mathcal{O}\langle S(\mathcal{D}'), G, R^0 \rangle = F(R^0)$ ; the pair  $\langle \mathbf{S}(\mathcal{D}')^N, \mathbf{G} \rangle$  implements  $F$  in strong Nash equilibria on  $\mathcal{D}$  if for each  $R^0 \in \mathcal{D}$ ,  $\mathcal{O}^*\langle S(\mathcal{D}'), G, R^0 \rangle = F(R^0)$ .<sup>27</sup>

Theorem 2 has implications for the implementation of the equal-division constrained Walrasian solution on  $\mathcal{R}$ .

**Proposition 1.** Let  $\mathcal{D} \subseteq \mathcal{R}_{\mathcal{I}}$  be such that  $\mathcal{R}_{\mathcal{H}} \subseteq \mathcal{D}$ . Then  $\langle \mathcal{D}, E \rangle$  implements  $W_{ed}^c$  in Nash and strong Nash equilibria on  $\mathcal{R}$ .

*Proof.* Let  $e \in E$ . Suppose  $\mathcal{R}_{\mathcal{H}} \subseteq \mathcal{D} \subseteq \mathcal{R}_{\mathcal{I}}$ . Thus,  $\mathcal{D} \supseteq \mathcal{R}_{\mathcal{L}}$ , and then from Lemmas 7 and 8,  $W_{ed}^c(R^0) \subseteq \mathcal{O}^*\langle S(\mathcal{D})^N, E^e, R^0 \rangle \subseteq \mathcal{O}\langle S(\mathcal{D})^N, E^e, R^0 \rangle \subseteq W_{ed}^c(R^0)$ . Thus,  $\mathcal{O}^*\langle S(\mathcal{D})^N, E^e, R^0 \rangle = \mathcal{O}\langle S(\mathcal{D})^N, E^e, R^0 \rangle = W_{ed}^c(R^0)$ .  $\square$

Let us remark that this result implies that each game form associated with  $E$  doubly implements  $W_{ed}^c$  in Nash and strong Nash equilibria, even if strategy spaces are restricted to be the space of homothetic preferences.

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<sup>27</sup>The equal-sacrifice solution is not Nash implementable, for it is not monotonic in the sense of Maskin (1999). Nevertheless it is implementable in subgame perfect Nash equilibria (Maniquet, 2003).

## Appendix

**Proof of Theorem 1.** Let  $R \in \mathcal{U}$  and  $\Theta \equiv \prod_{t=1}^K \Omega^t$ . For each  $i \in N$ , the function  $u_i$ , defined by  $x_i \in \mathbb{R}_+^K \mapsto u_i(x_i) \equiv \Theta - a(R_i, x_i)$ , is a continuous representation of  $R_i$  on  $\text{rec}\{0, \Omega\}$ . Let  $Z_{\leq} \equiv \{z \equiv (z_i)_{i \in N} \in \prod_N \mathbb{R}_+^K : \sum_{i \in N} z_i \leq \Omega\}$  and  $\Phi(R) \equiv \{\nu \in \mathbb{R}_+ : \text{there is } z \in Z_{\leq} \text{ s.t. for each } i \in N, a(R_i, z_i) = \nu\}$ . Since for each  $i \in N$ ,  $a(R_i, 0) = \Theta$ , then  $\Theta \in \Phi(R)$ . Thus,  $\Phi(R)$  is non-empty. Clearly,  $\Phi(R) \equiv \{\nu \in \mathbb{R}_+ : \text{there is } z \in Z_{\leq} \text{ s.t. for each } i \in N, \Theta - u_i(z_i) = \nu\}$ . Continuity of  $u$  implies that  $\Phi(R)$  is closed. Since  $\Phi(R)$  is bounded, then it is compact. Let  $\nu^* \equiv \min \Phi(R)$ . Since  $\nu^* \in \Phi(R)$ , then there is  $z \in Z_{\leq}$  such that for each  $i \in N$ ,  $a(R_i, z_i) = \nu^*$ . Moreover, for each pair  $\{i, j\} \subseteq N$ ,  $a(R_i, z_i) = a(R_i, z_j)$ . Let  $z' \in Z_{\leq}$  be such that for each pair  $\{i, j\} \subseteq N$ ,  $a(R_i, z'_i) = a(R_i, z'_j)$ . Then, the common value of  $a(R_i, z'_i)$  for  $i \in N$ , is in  $\Phi(R)$ . Thus, for each  $i \in N$ ,  $a(R_i, z_i) = \nu^* \leq a(R_i, z'_i)$ . In particular, for each  $z' \in \Psi(R)$  and each  $i \in N$ ,  $a(R_i, z_i) = \nu^* \leq a(R_i, z'_i)$ .

Let  $\bar{z} = \sum_{i \in N} z_i$ . Let  $i \in N$  and  $\alpha_i \equiv \max\{\alpha \in [0, 1] : z_i + \alpha(\Omega - \bar{z})I_i z_i\}$ . Since preferences are continuous, then each  $\alpha_i$  is well-defined. We claim that  $\sum_{i \in N} \alpha_i \geq 1$ . Suppose by contradiction that  $\sum_{i \in N} \alpha_i < 1$ . Since preferences are monotone, there is  $(\alpha'_i)_{i \in N} \gg (\alpha_i)_{i \in N}$  such that  $\sum_{i \in N} \alpha'_i \leq 1$  and for each  $i \in N$ ,  $z_i + \alpha'_i(\Omega - \bar{z})P_i z_i$ . Since preferences are continuous, we can suppose without loss of generality that for each pair  $\{i, j\} \subseteq N$ ,  $a(R_i, z_i + \alpha'_i(\Omega - \bar{z})) = a(R_j, z_j + \alpha'_j(\Omega - \bar{z}))$ . For each  $i \in N$ , let  $\tilde{z}_i \equiv z_i + \alpha'_i(\Omega - \bar{z})$ . Then,  $\tilde{z} \in Z_{\leq}$ . Since the common value  $a(R_i, \tilde{z}_i)$  for  $i \in N$  is less than  $\nu^*$ , then  $\min \Phi(R) < \nu^*$ . This is a contradiction.

Let  $(\beta_i)_{i \in N} \in \mathbb{R}_+^N$  be such  $\sum_{i \in N} \beta_i = 1$  and  $(\beta_i)_{i \in N} \leq (\alpha_i)_{i \in N}$ . For each  $i \in N$ , let  $z'_i \equiv z_i + \beta_i(\Omega - \bar{z})$ . Let  $i \in N$ . Since,  $z'_i I_i z_i$ , then  $a(R_i, z'_i) = \nu^*$ . Since  $z' \in Z$ , then  $z' \in E(R)$ .  $\square$

**Proof of Lemma 1.** Let  $R \in \mathcal{U}^N$  and  $z \in E(R)$ .

**(i-a)  $z \in P^w(R)$ .** Suppose by contradiction that there is  $z' \in Z$  such that for each  $i \in N$ ,  $z'_i P_i z_i$ . Let  $\nu \equiv \max_{j \in N} a(R_j, z'_j)$ . Thus, for each  $i \in N$ ,  $a(R_i, z'_i) \leq \nu < a(R_i, z_i)$ . Let  $Z_{\leq}$  and  $\Phi(R)$  be defined as in the proof of Theorem 1. Since for each  $i \in N$ ,  $a(R_i, 0) \geq \nu$  and preferences are monotone and continuous, then by the Intermediate Value Theorem, there is  $z'' \in Z_{\leq}$  such that for each  $i \in N$ ,  $a(R_i, z''_i) = \nu$ . Thus,  $\min \Phi(R) \leq \nu$ . By the same argument as in the proof of Theorem 1, the common sacrifice at each allocation in  $E(R)$  is at most  $\nu$ . Thus,  $z \notin E(R)$ . This is a contradiction.

**(i-b) for each  $i \in N$ ,  $z_i P_i 0$ .** Let  $m \equiv \max_{i \in N} a(R_i, \frac{1}{n}\Omega)$ . Since preferences are monotone,  $m > 0$ . Since preferences are continuous, then by

the Intermediate Value Theorem, there is  $z' \in Z_{\leq}$  such that for each  $i \in N$ ,  $a(R_i, z'_i) = m$ . Thus, for each  $i \in N$ ,  $z_i R_i z'_i$ .

**(ii) If  $|N| = 2$ , then,  $E(R) \subseteq P(R)$ .** Suppose by contradiction that there is  $z \in E(R)$  such that  $z \notin P(R)$ . Then, there are  $z' \in Z$ ,  $i \in N$  and  $j \in N$  such that  $z'_i R_i z_i$  and  $z'_j P_j z_j$ . Recall that by feasibility,  $z'_i + z'_j = \Omega$ . We claim that  $\Omega P_i z'_i$ . Suppose by contradiction that  $z'_i I_i \Omega$ . Since  $z \in P^w(R)$ , then  $z'_i I_i z_i$ . Then,  $a(R_i, z'_i) = 0$  and thus,  $a(R_i, z_i) = 0$ . Since  $z \in E(R)$ , then  $a(R_j, z_j) = 0$ . But, since  $z'_j \in U^c(P_j, z_j)$ , then  $a(R_j, z_j) > 0$ . This is a contradiction. Now, since preferences are continuous, then there is  $\alpha \in (0, 1)$  such that  $(1 - \alpha)z'_j P_j z_j$ . Since  $z'_i + z'_j = \Omega$ , then  $z'_i + \alpha z'_j = (1 - \alpha)z'_i + \alpha \Omega$  (we are able to make this claim because  $|N| = 2$ ). Since preferences are monotone, convex, and continuous and  $\Omega P_i z'_i$ , then  $(z'_i + \alpha z'_j) P_i z'_i$ . Let  $z''$  be the allocation defined by:  $z''_i \equiv z'_i + \alpha z'_j$  and  $z''_j \equiv (1 - \alpha)z'_j$ . Since  $z' \in Z$ , then  $z'' \in Z$ . Since  $z''_i P_i z_i$  and  $z''_j P_j z_j$ , then  $z \notin P^w(R)$ . This is a contradiction.

**(iii-b) If  $R \in \mathcal{R}^N$ , then,  $\Omega P_i z_i$ .**

Let  $i \in N$ . Since preferences are strictly monotone in the interior of the consumption space and  $z_i \leq \Omega$ , then  $z_i I_i \Omega$  implies  $z_i = \Omega$ . Thus  $\Omega P_i z_i$ , for otherwise for each  $j \in N \setminus \{i\}$ ,  $z_j = 0$ , which contradicts (i-b).

**(iii-a) If  $R \in \mathcal{R}^N$ , then,  $E(R) \subseteq P(R)$ .**

Suppose by contradiction that there is  $z \in E(R)$  such that  $z \notin P(R)$ . Then, there is  $z' \in Z$  such that for each  $i \in N$ ,  $z'_i R_i z_i$  and for some  $j \in N$ ,  $z'_j P_j z_j$ . Now, since preferences are continuous, then there is  $\alpha \in (0, 1)$  such that  $(1 - \alpha)z'_j P_j z_j$ . Since  $z'_j P_i z_j P_j 0$ , then  $z'_j \neq 0$ . Let  $i \in N \setminus \{j\}$ . Since  $R \in \mathcal{R}^N$  and  $z'_i R_i z_i P_i 0$ , then  $(z'_i + \frac{\alpha}{|N|-1} z'_j) P_i z_i$ . Let  $z''$  be the allocation defined by:  $z''_j \equiv (1 - \alpha)z'_j$ , and for each  $i \in N \setminus \{j\}$ ,  $z''_i \equiv z'_i + \frac{\alpha}{|N|-1} z'_j$ . Since  $z' \in Z$ , then  $z'' \in Z$ . Since for each  $i \in N$ ,  $z''_i P_i z_i$ , then  $z \notin P^w(R)$ . This is a contradiction.  $\square$

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