

# Reflecting inequality of claims in gains and losses

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\* University of Rochester, Department of Economics

\*\* Texas A&M University, Department of Economics

# Resource allocation problems

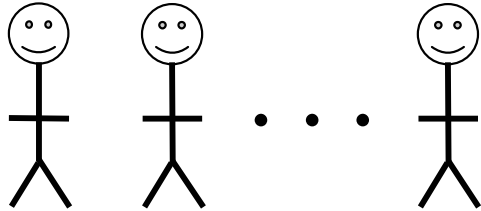
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- General discussion

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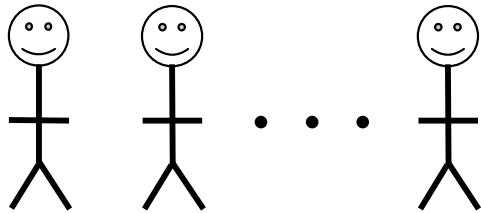
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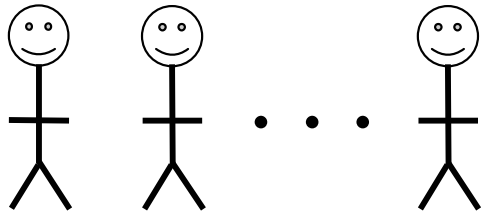
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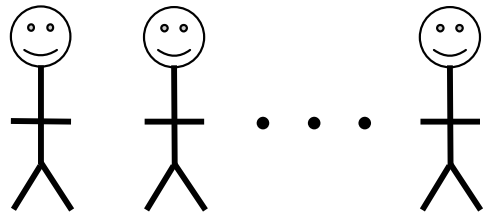


How to divide the resource?

# Resource allocation problems

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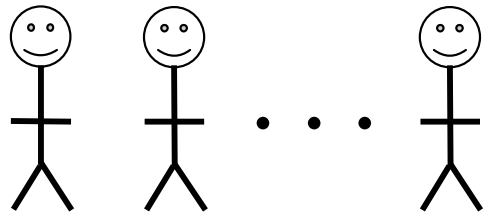
How to divide the resource?

How to choose the social outcome?

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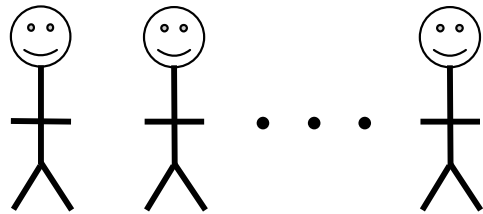
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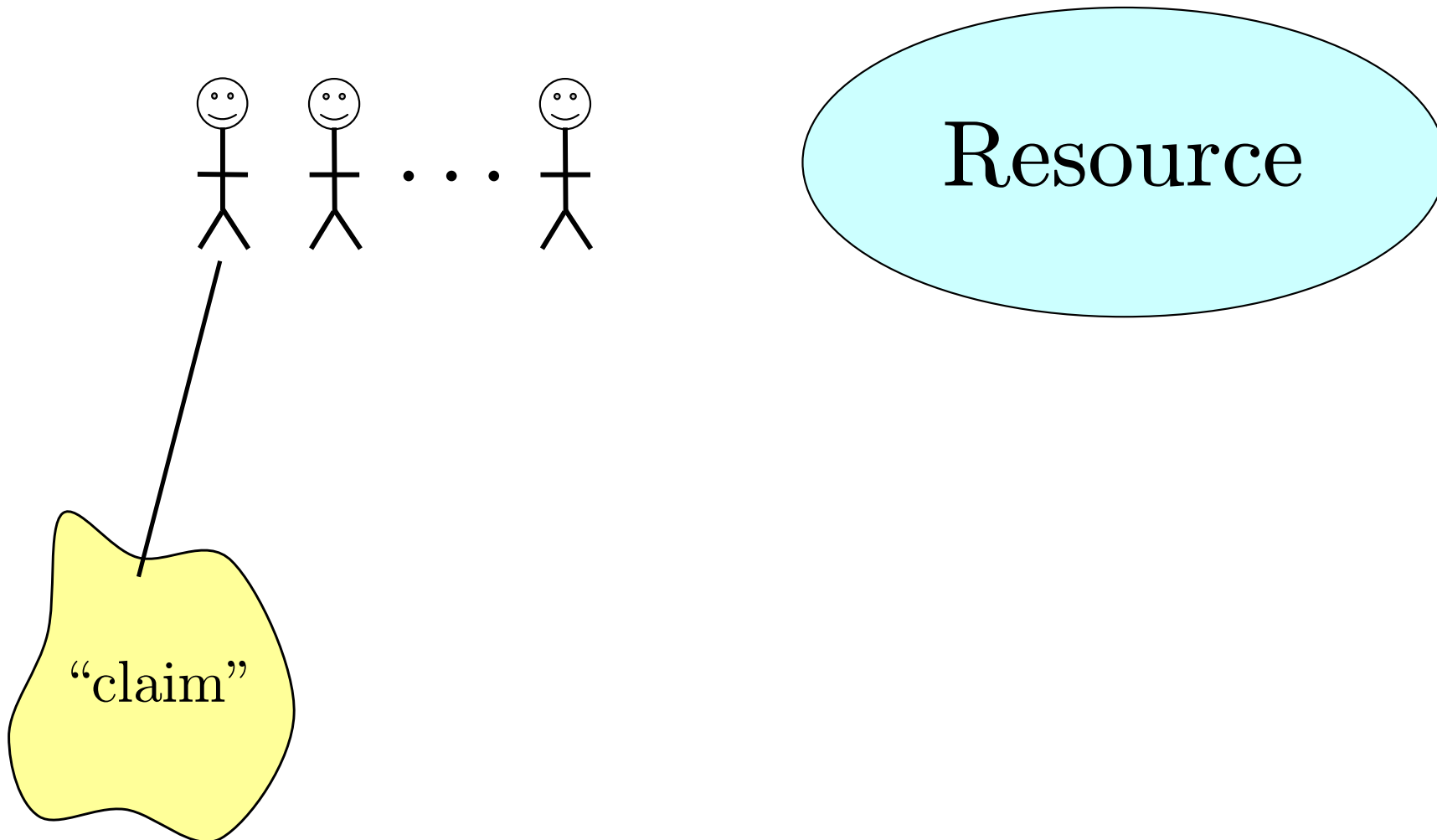
How to choose the social outcome?

Efficiently      Fairly

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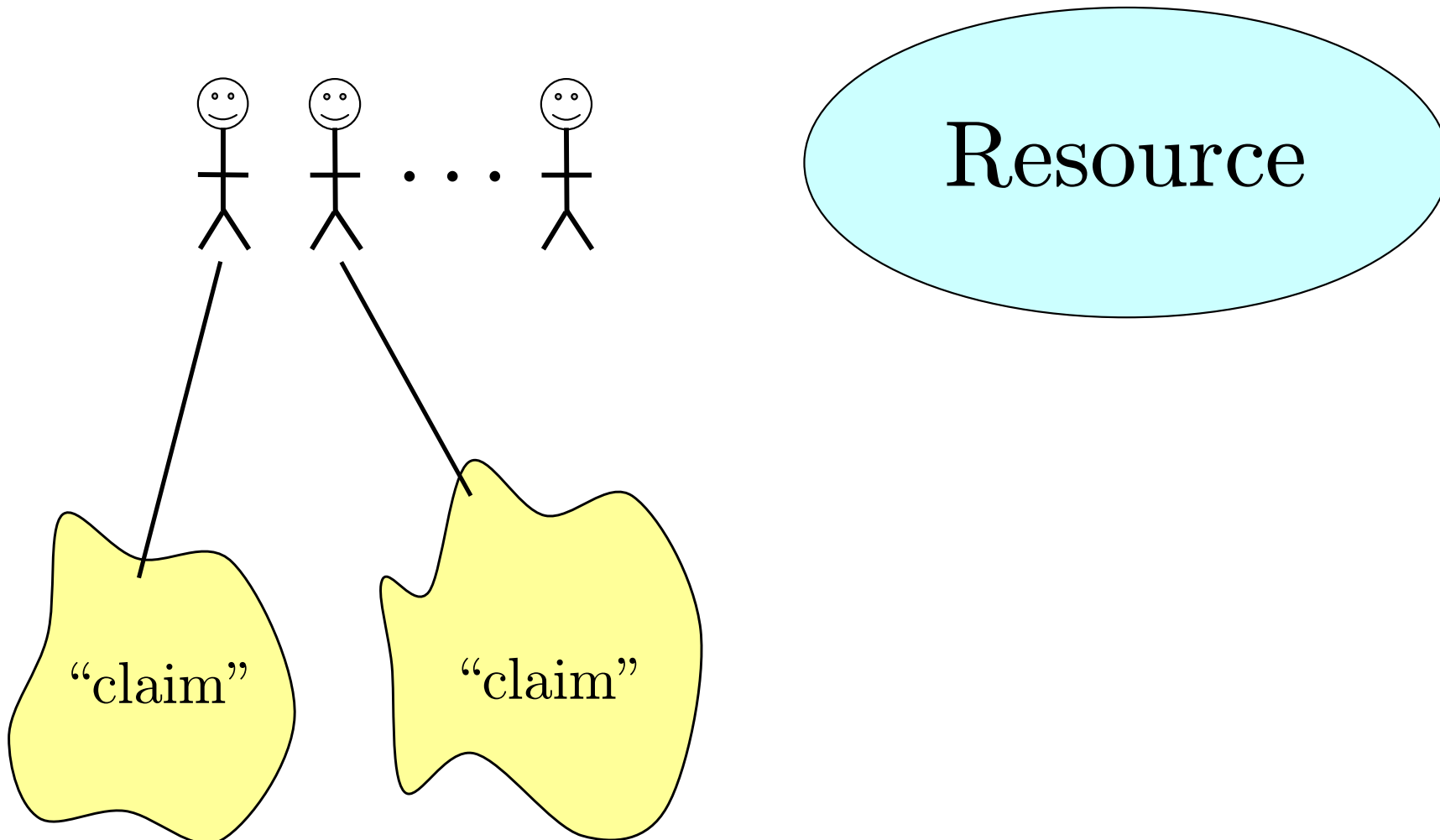
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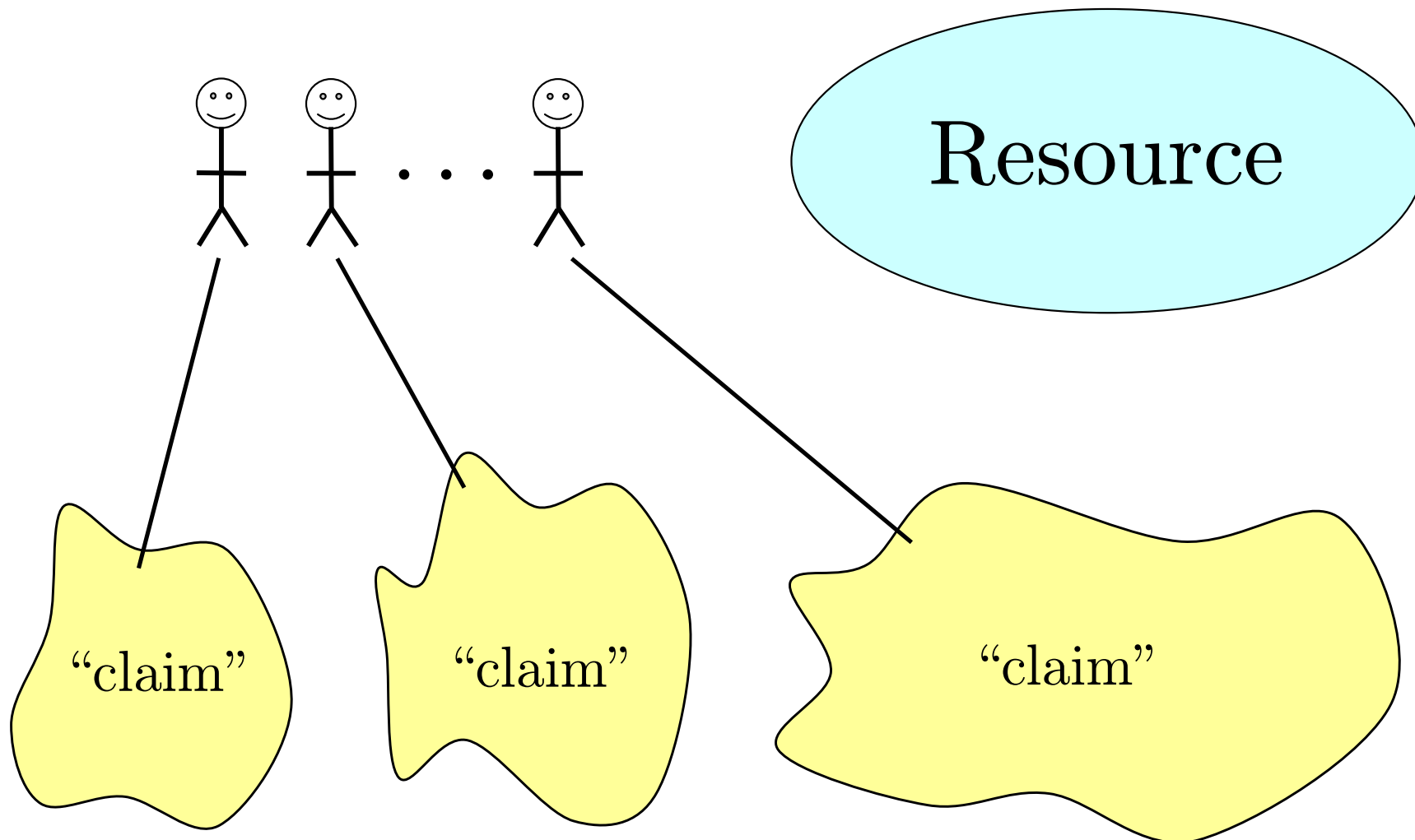
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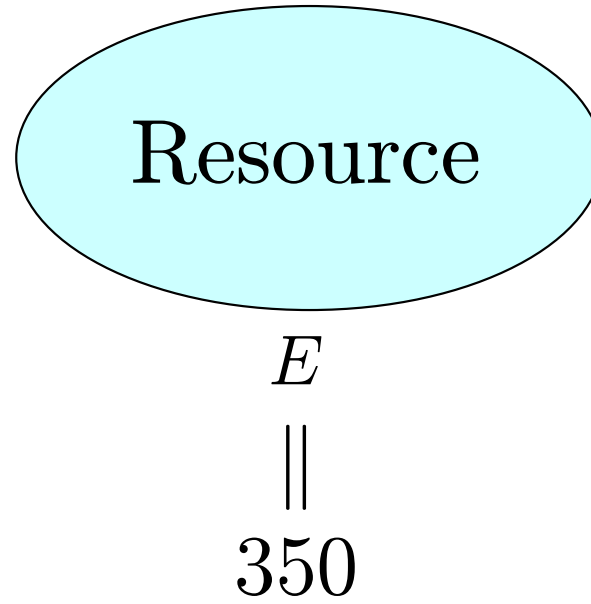
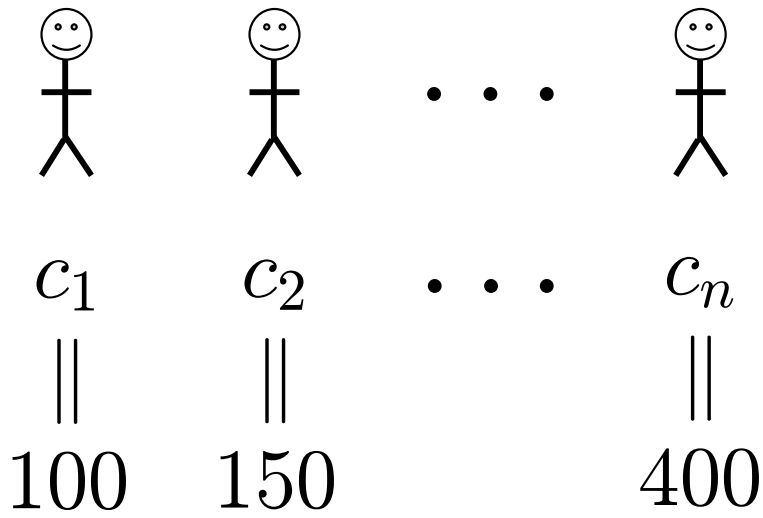


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Resource

$c_1$

$c_2$

...

$c_n$

$E$

$$\sum_{i \in N} c_i \geq E$$



100

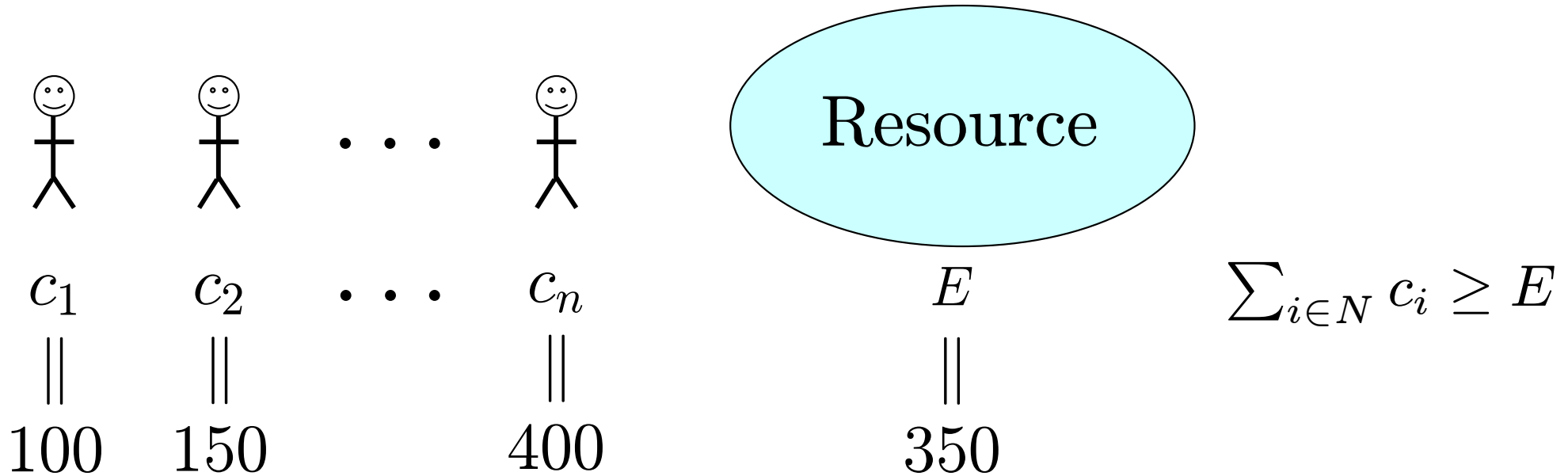
150

400

350

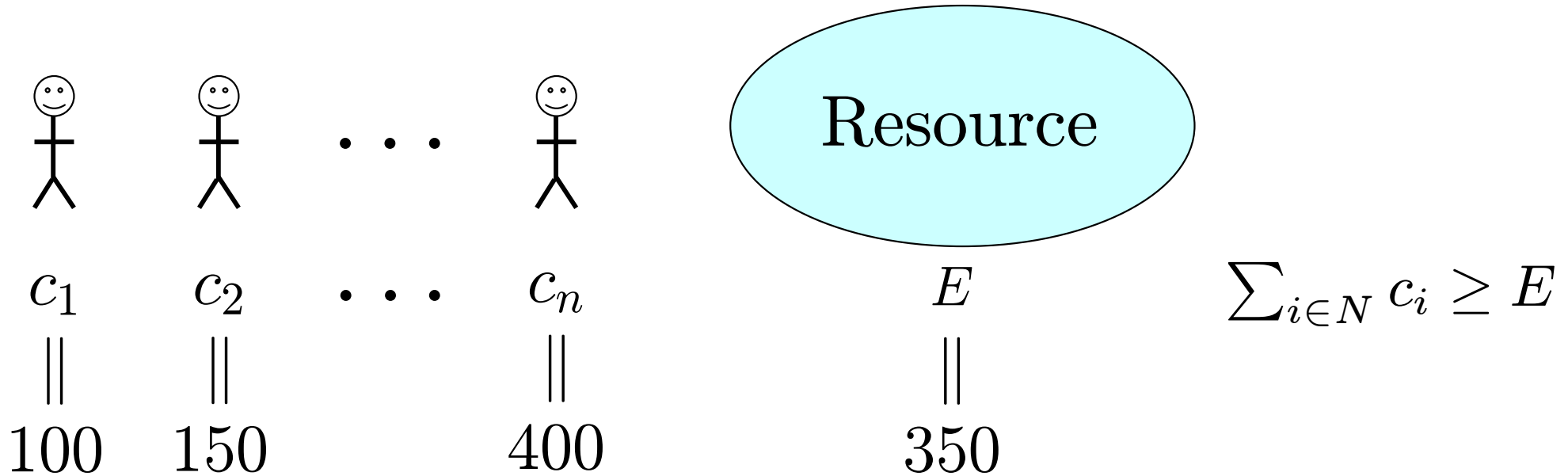
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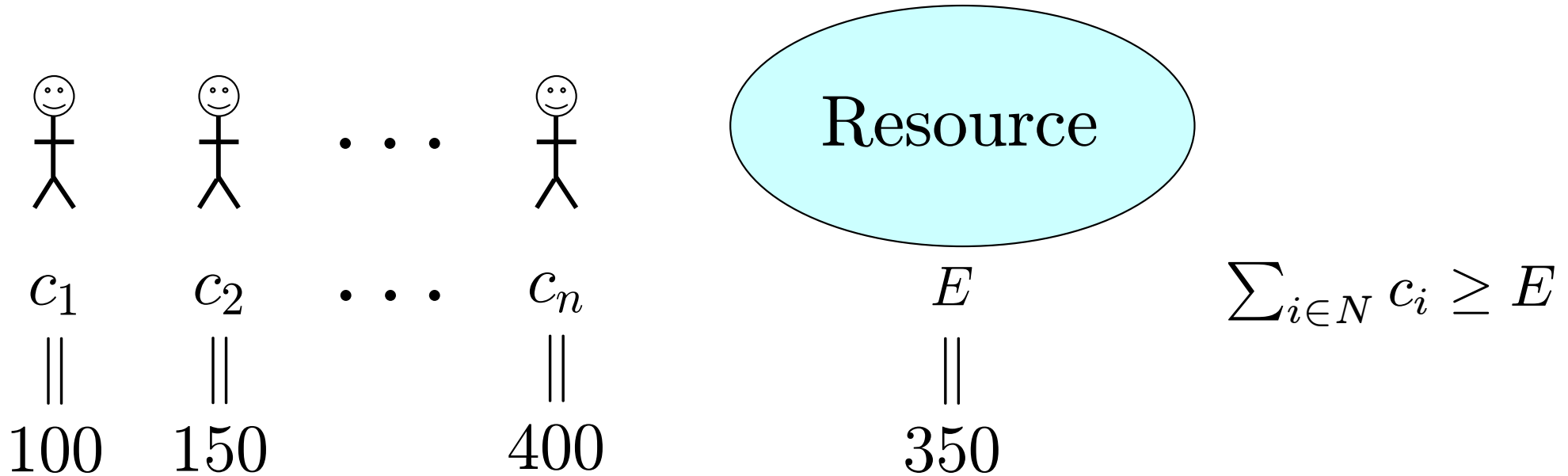
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- Bankruptcy

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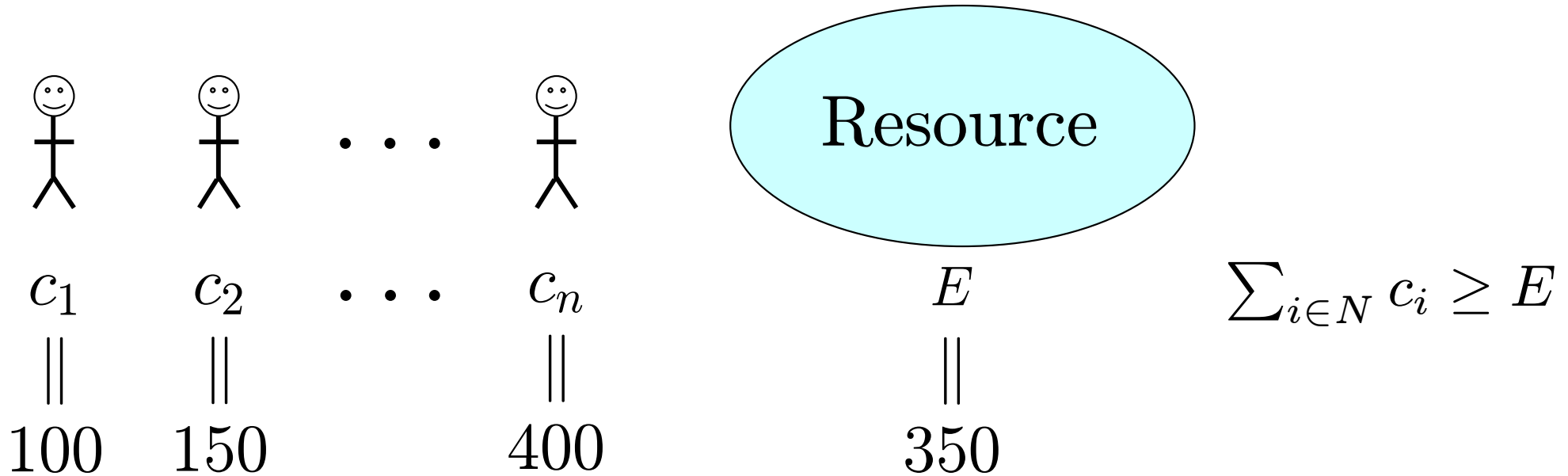
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- Bankruptcy
- Estate division

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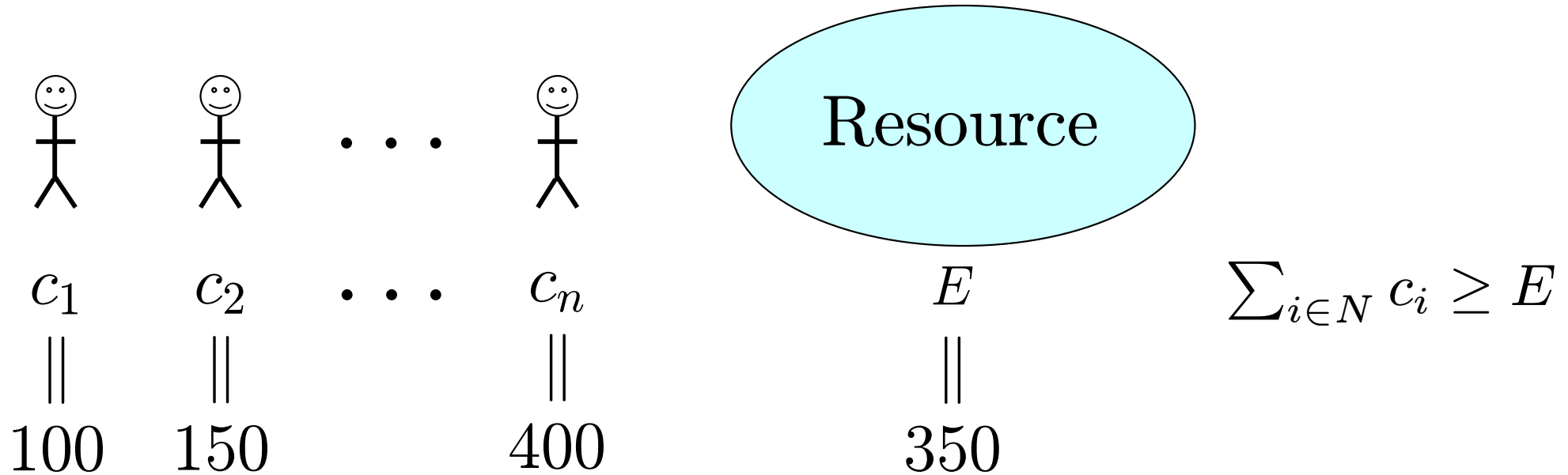
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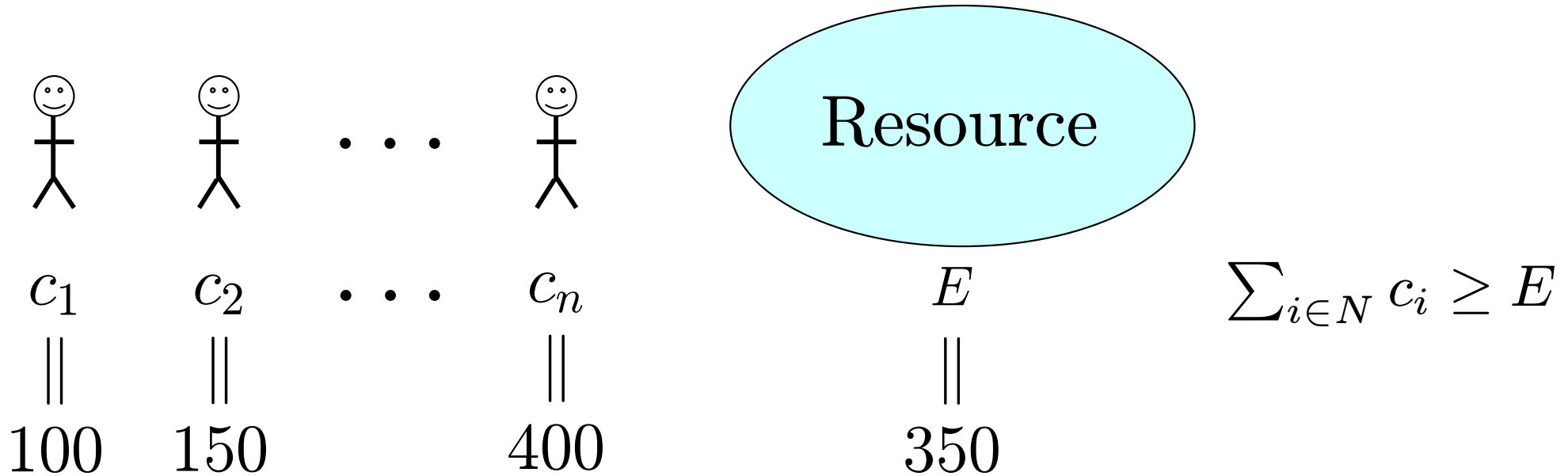


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$$\text{“} \sum_{i \in N} c_i - T \text{”}$$

# Claims problems (O'Neill, 1982)

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- Bankruptcy
- Estate division
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# Side note: Examples from Talmud

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- **Example 1 (Contested garment problem):**

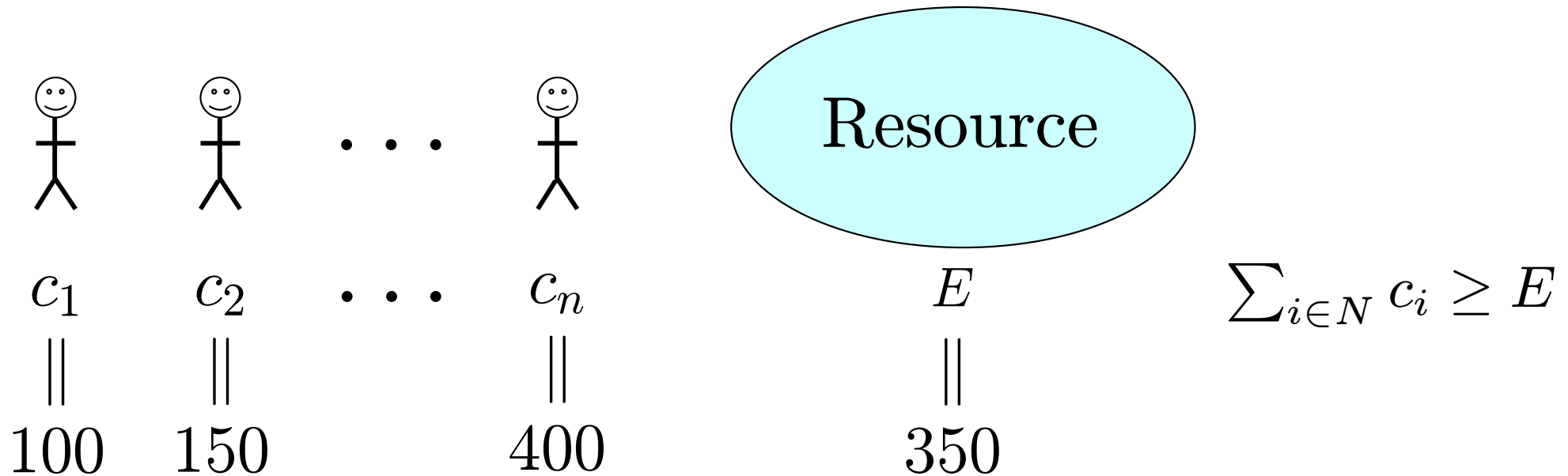
	$c_1 = \frac{1}{2}$	$c_2 = 1$
$E = 1$	$\frac{1}{4}$	$\frac{3}{4}$

- **Example 2 (Marriage contract problem):**

	$c_1 = 100$	$c_2 = 200$	$c_3 = 300$
$E = 100$	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
$E = 200$	50	75	75
$E = 300$	50	100	150

# Claims problems (O'Neill, 1982)

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- Bankruptcy
- Estate division
- Taxation
- Examples from Talmud

# Our paper

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**Axiomatic approach.**

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# Model (Formal)

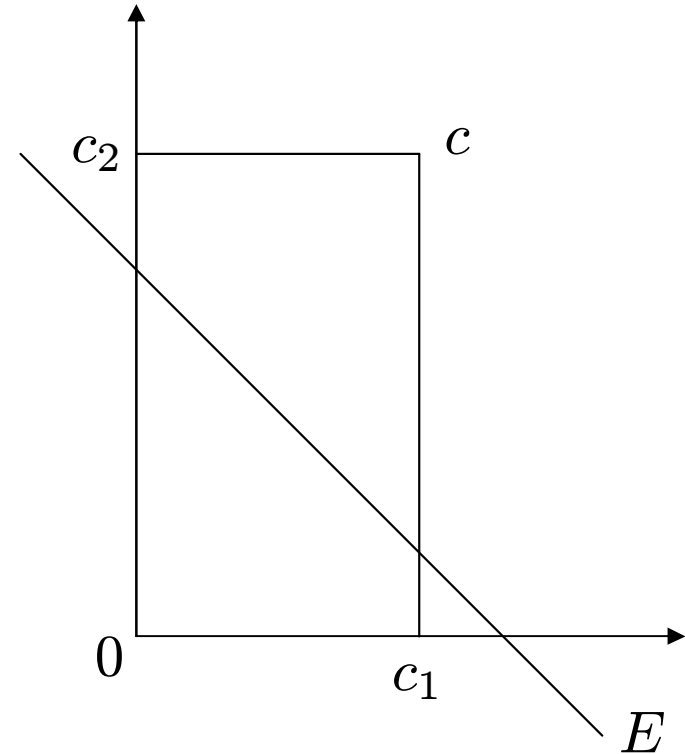
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- $N = \{1, 2, \dots, n\}$ : the set of agents
- $i \in N$
- $c_i \in \mathbb{R}_+$ : the claim of agent  $i \in N$
- $c = (c_i)_{i \in N}$ : the claims vector
- $E \in \mathbb{R}_+$ : endowment
- A **problem for  $N$** :  $(c, E) \in \mathbb{R}_+^N \times \mathbb{R}_+$  with  $\sum_{i \in N} c_i \geq E$

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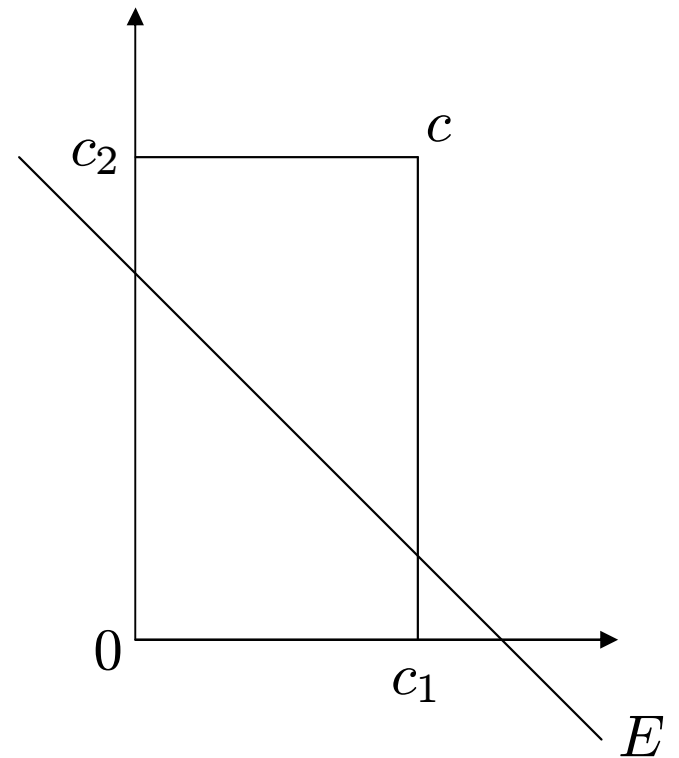
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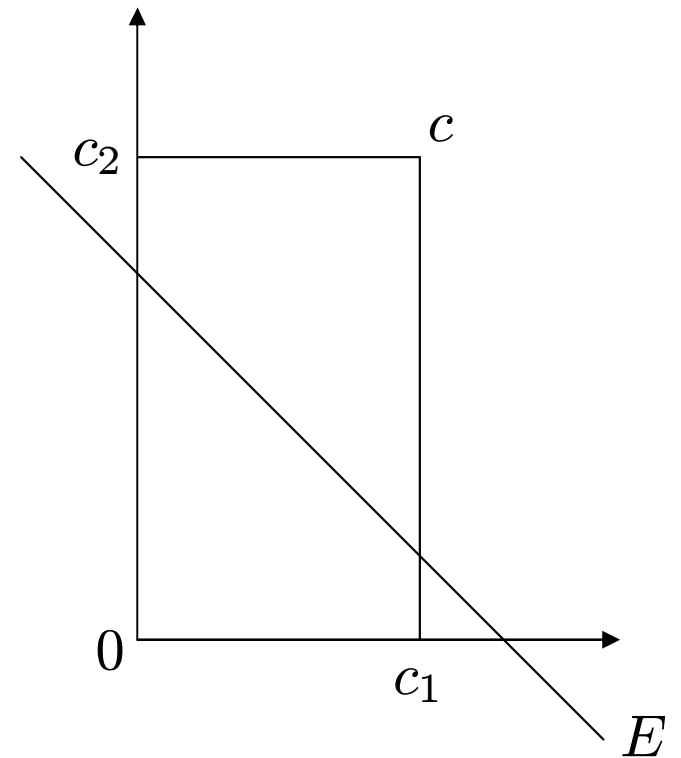
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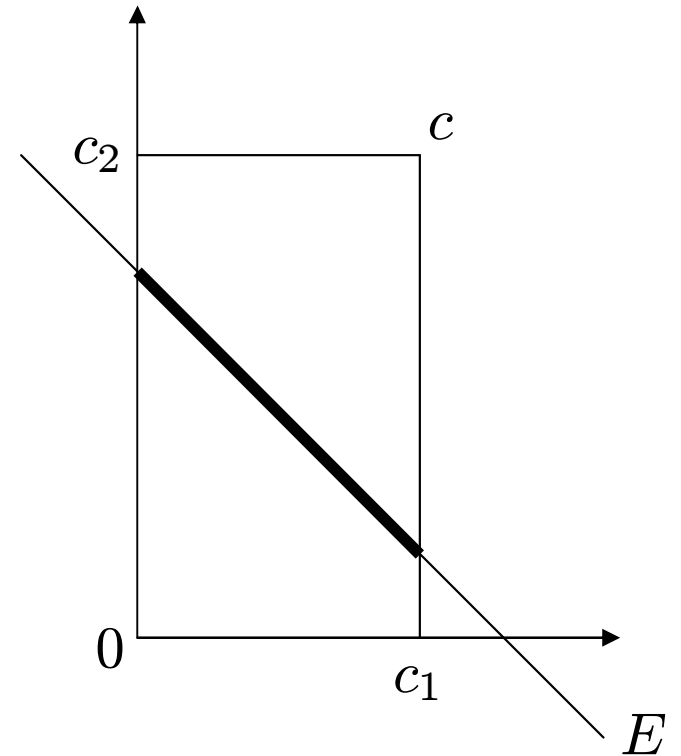
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- An awards vector for  $(c, E) \in \mathcal{C}$ :  
 $x \in \mathbb{R}^N$  such that for each  $i \in N$ ,  
 $0 \leq x_i \leq c_i$  and  $\sum_{i \in N} x_i = E$



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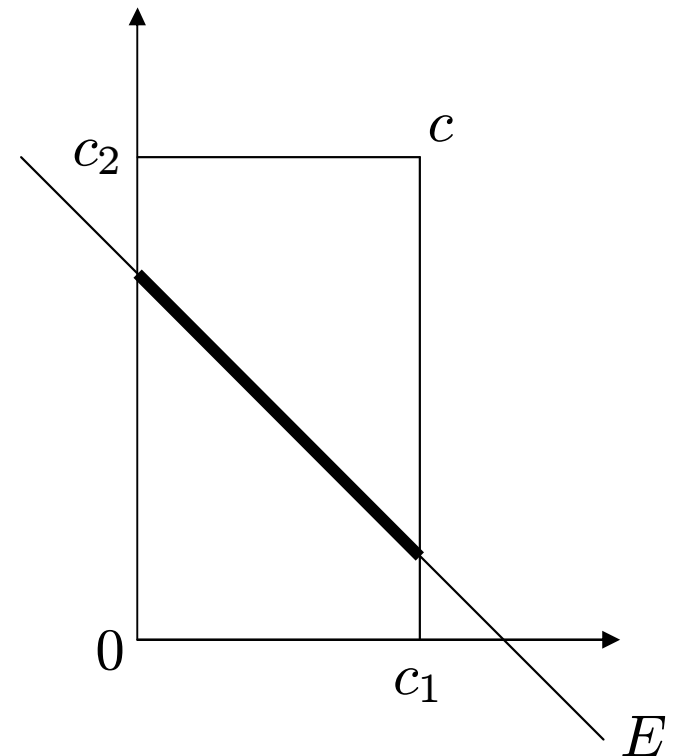
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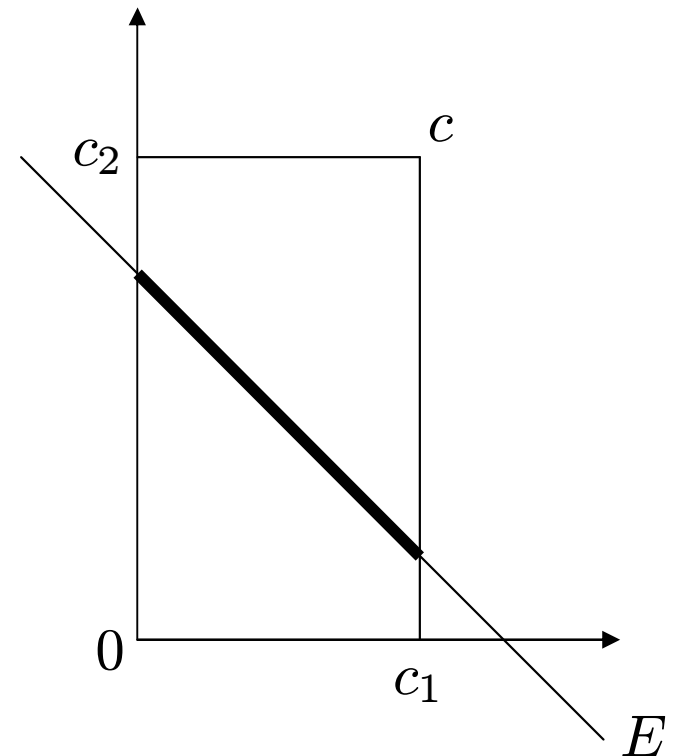
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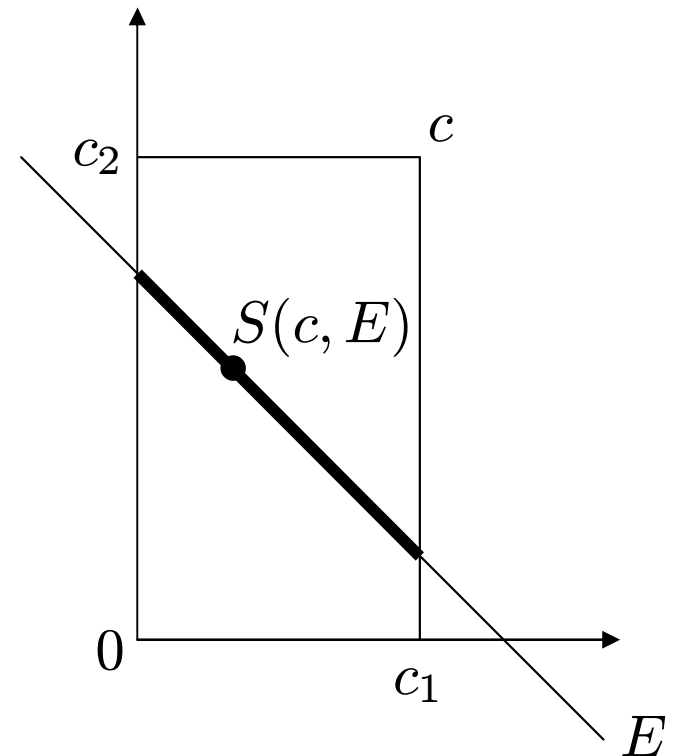
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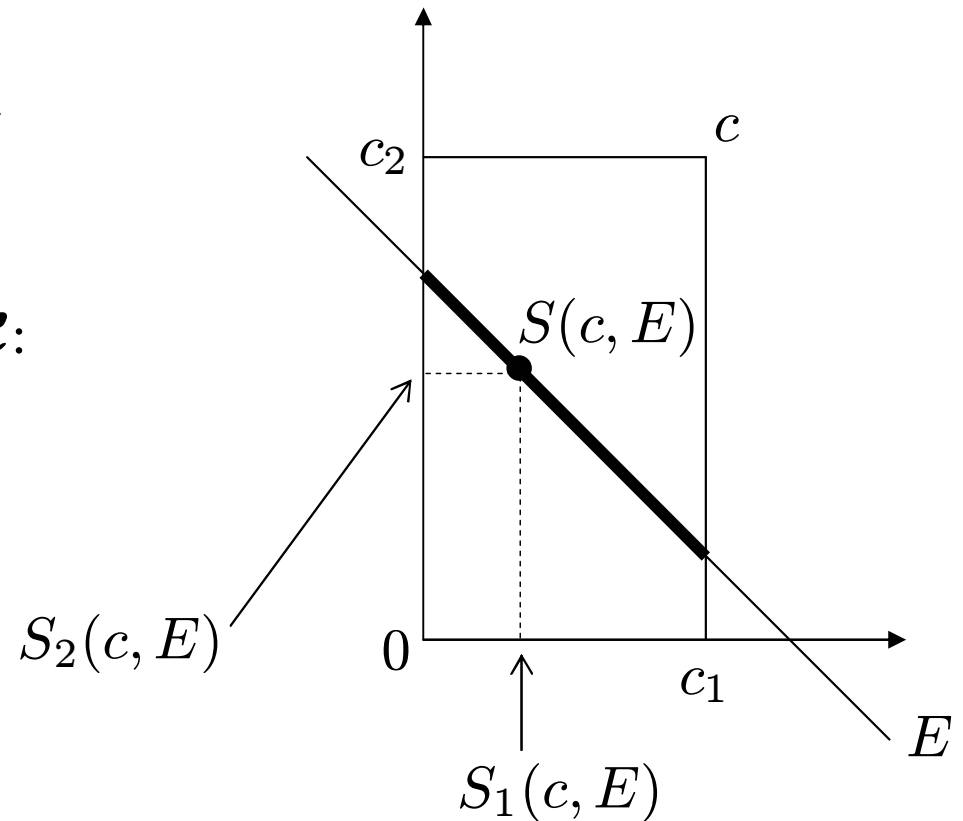
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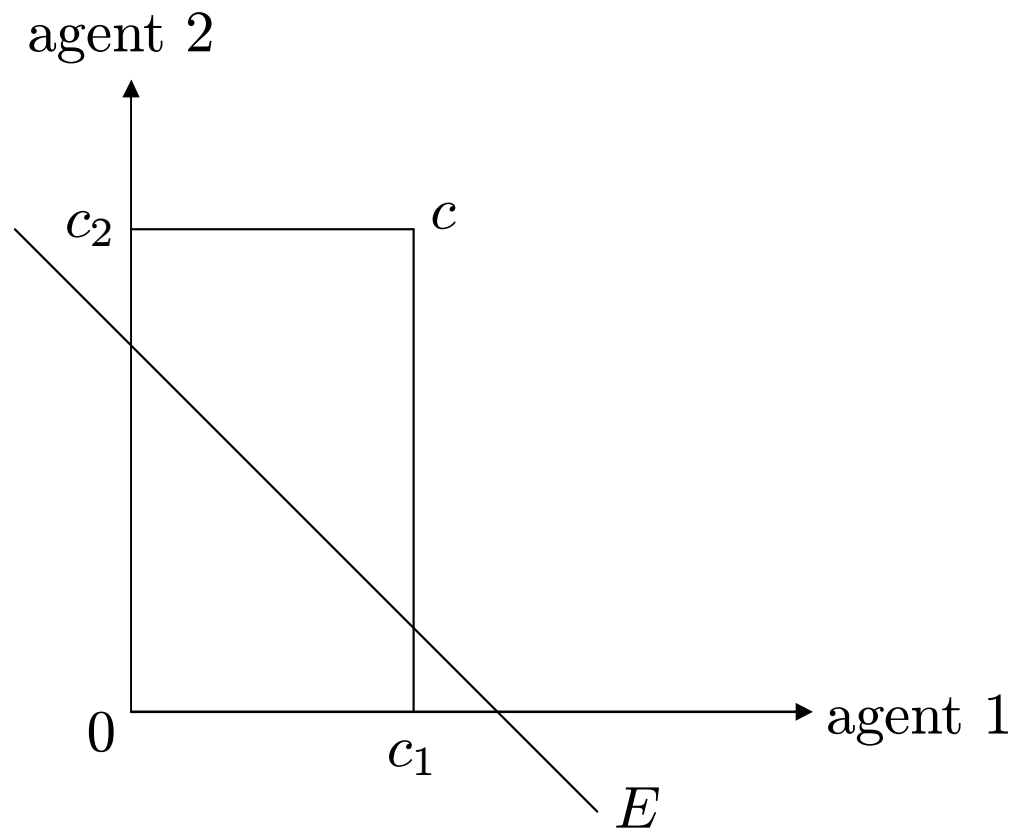
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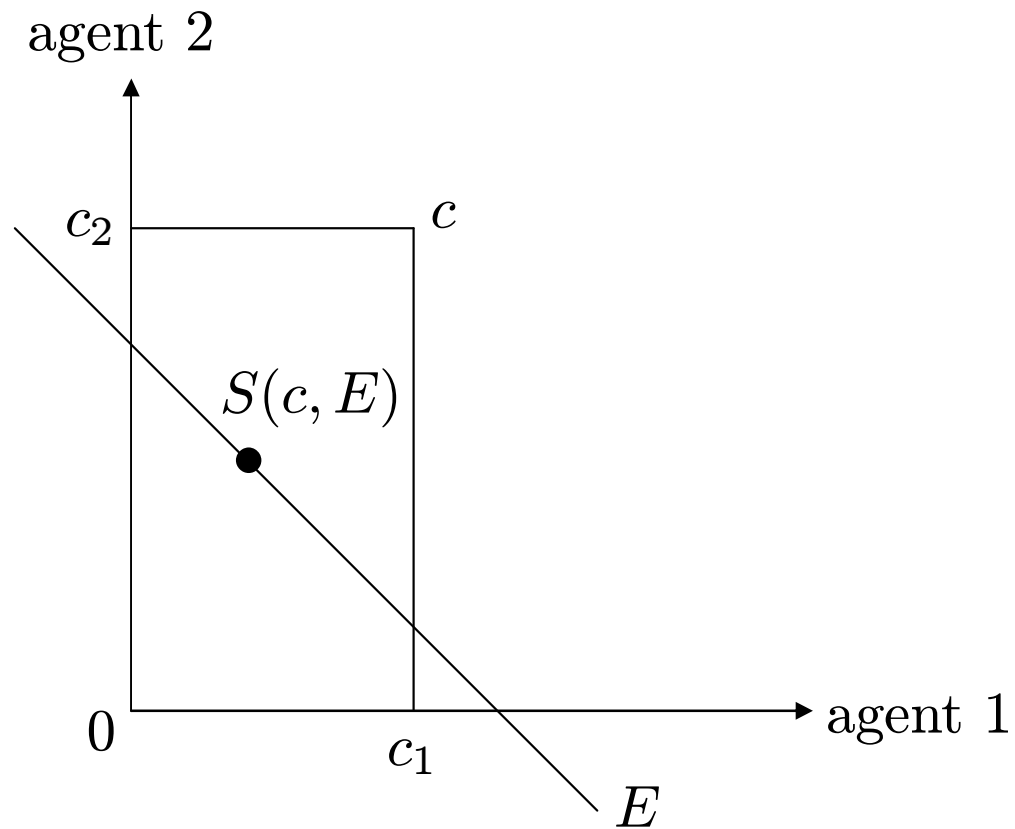
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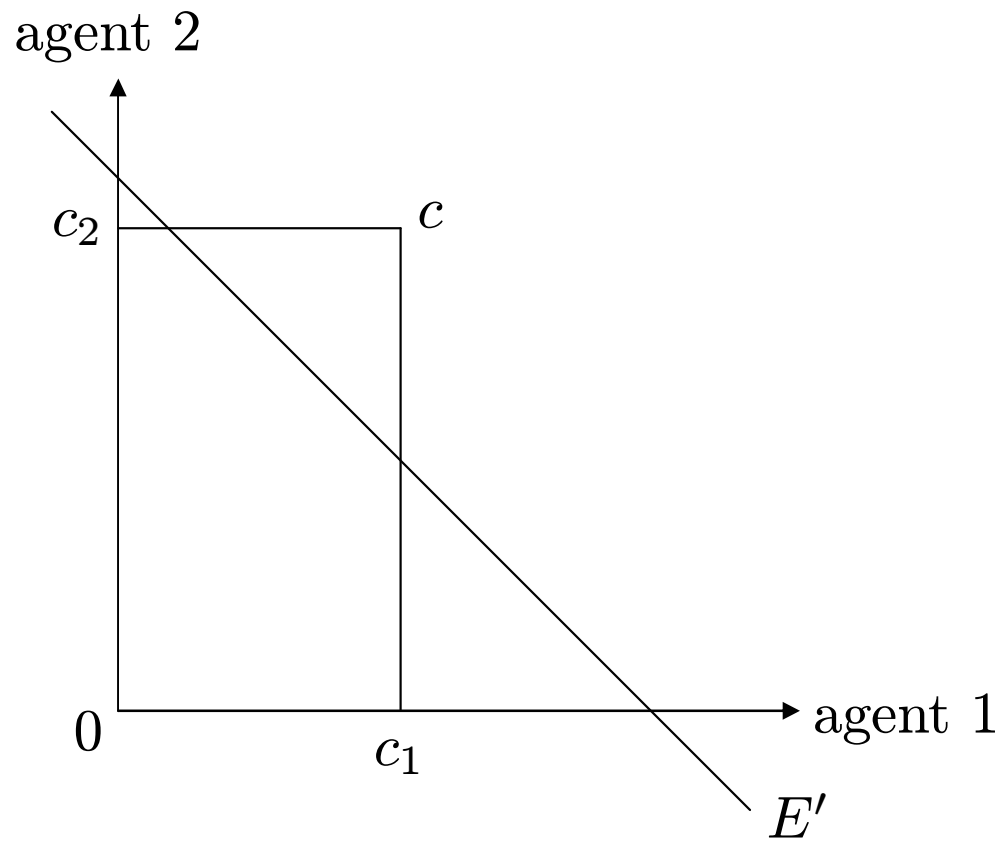
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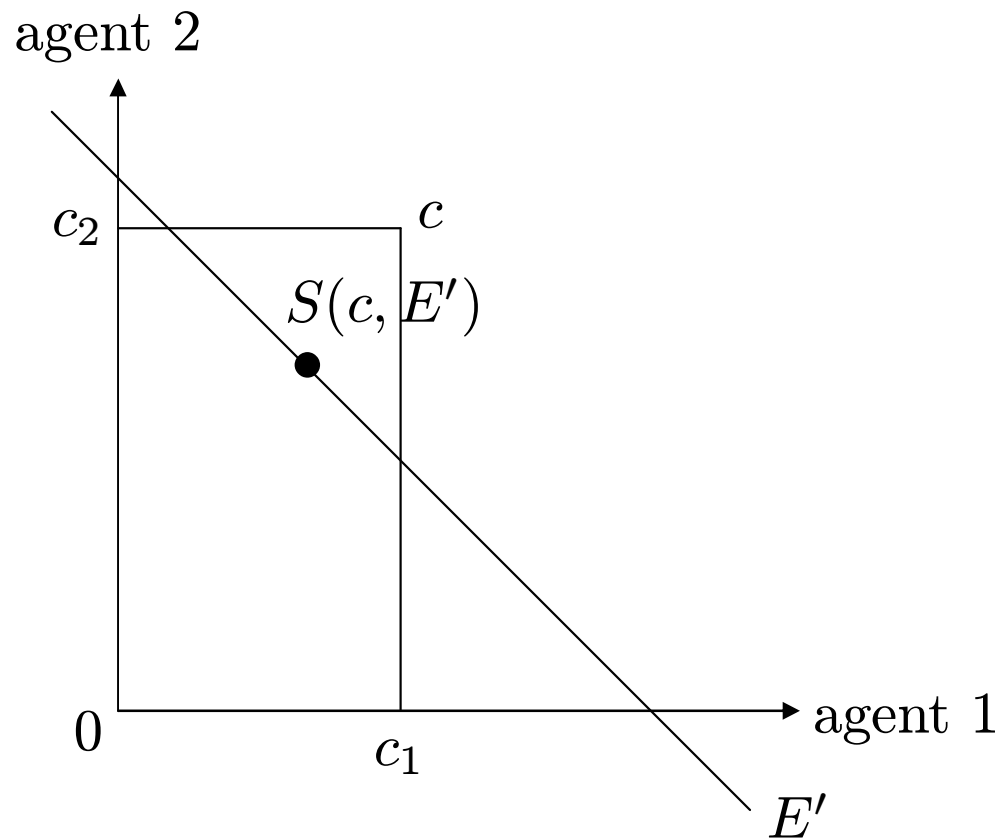
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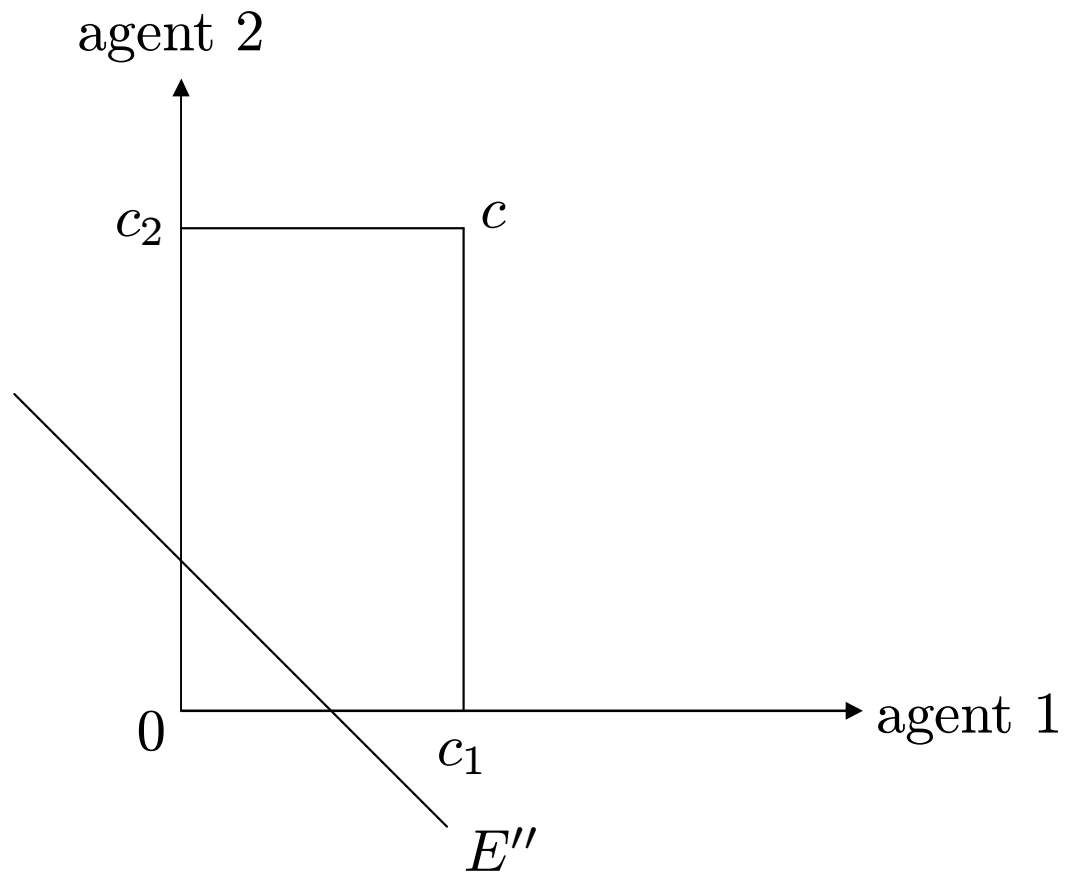
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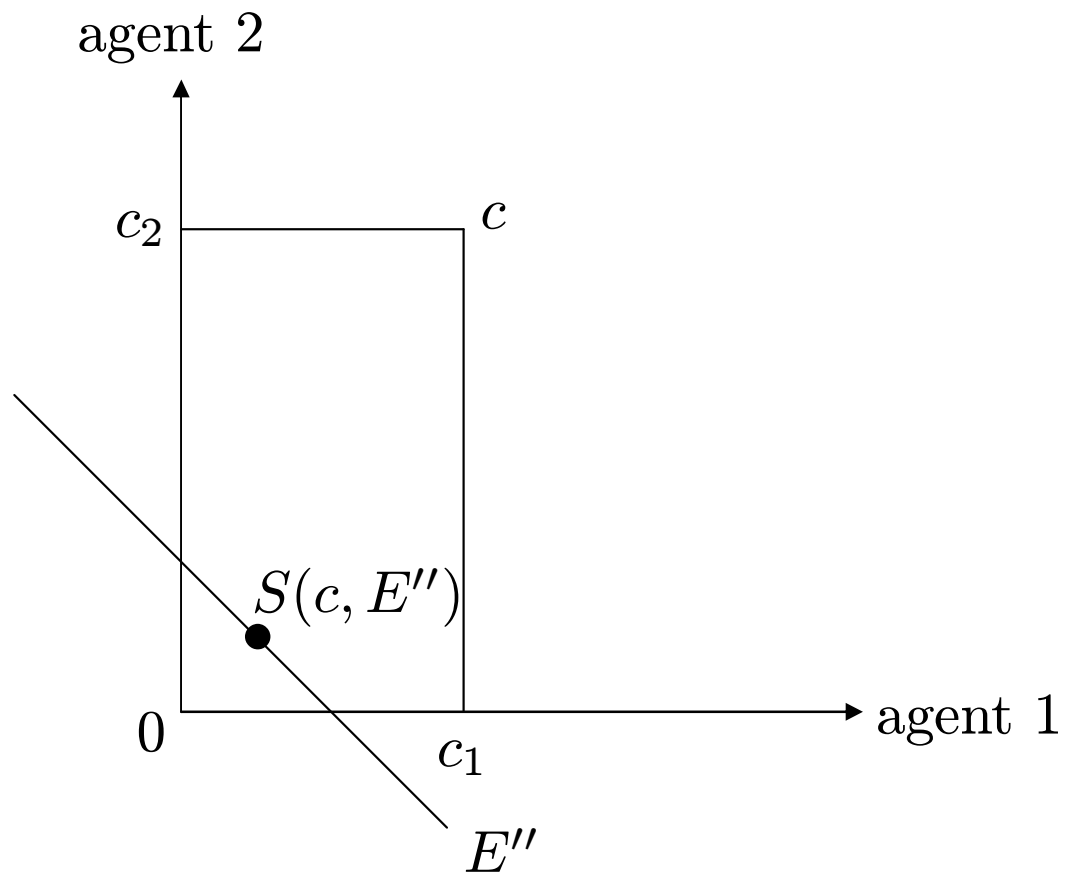
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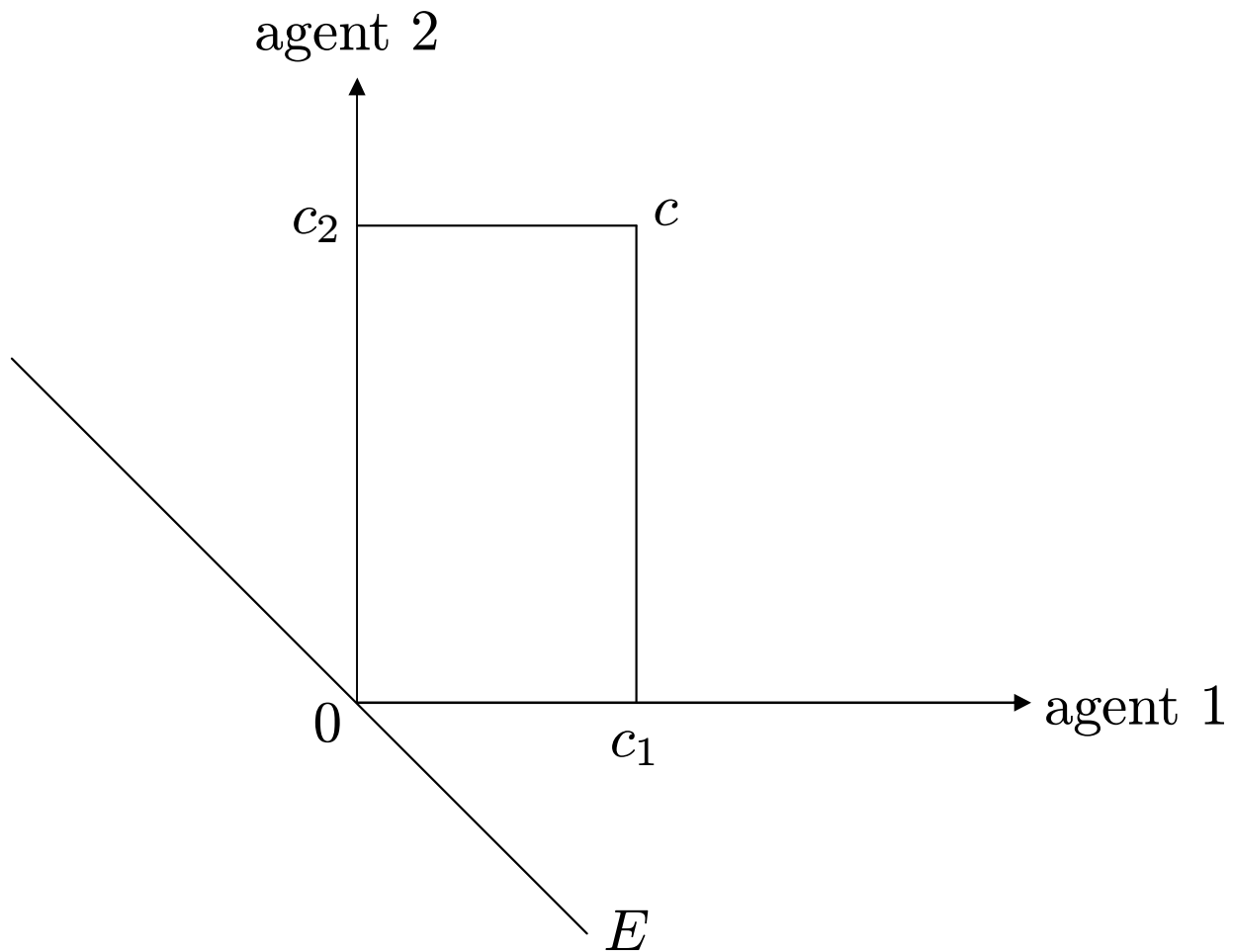
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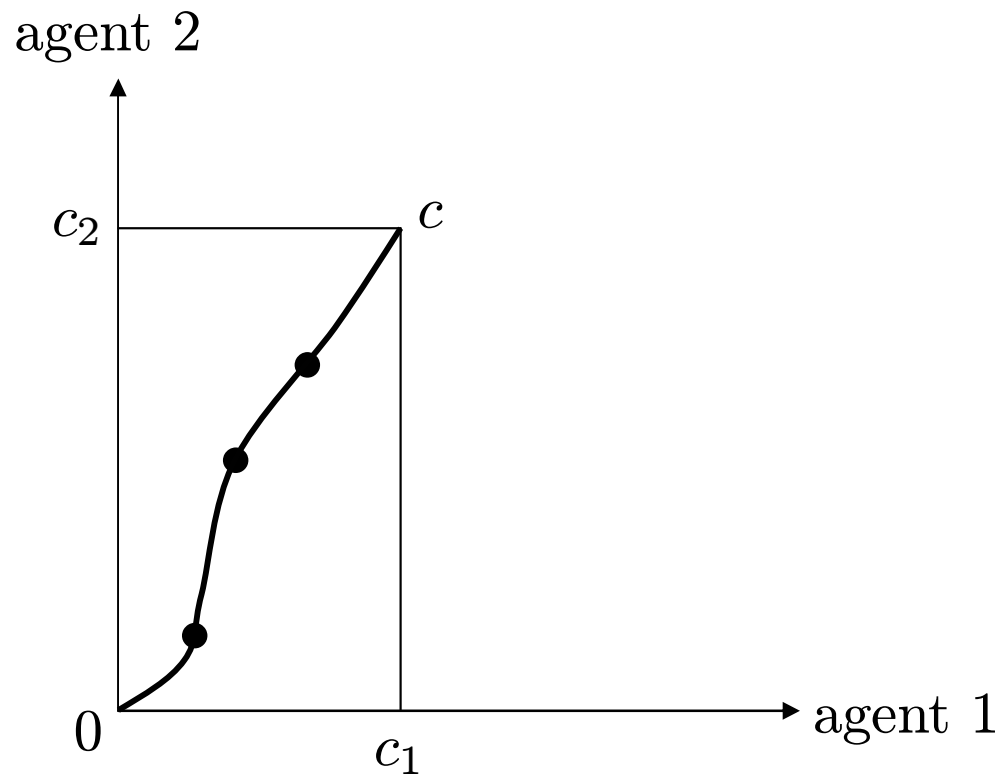
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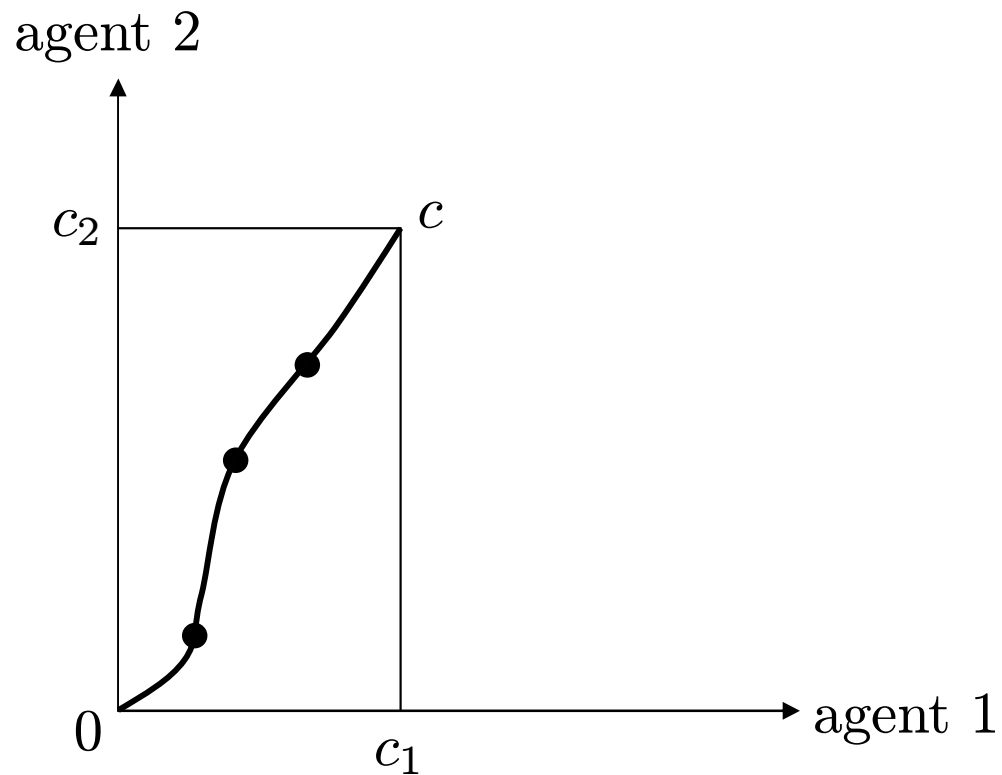
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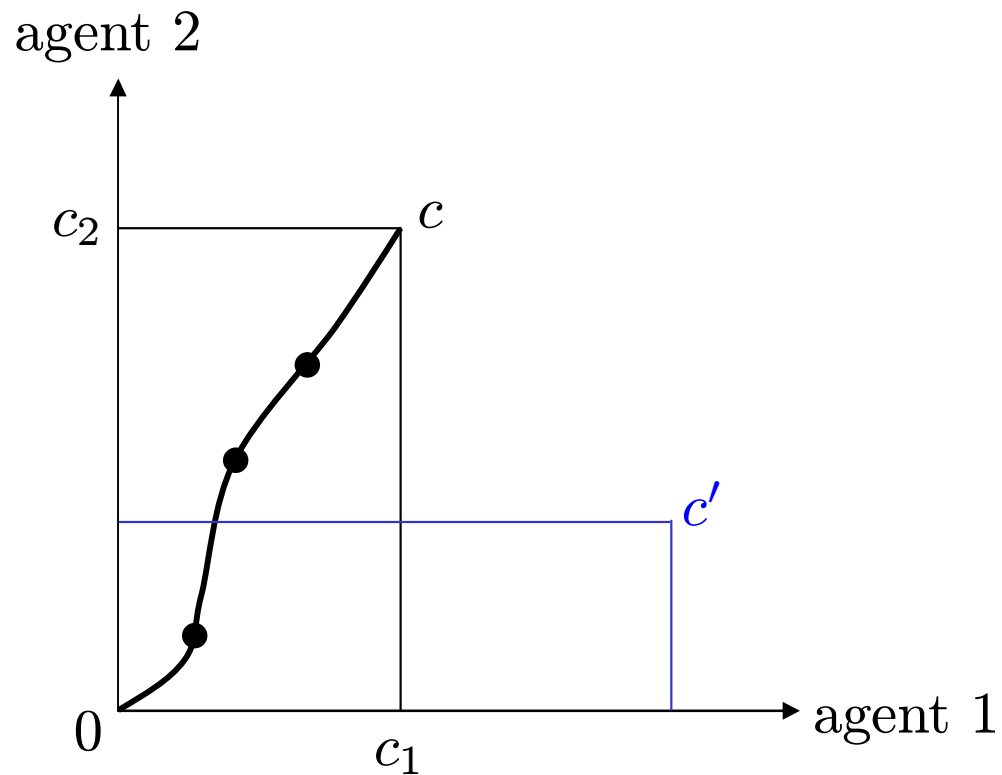
- **Path** of awards of the rule for a claims vector



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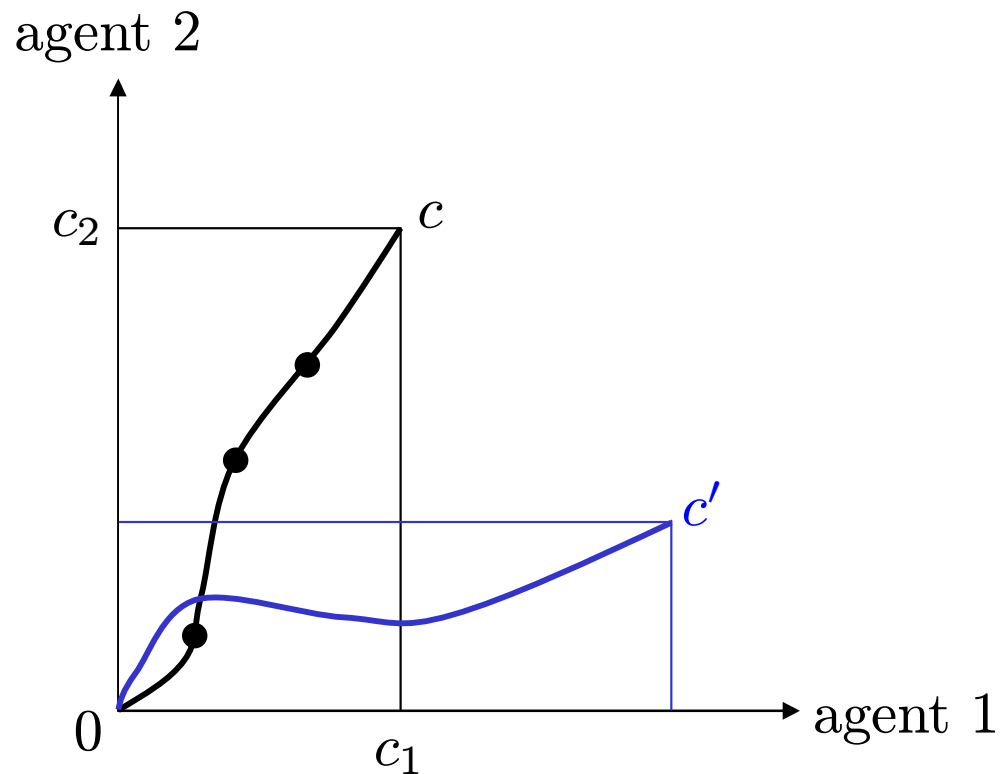
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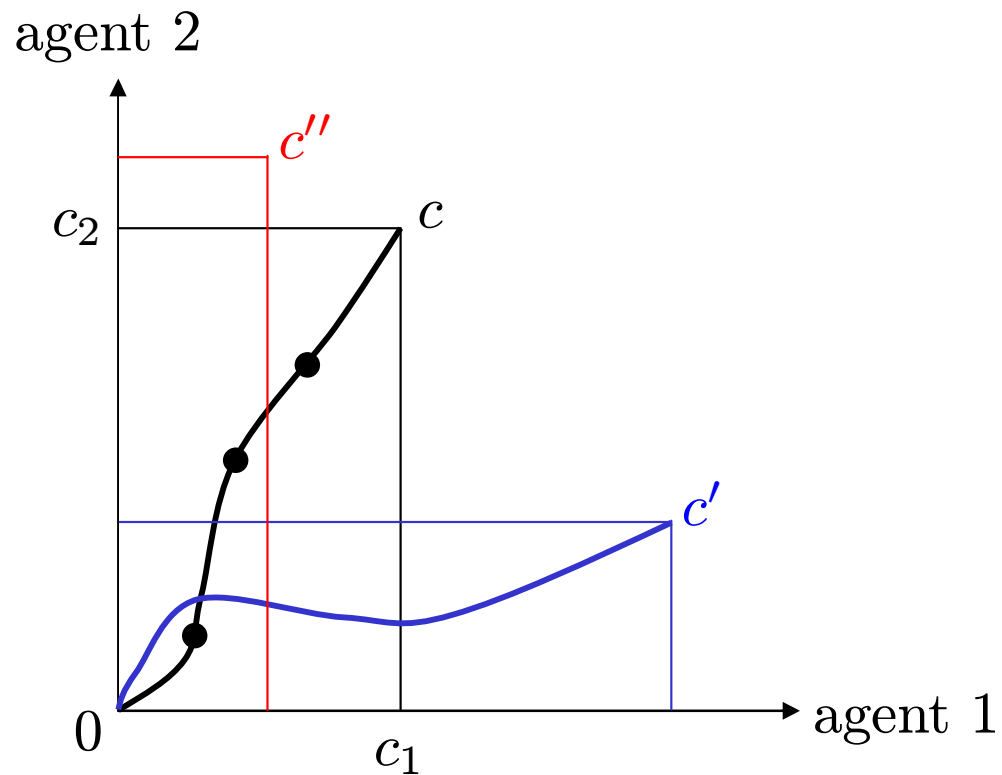
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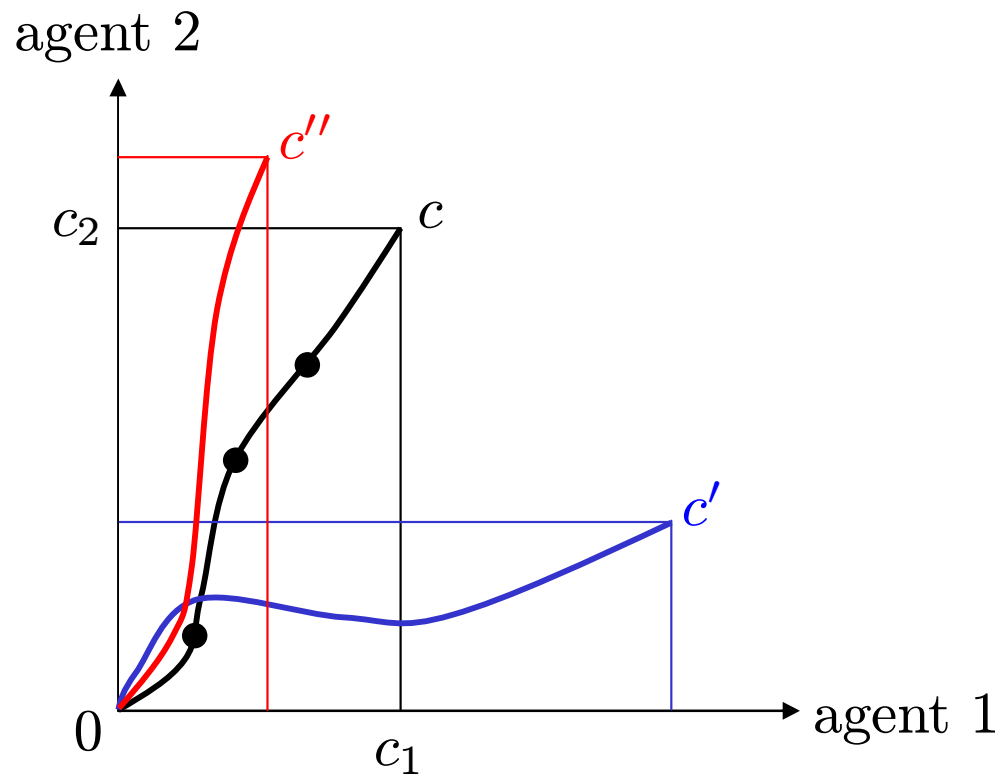
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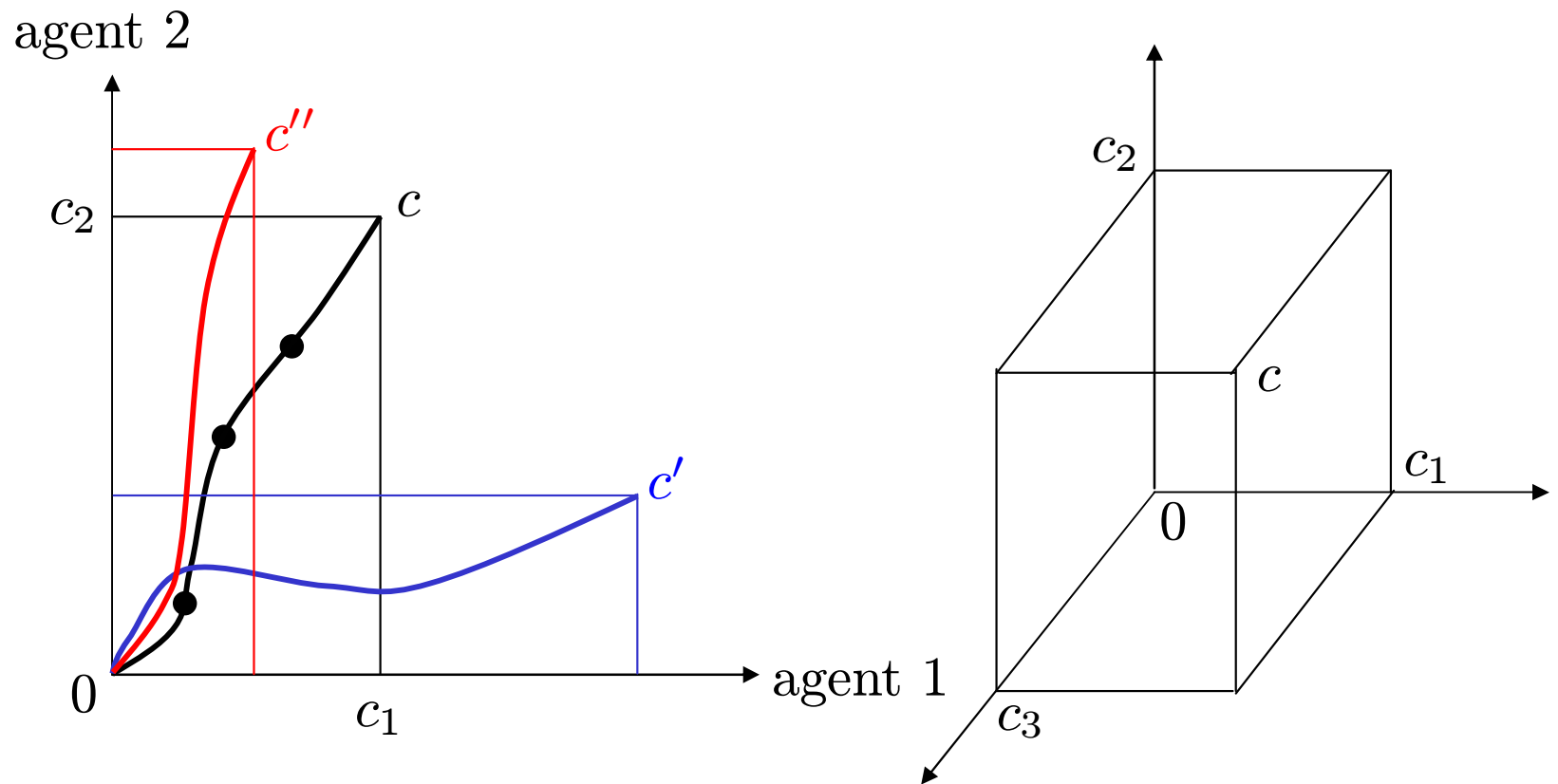
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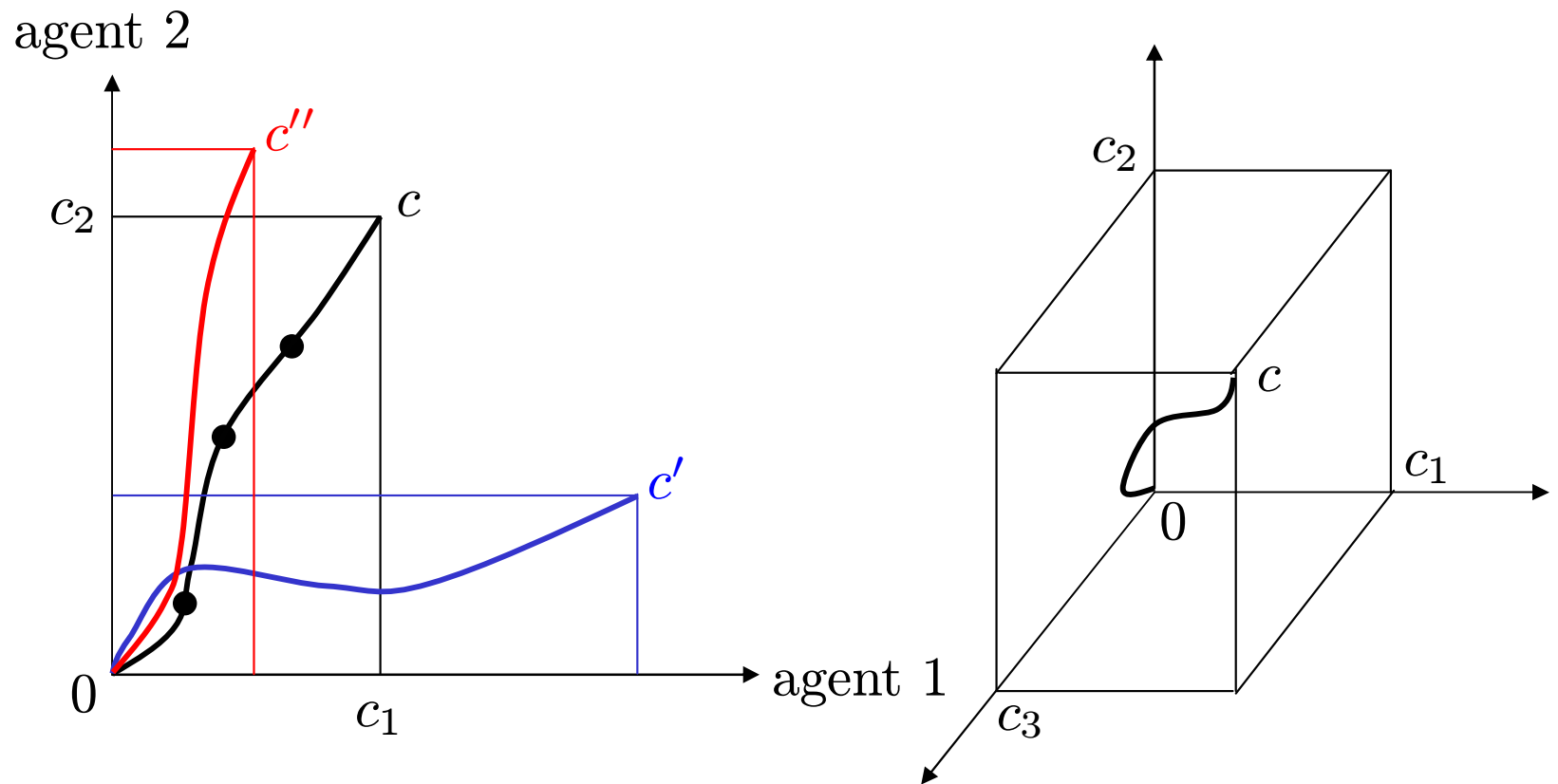
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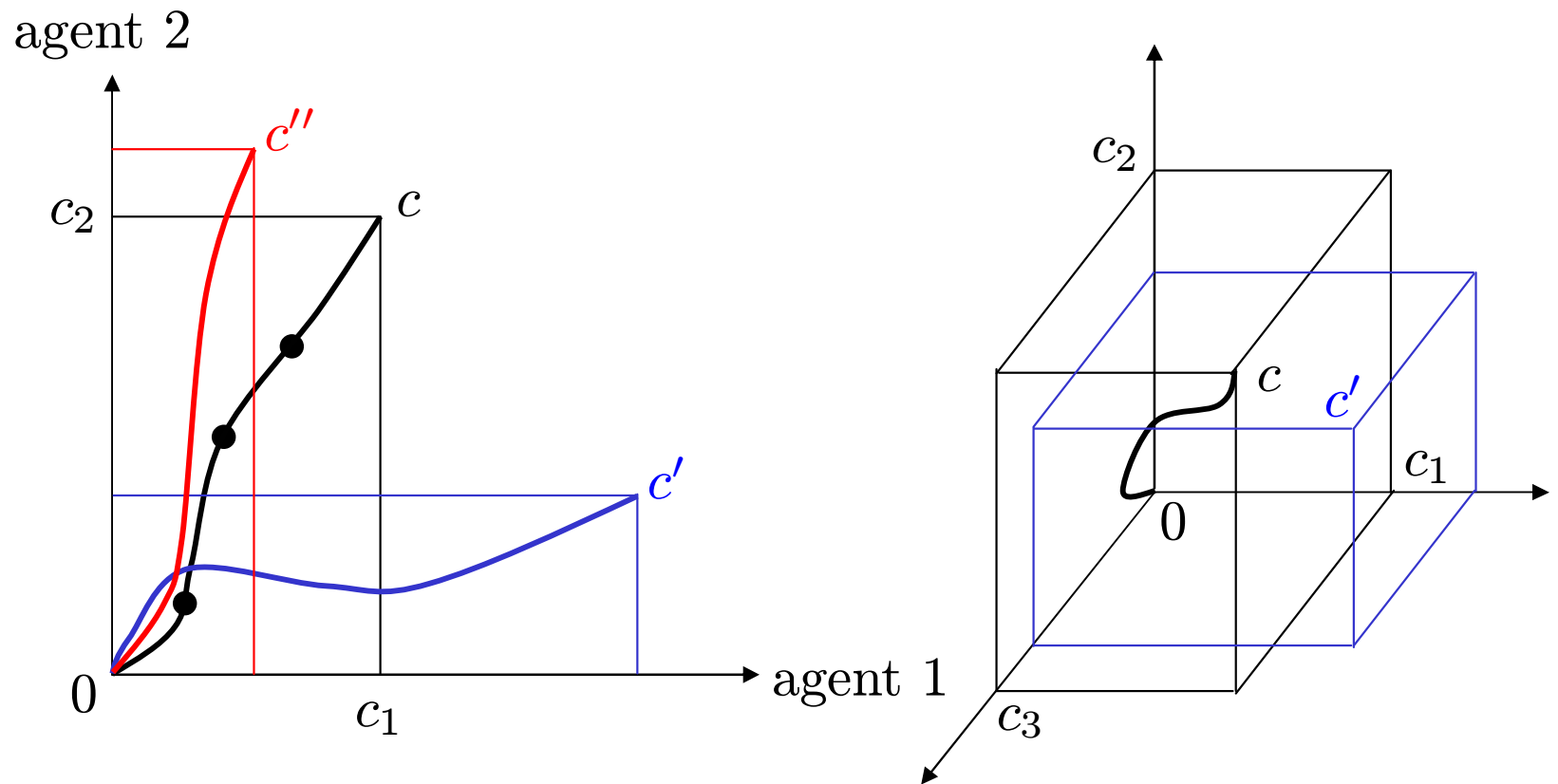
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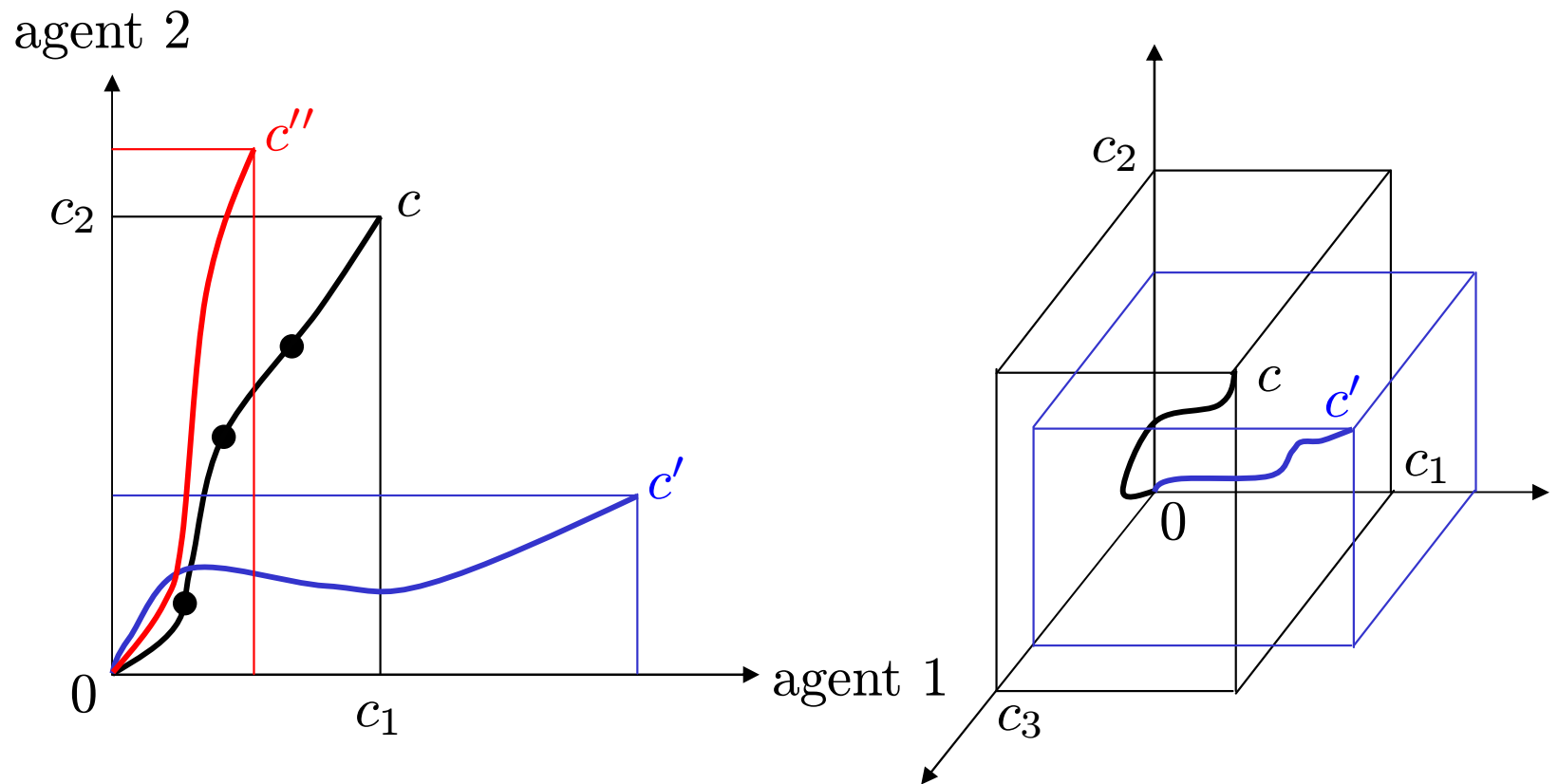
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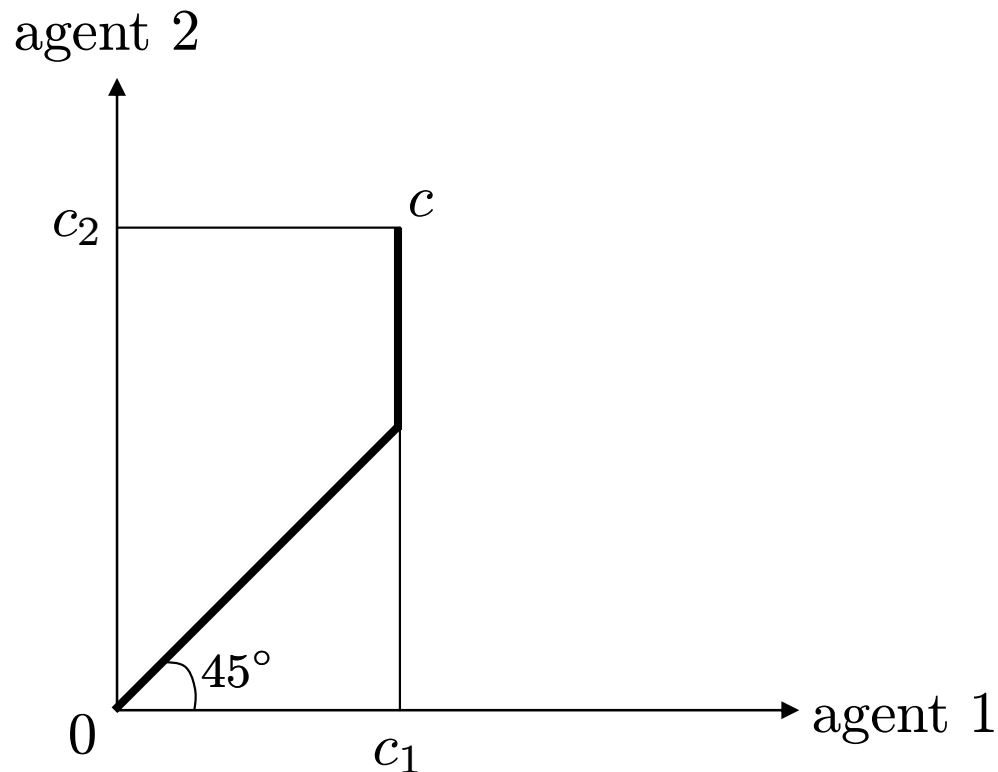
# Examples of rules

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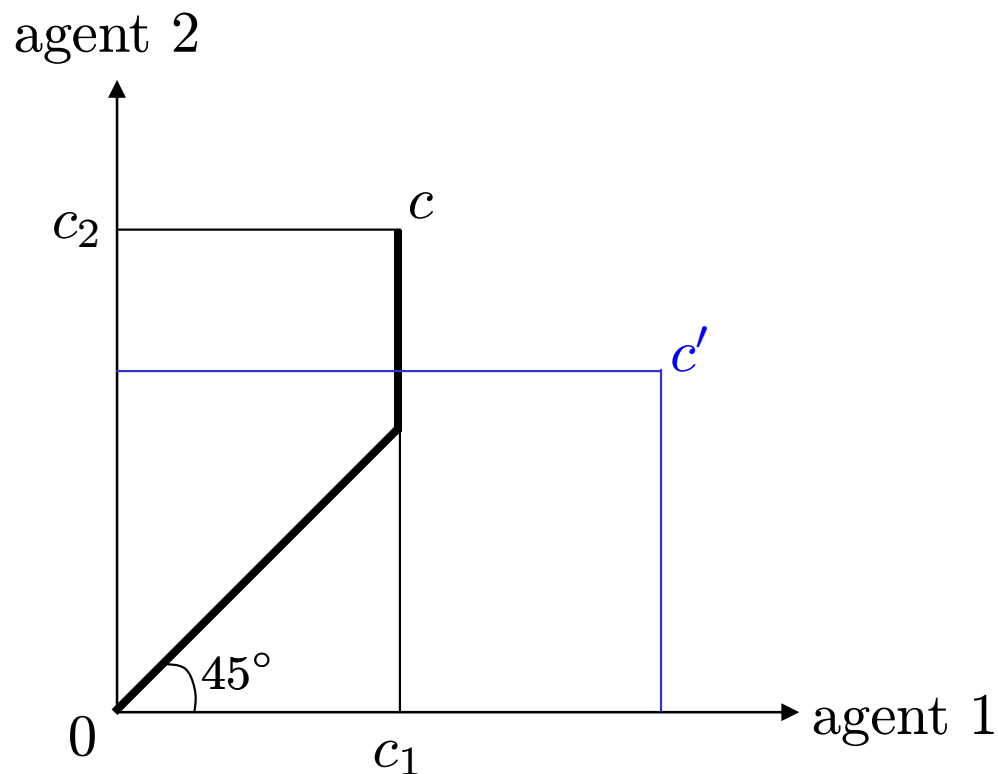
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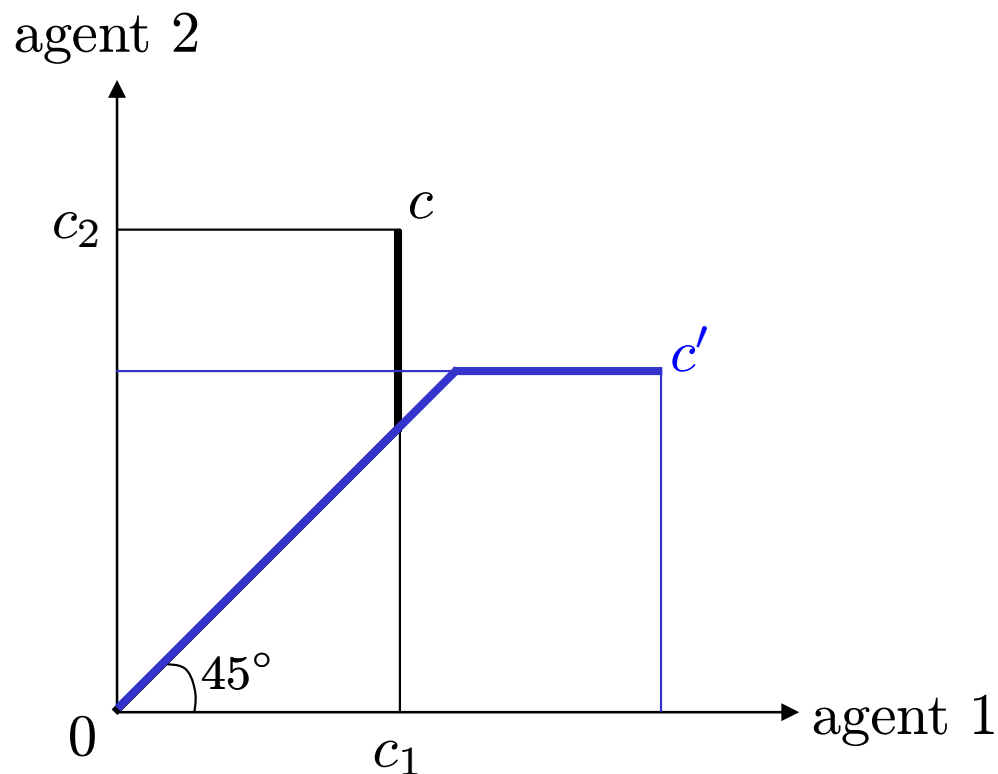
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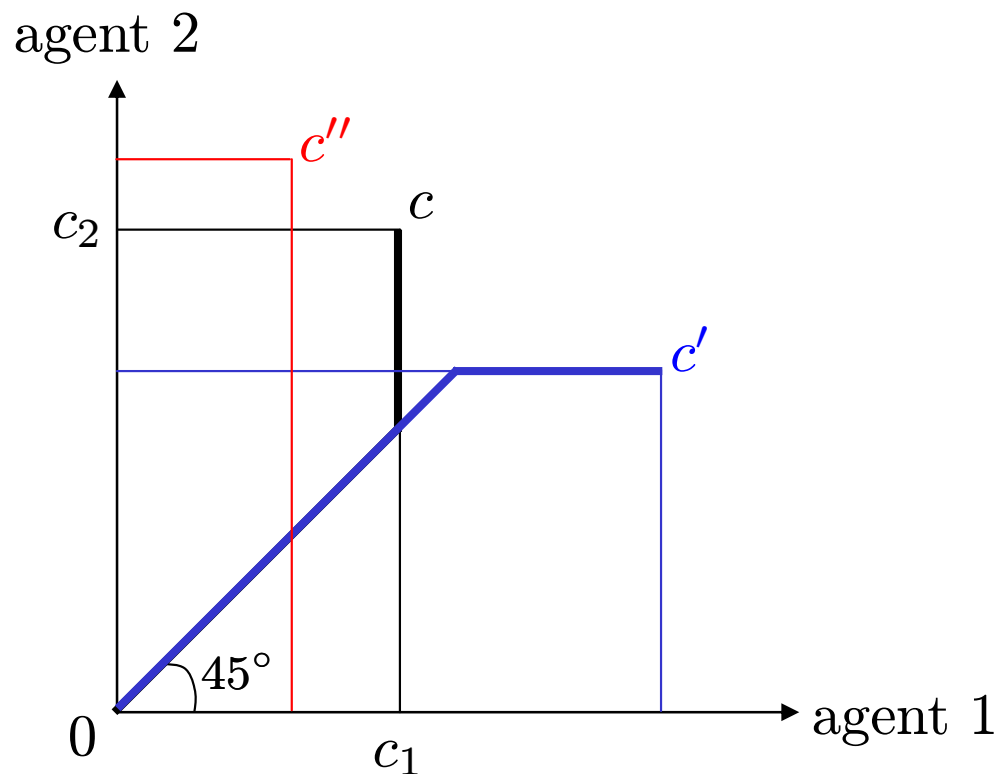
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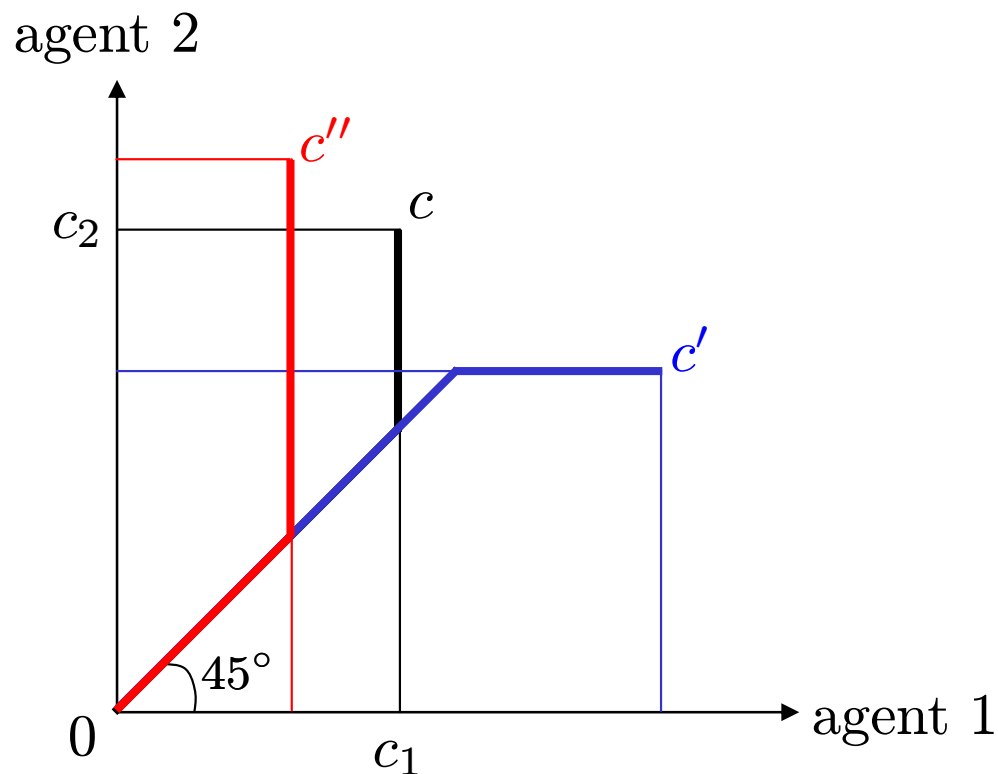
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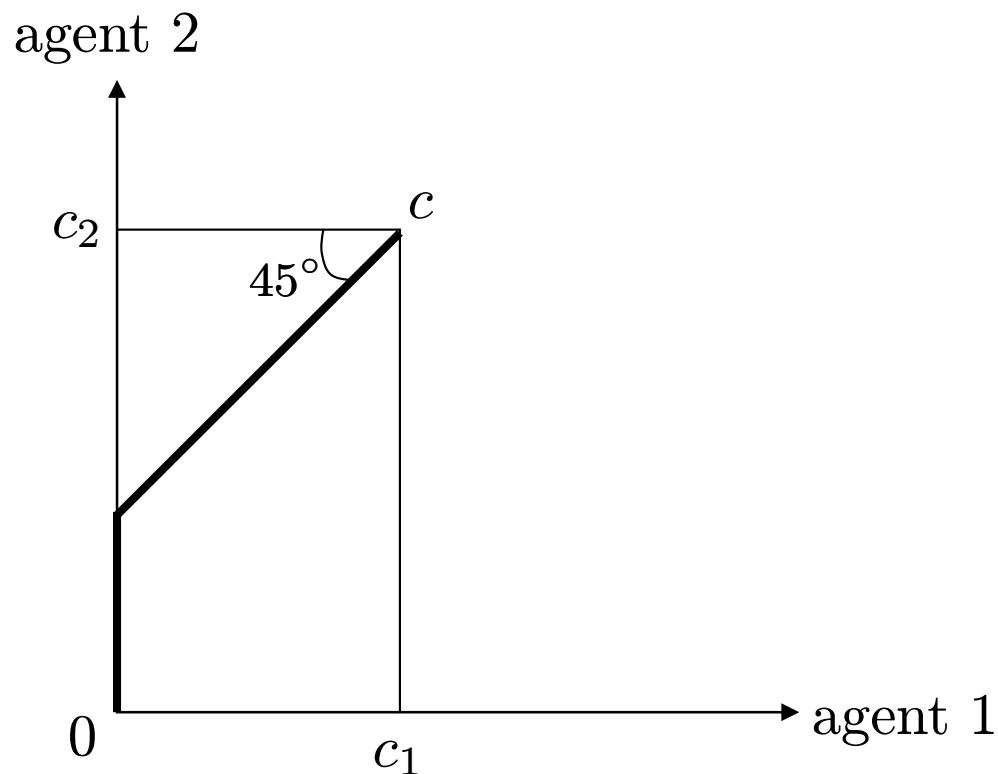
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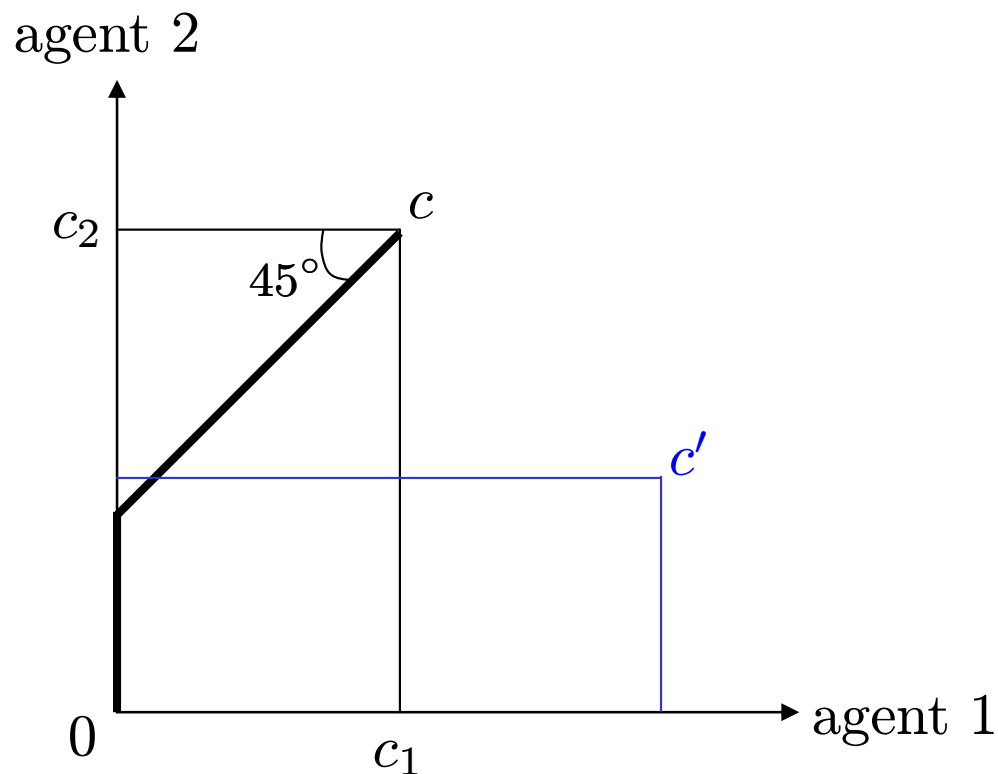
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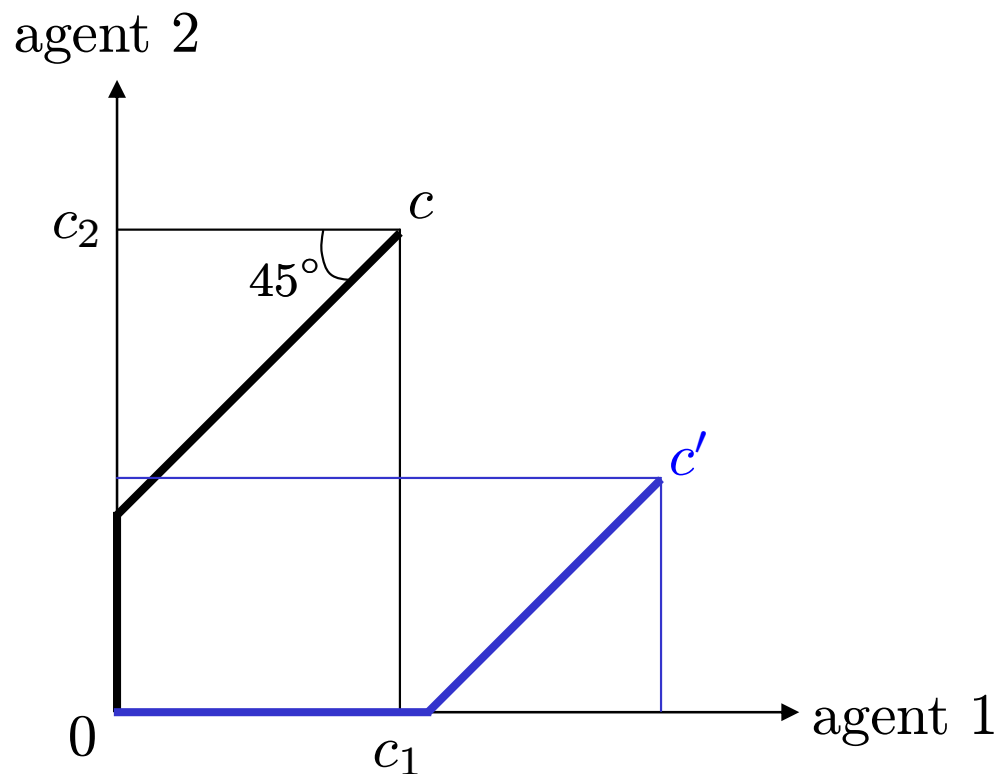
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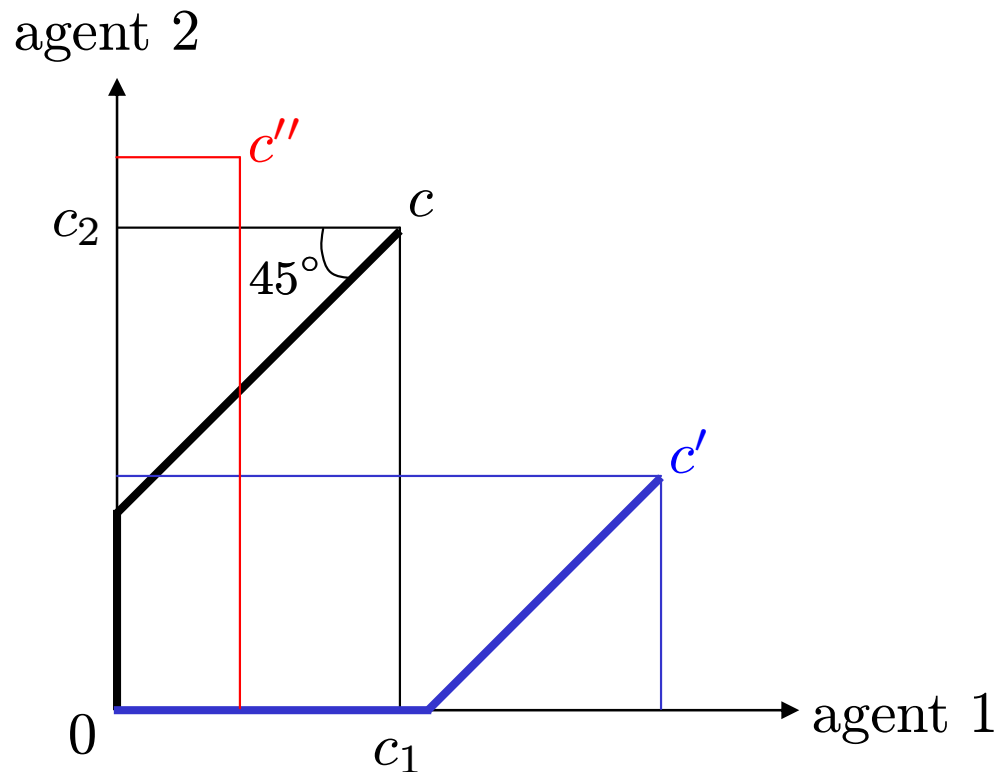
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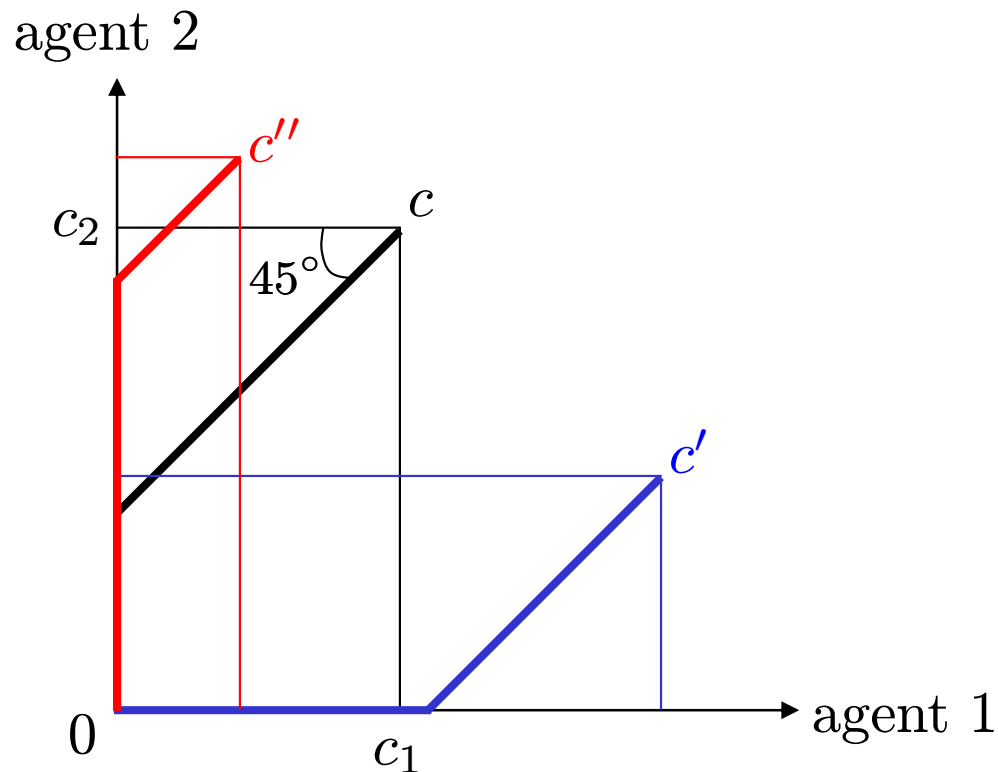
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**Constrained equal losses rule, CEL:**

$$CEL_i(c, E) = \max\{c_i - \lambda, 0\},$$

where  $\lambda \in \mathbb{R}_+$  is chosen such that  $\sum_{i \in N} CEL_i(c, E) = E$ .



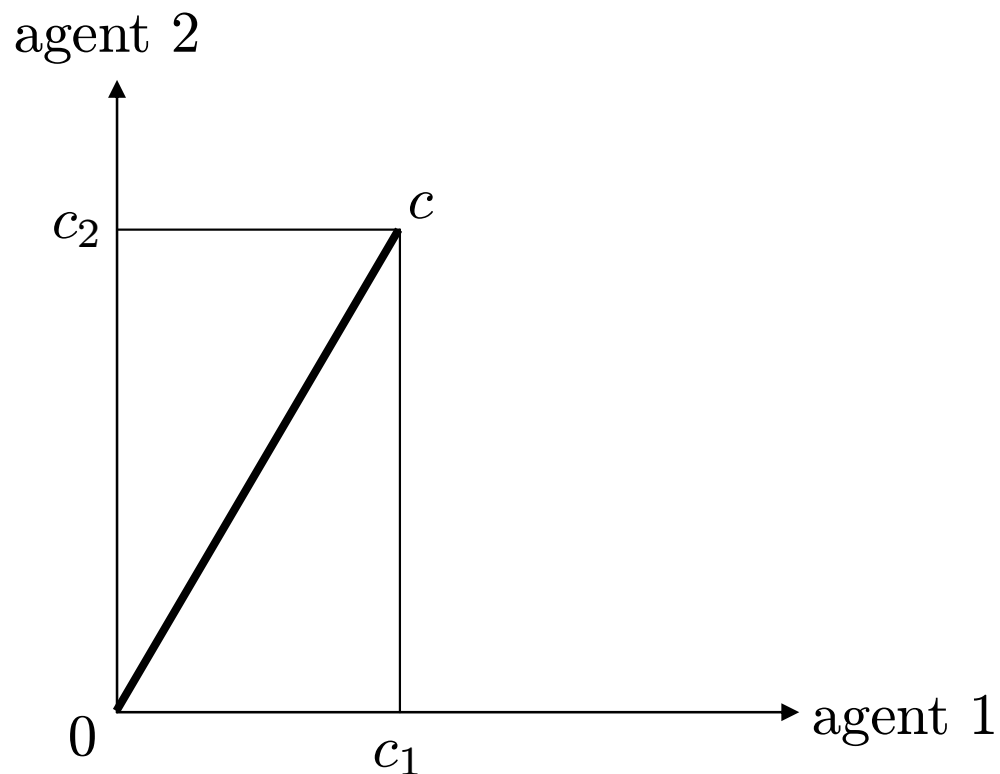
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**Proportional rule, P:**

$$P_i(c, E) = \lambda c_i,$$

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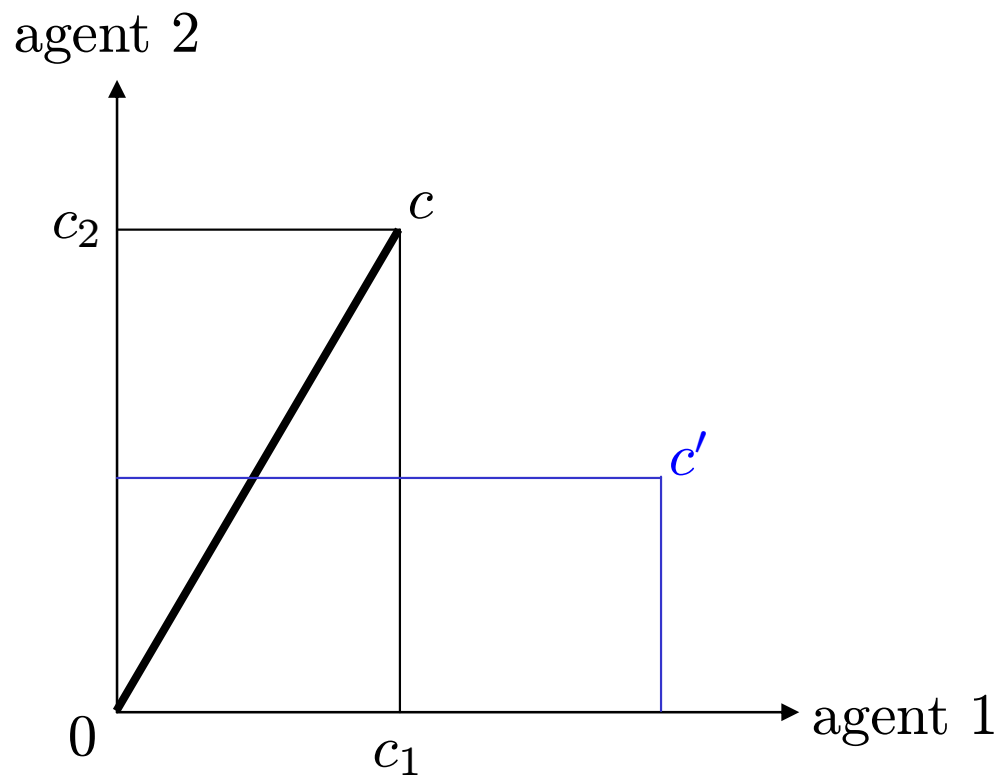
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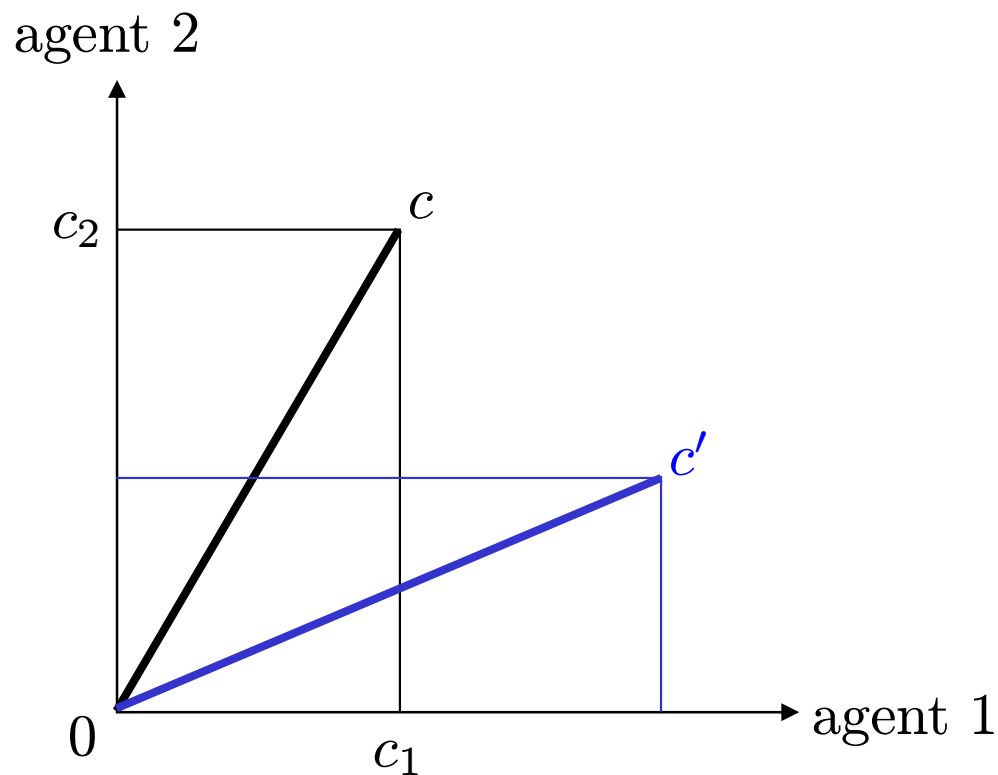
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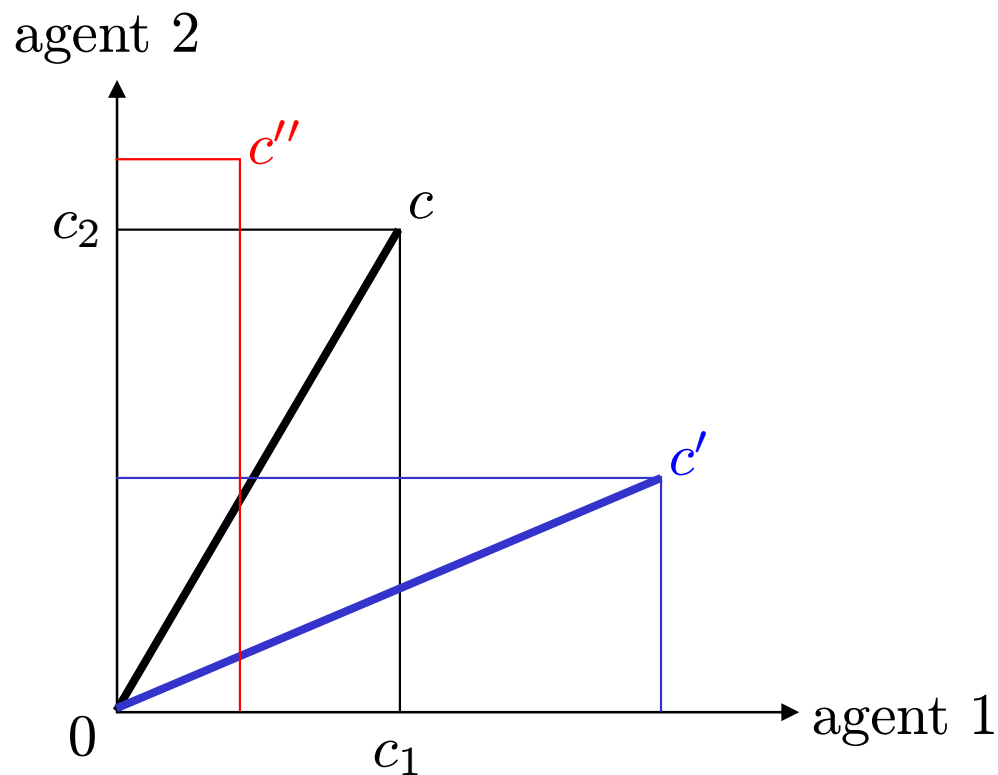
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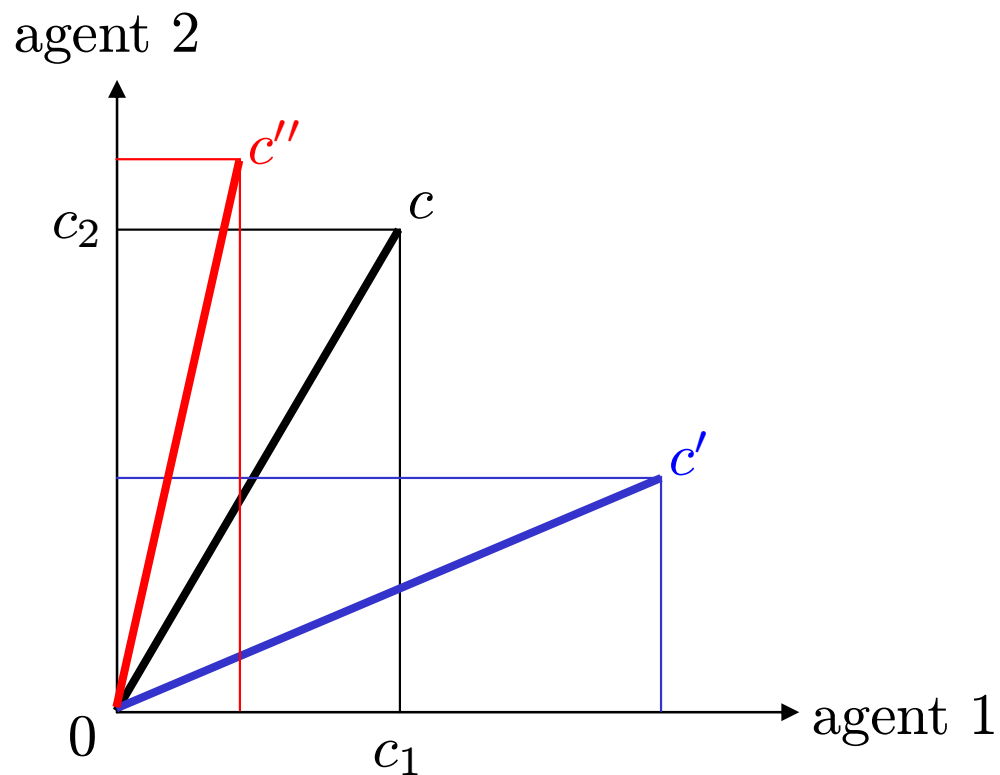
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# Side note: Examples from Talmud

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- **Example 1 (Contested garment problem):**

	$c_1 = \frac{1}{2}$	$c_2 = 1$
$E = 1$	$\frac{1}{4}$	$\frac{3}{4}$

- **Example 2 (Marriage contract problem):**

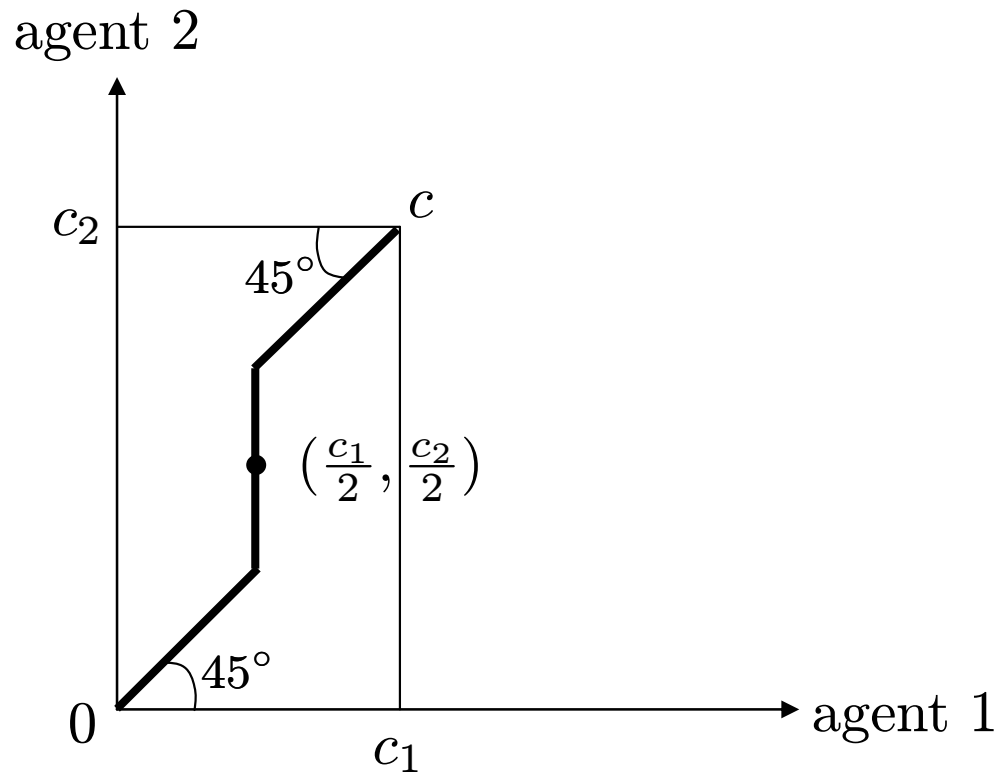
	$c_1 = 100$	$c_2 = 200$	$c_3 = 300$
$E = 100$	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
$E = 200$	50	75	75
$E = 300$	50	100	150

# Examples of rules

**Talmud rule** (Aumann and Maschler, 1985), **Tal**:

$$Tal_i(c, E) = \begin{cases} \min\{\frac{c_i}{2}, \lambda\} & \text{if } E \leq \sum_{j \in N} \frac{c_j}{2}, \\ c_i - \min\{\frac{c_i}{2}, \lambda\} & \text{otherwise,} \end{cases}$$

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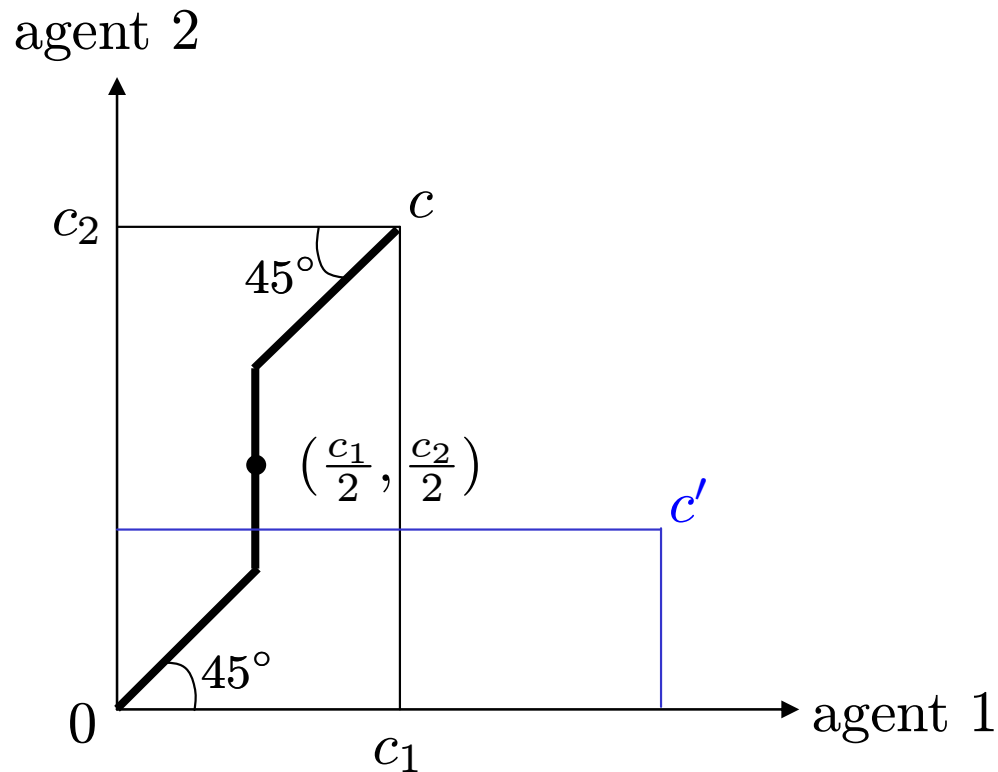


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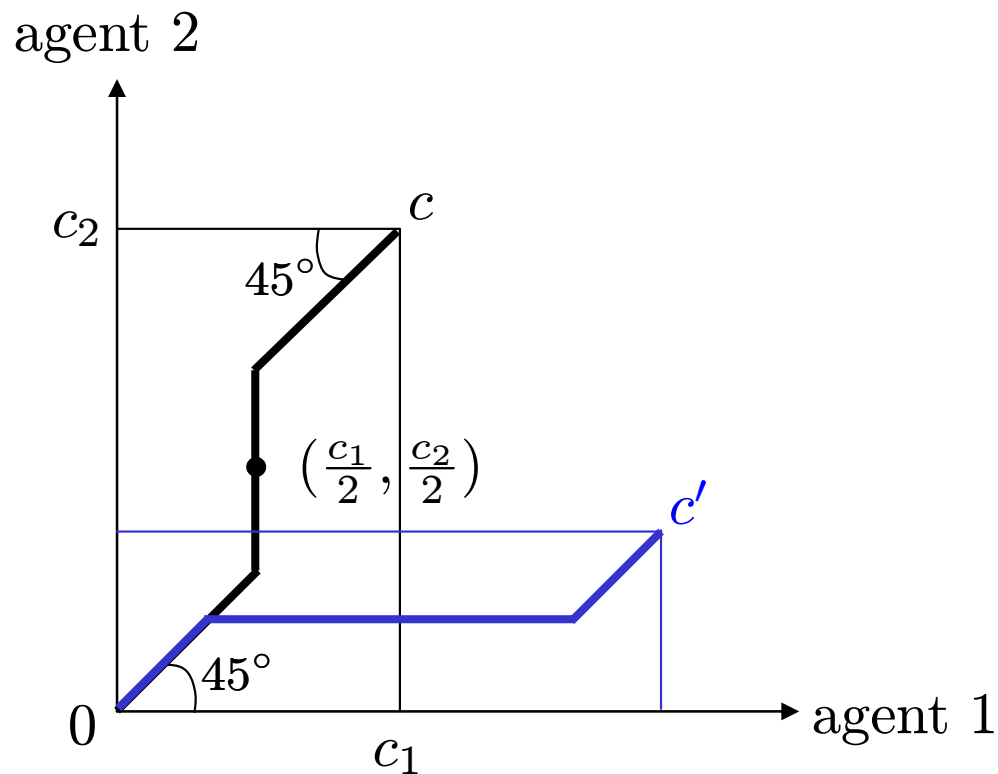


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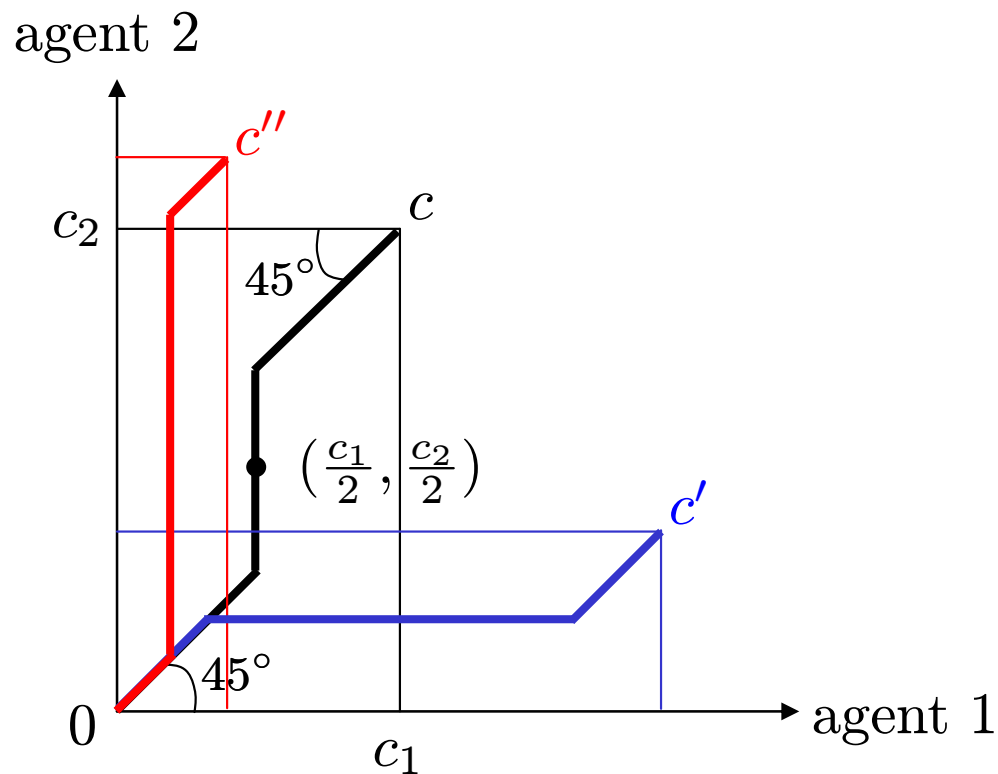


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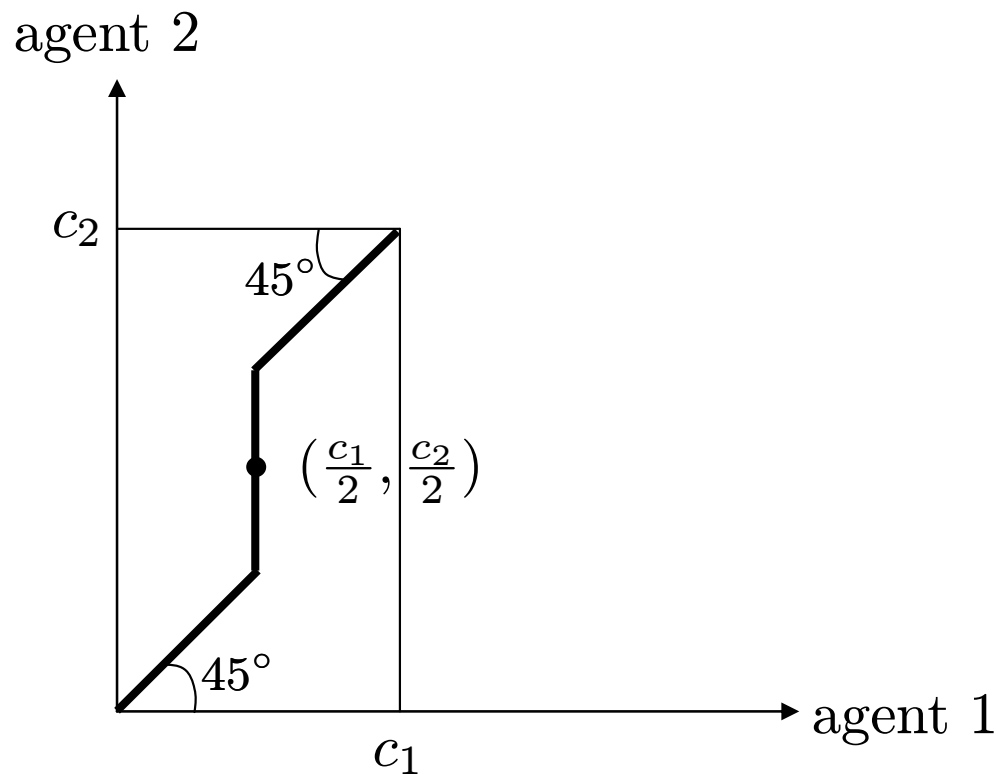


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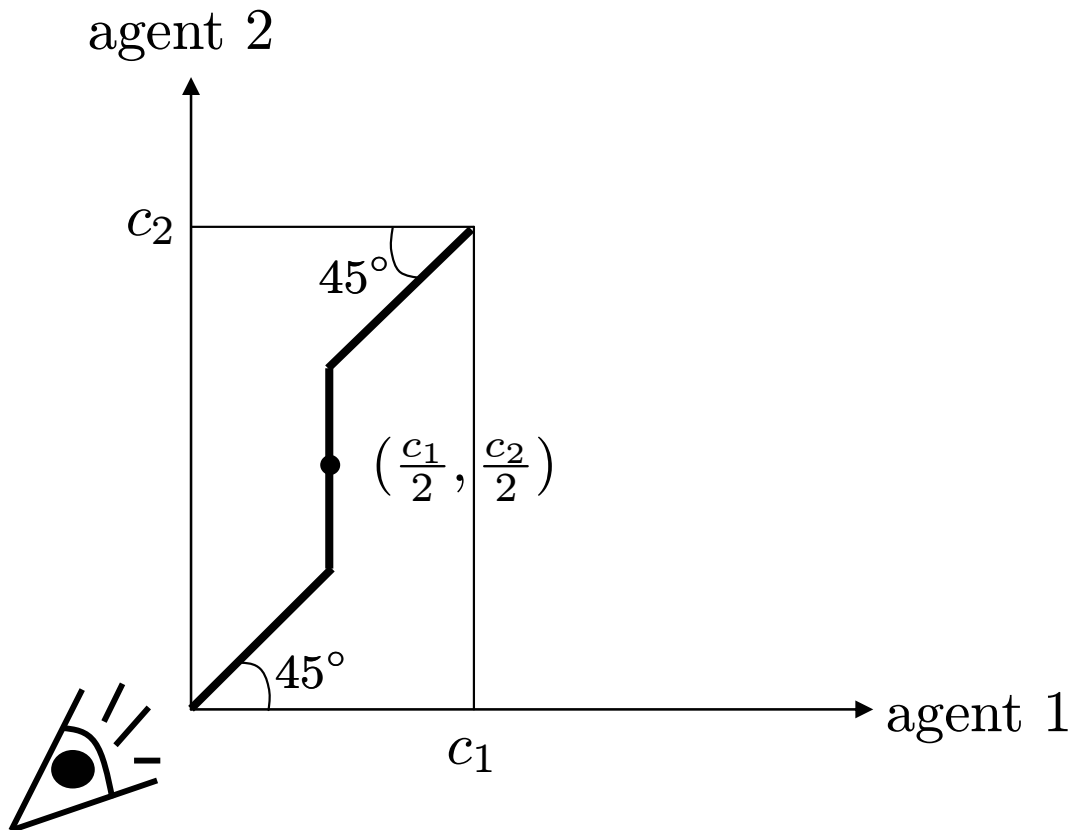


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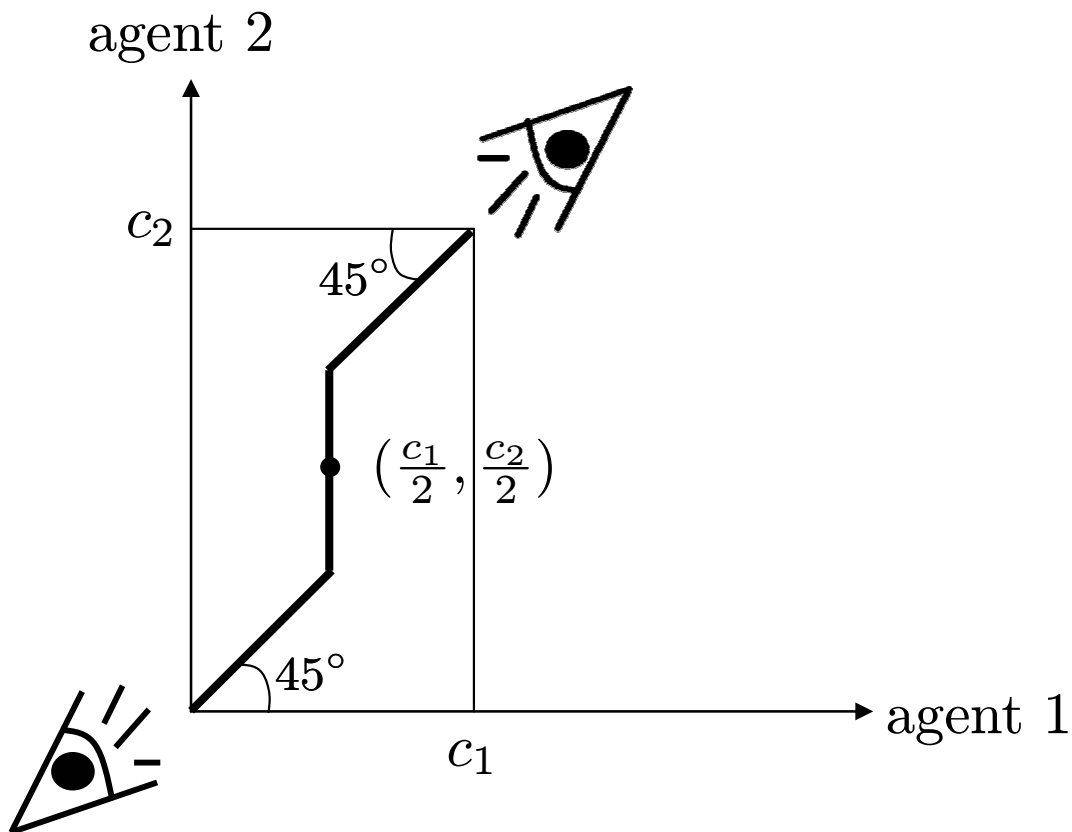


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# Our interest

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- “Reflecting inequality of claims in gains”
- “Reflecting inequality of claims in losses”

# Inequality comparison

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# Inequality comparison

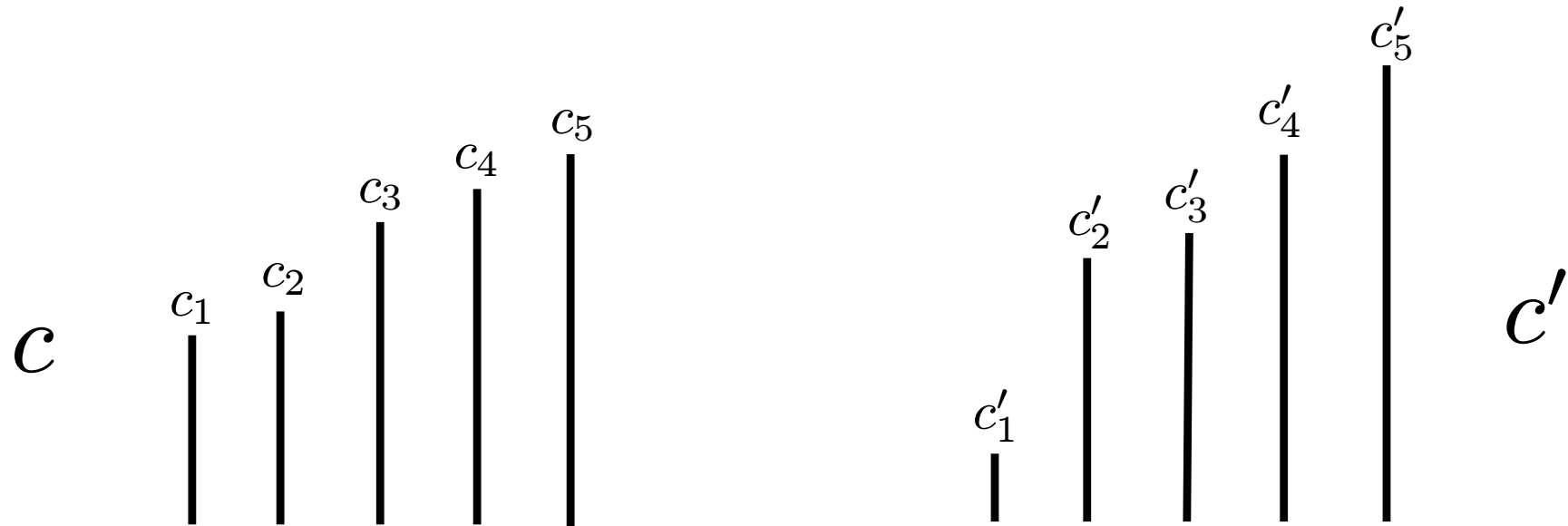
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- Inequality comparison: Lorenz criteria

# Inequality comparison

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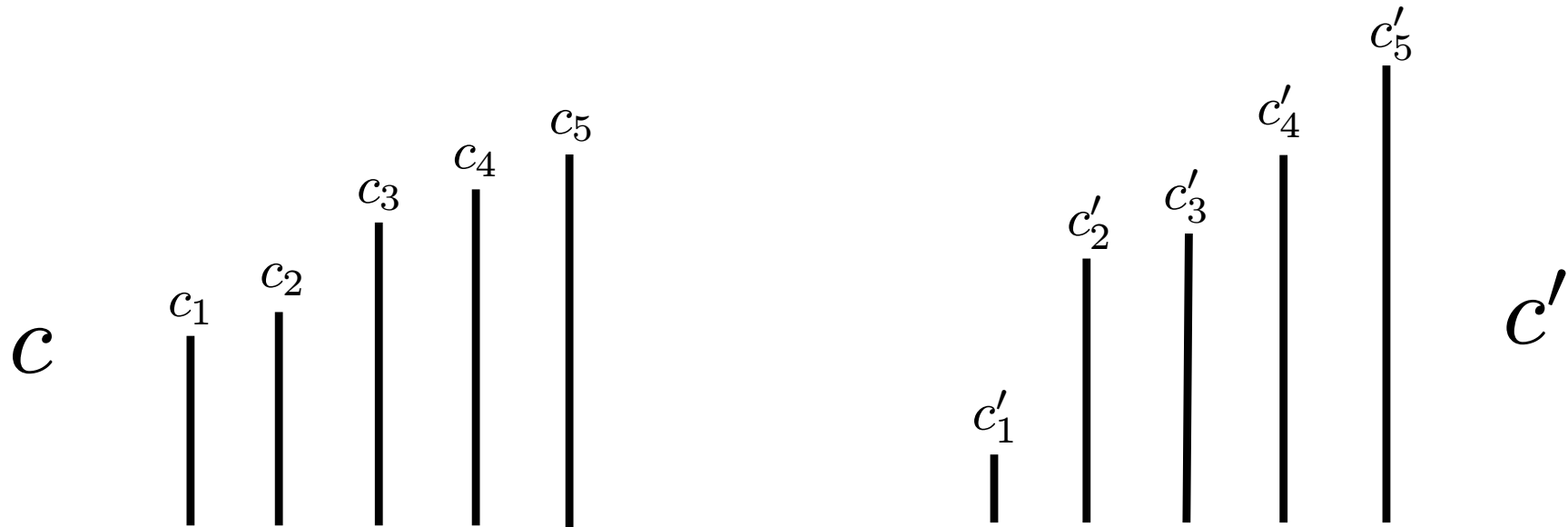
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# Inequality comparison

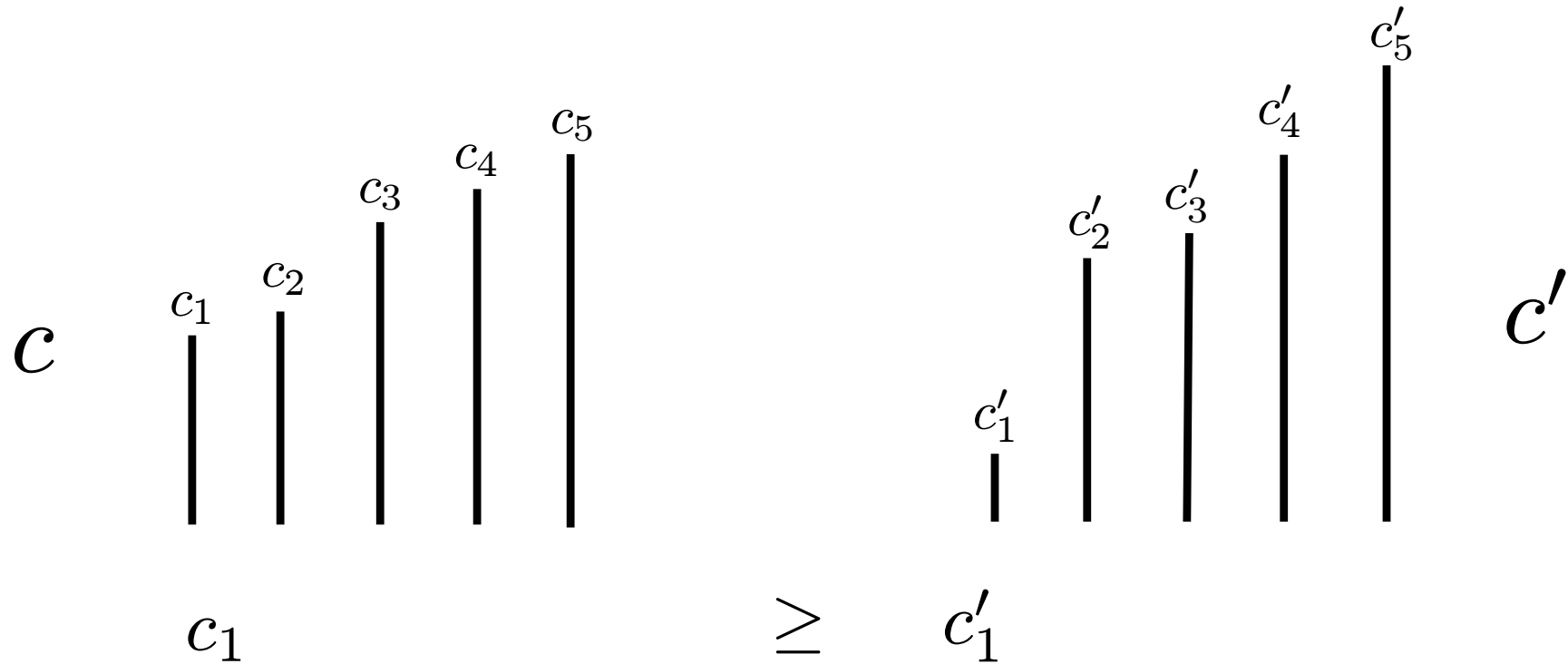
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- Inequality comparison: Lorenz criteria  $\sum c_i = \sum c'_i$



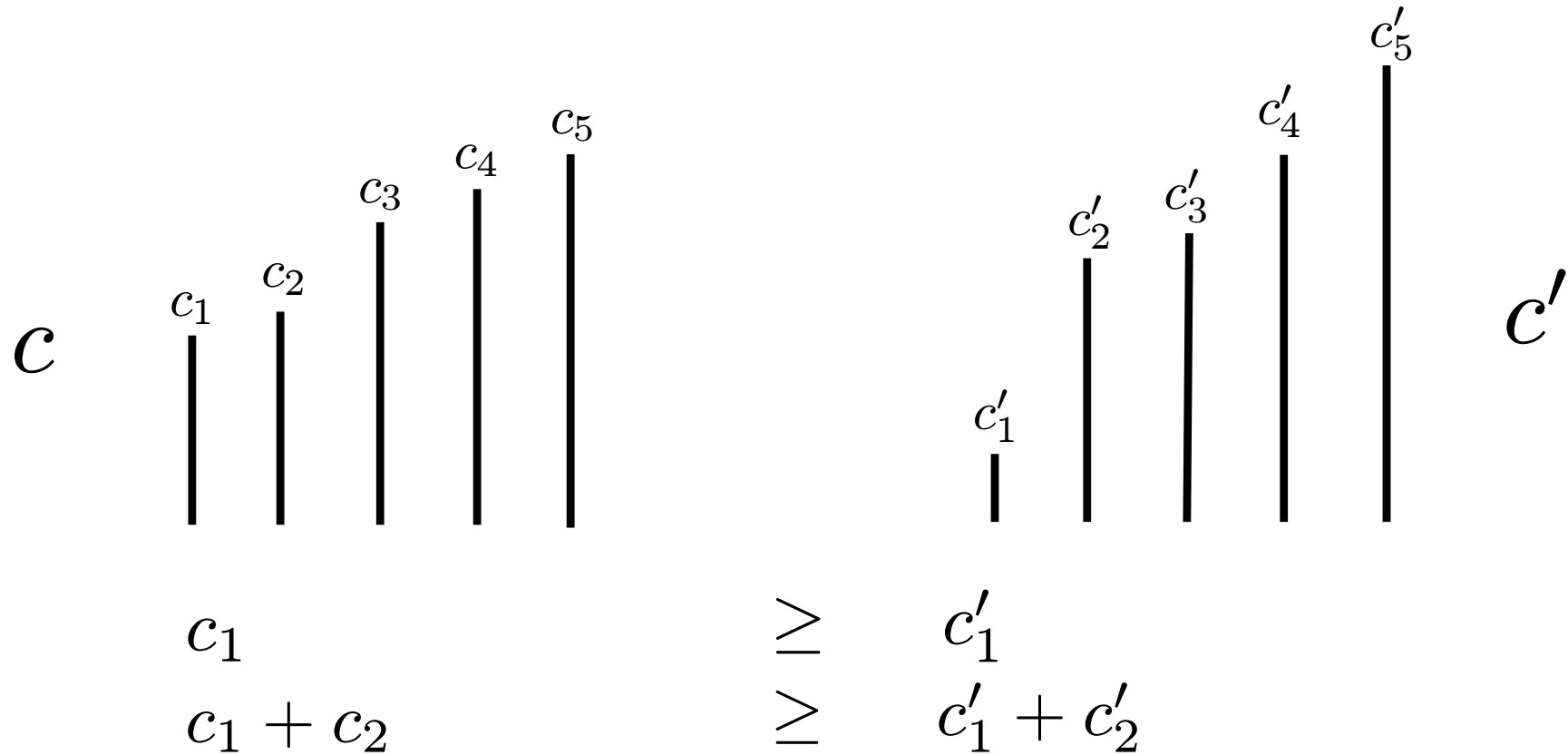
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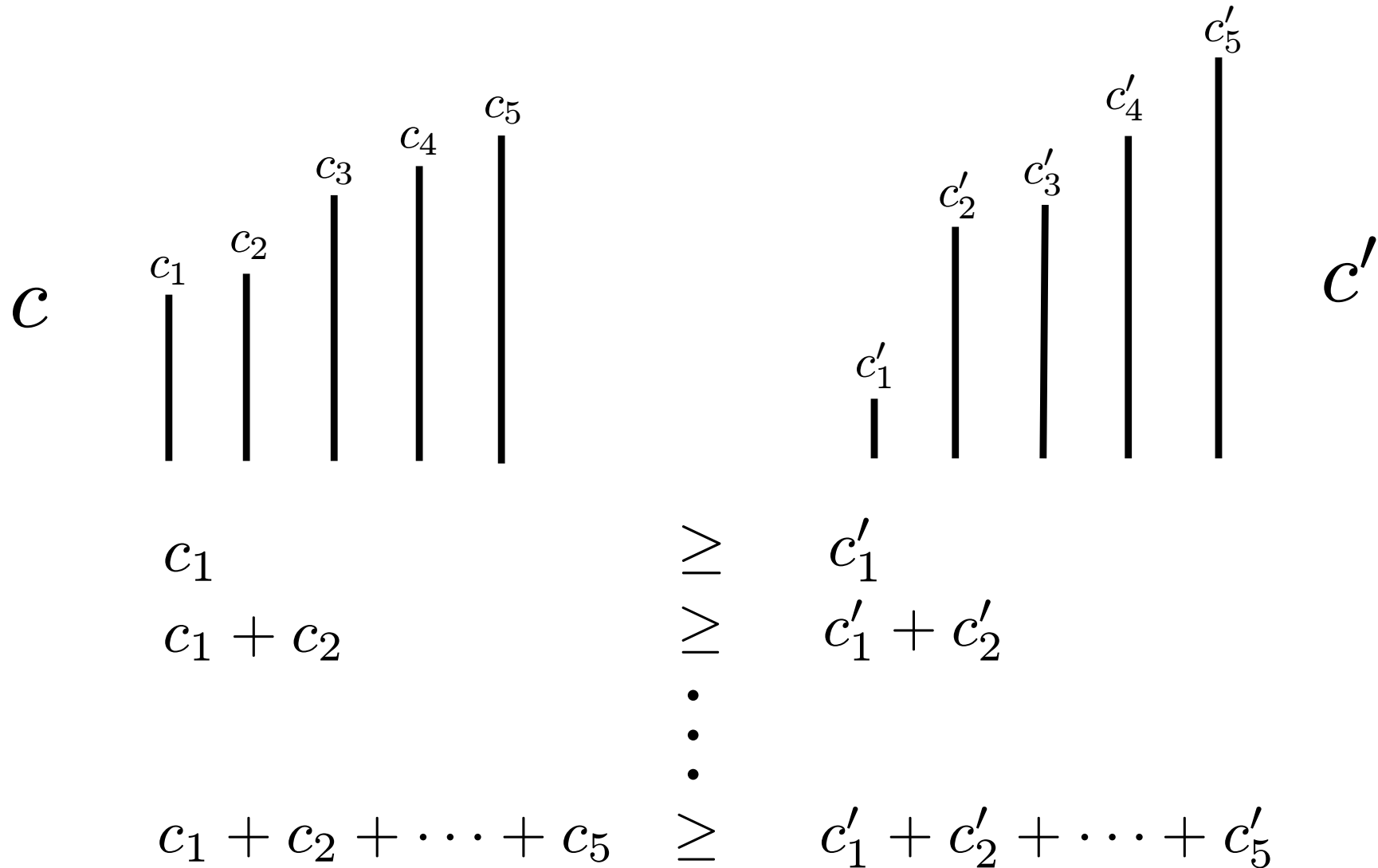
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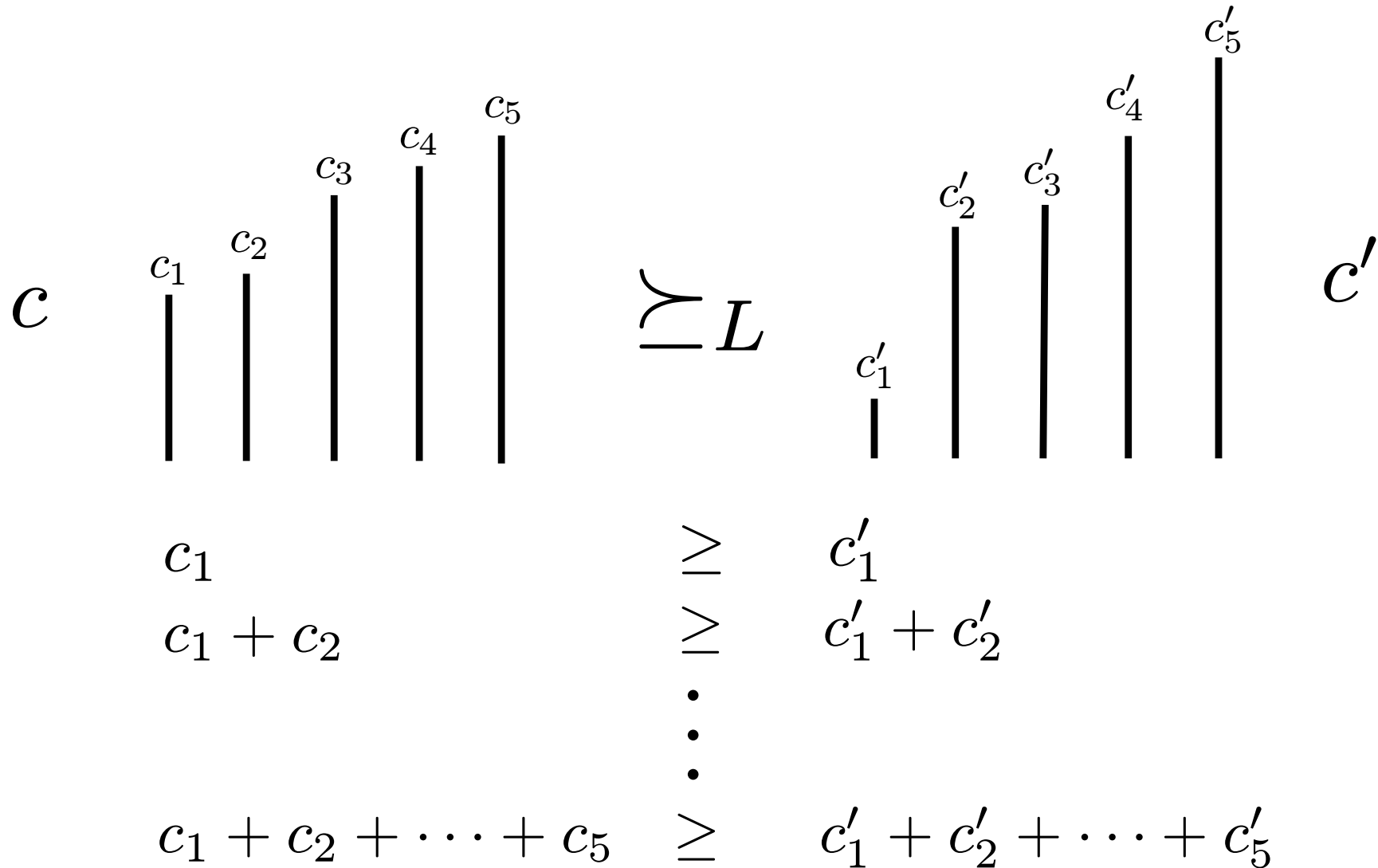
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# Inequality preservation in gains

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Hougaard and Thorlund-Petersen (2001), Hougaard and Østerdal (2005, EL)

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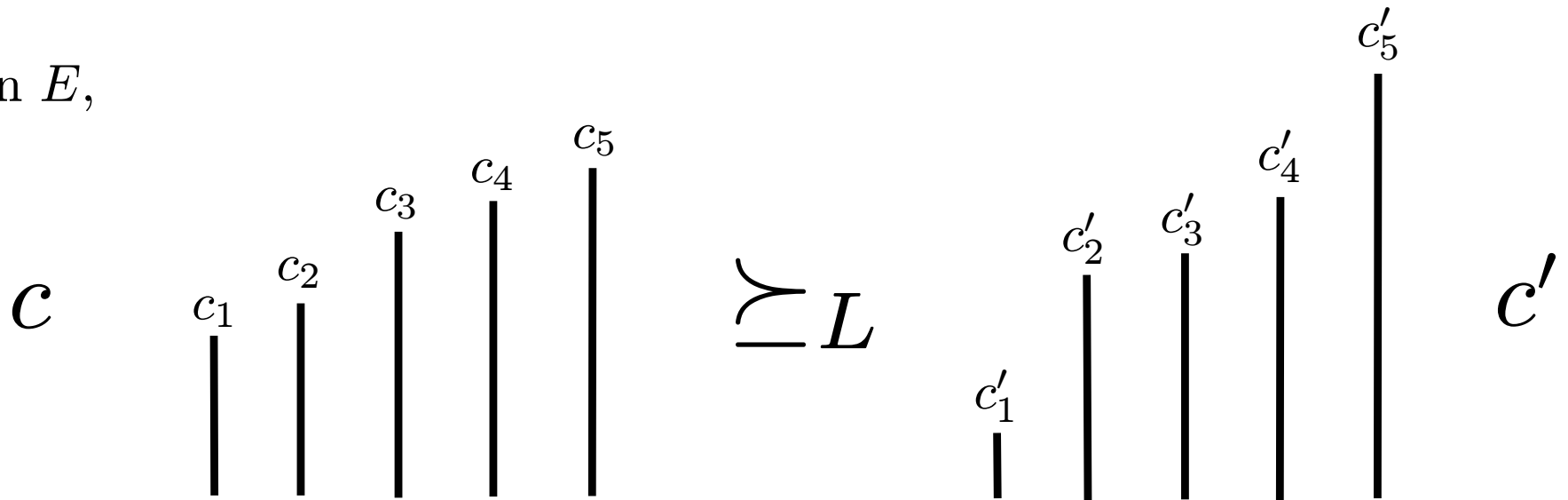
Given  $E$ ,

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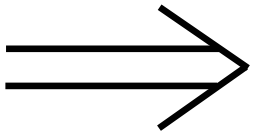
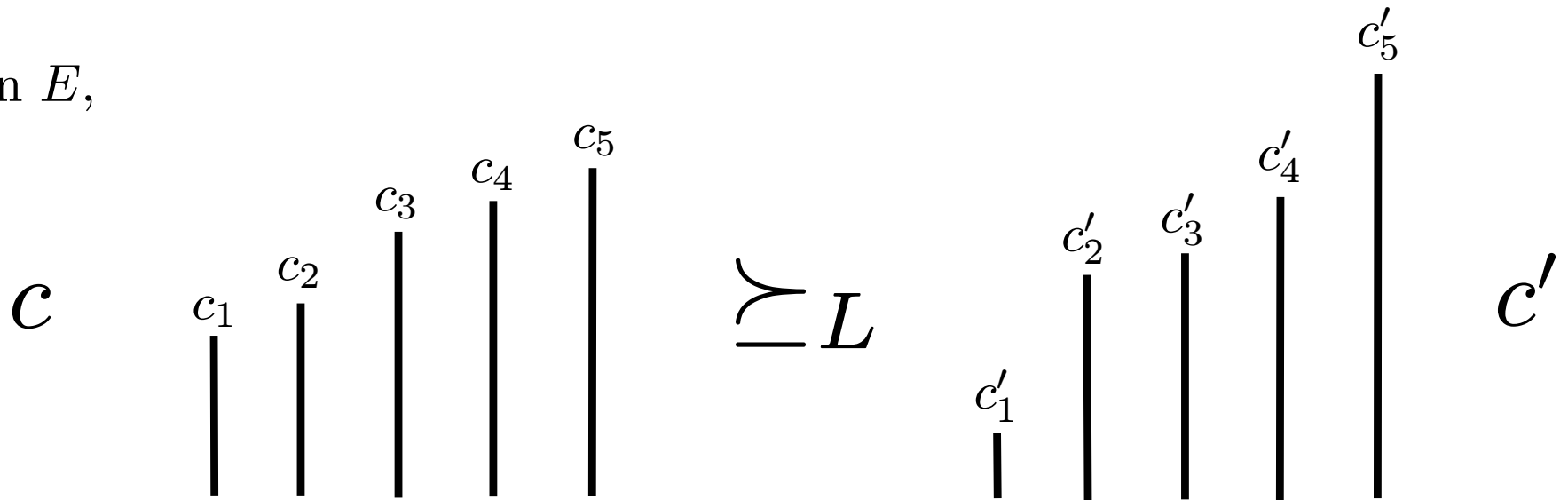
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Hougaard and Thorlund-Petersen (2001), Hougaard and Østerdal (2005, EL)

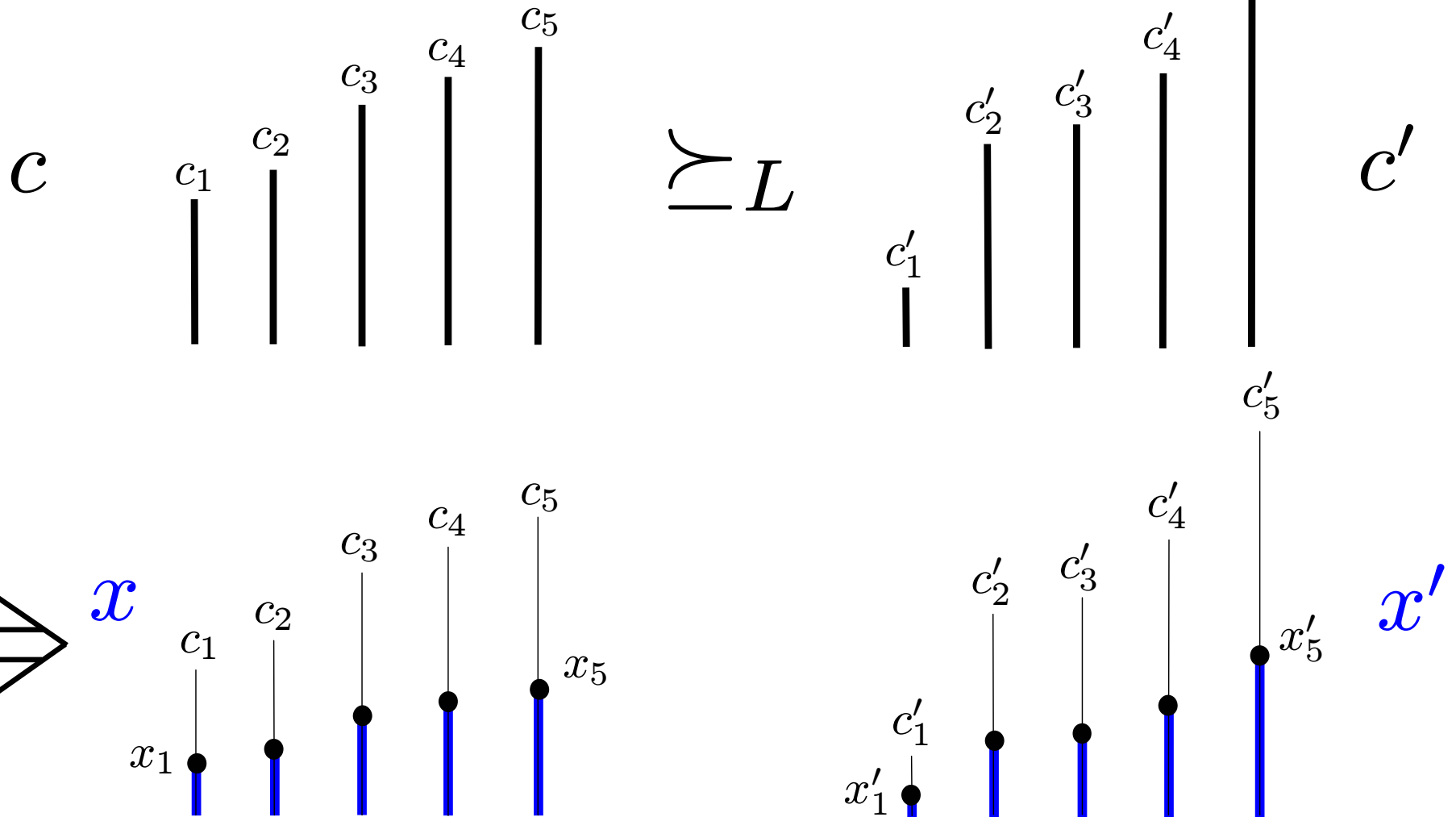
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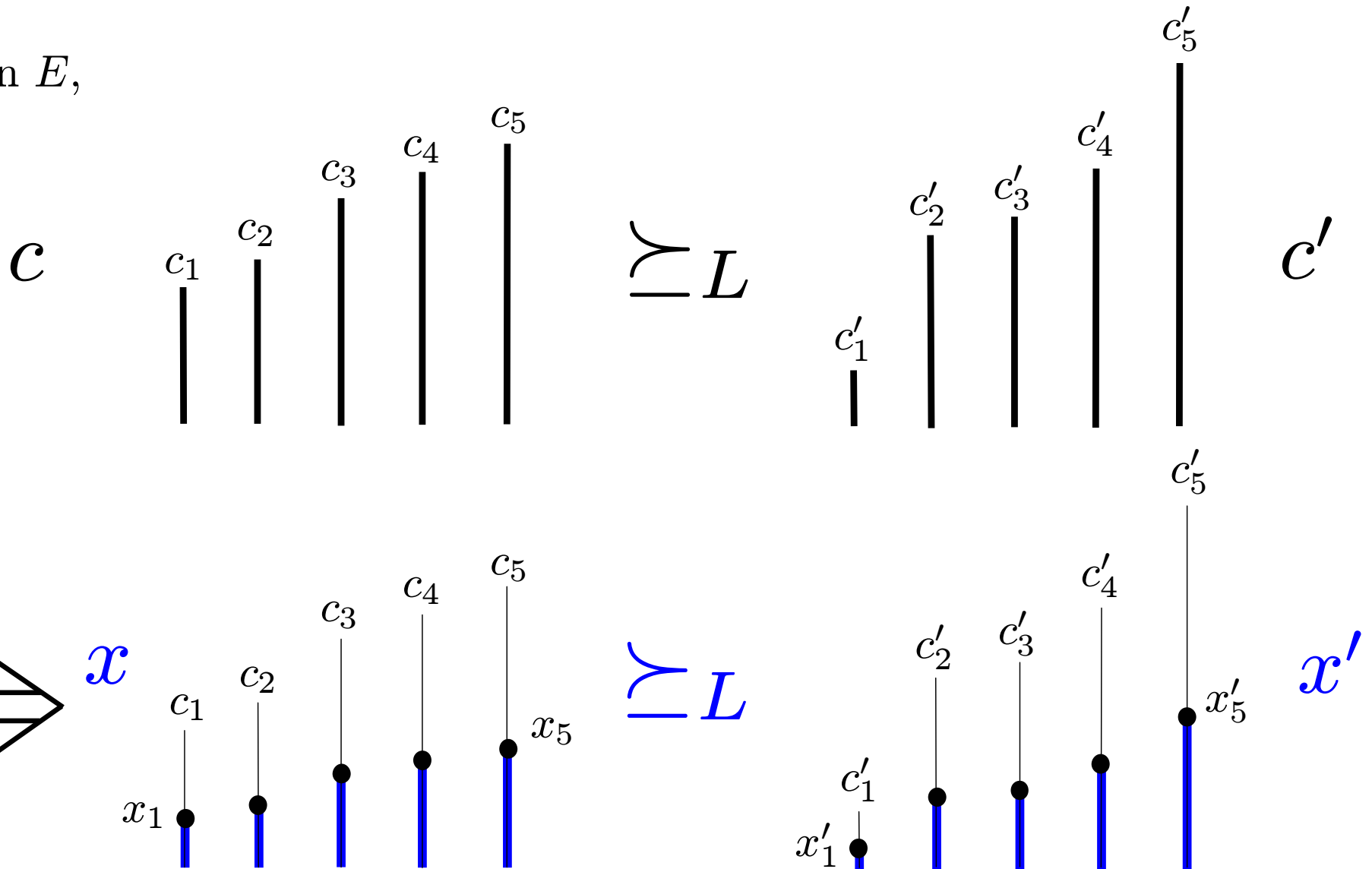
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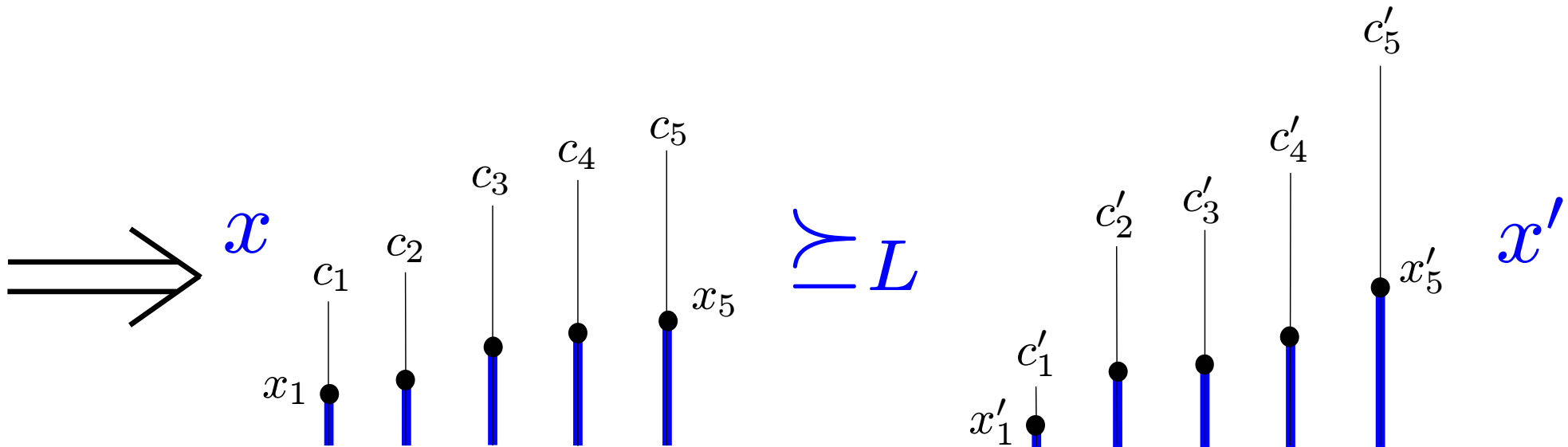
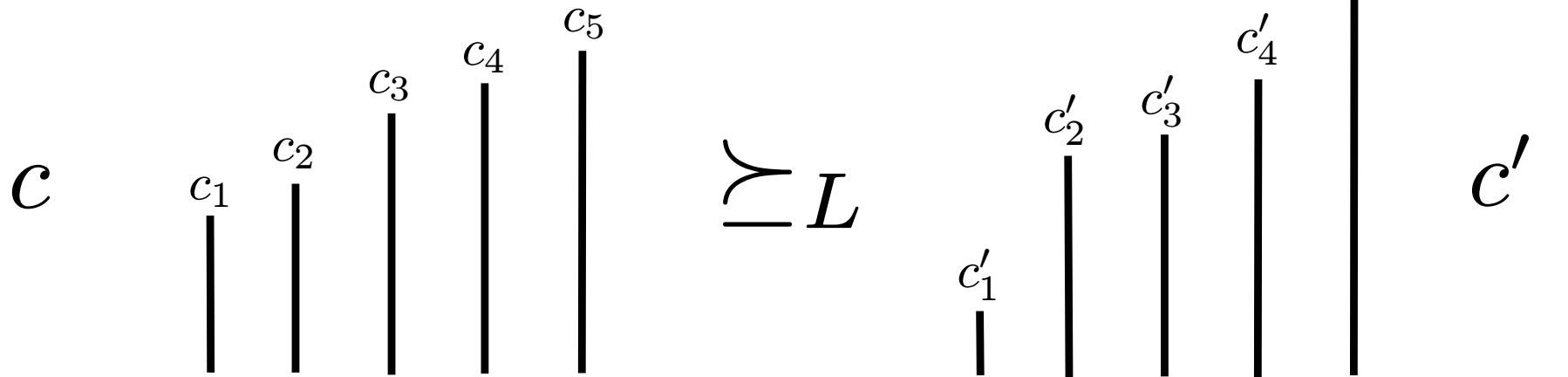
Given  $E$ ,



# Inequality preservation in gains

Hougaard and Thorlund-Petersen (2001), Hougaard and Østerdal (2005, EL)

Given  $E$ ,  $\leftarrow$  “ $\sum_{i \in N} c_i - T$ ”



# Inequality preservation in losses

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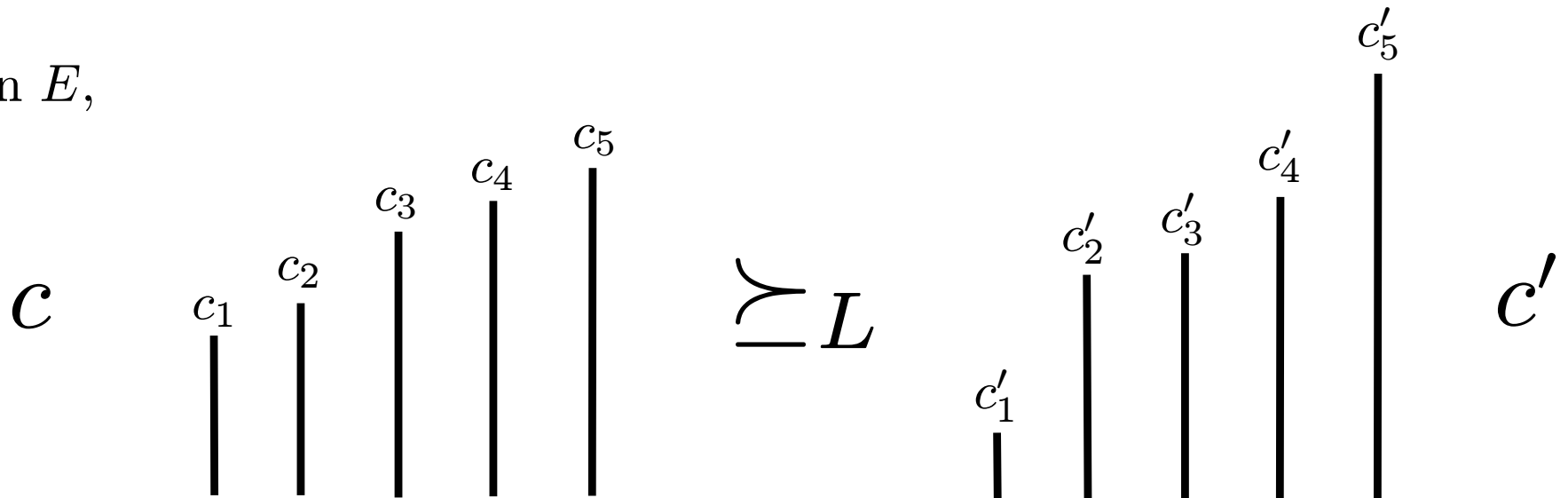
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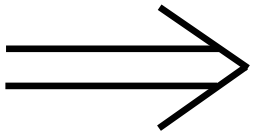
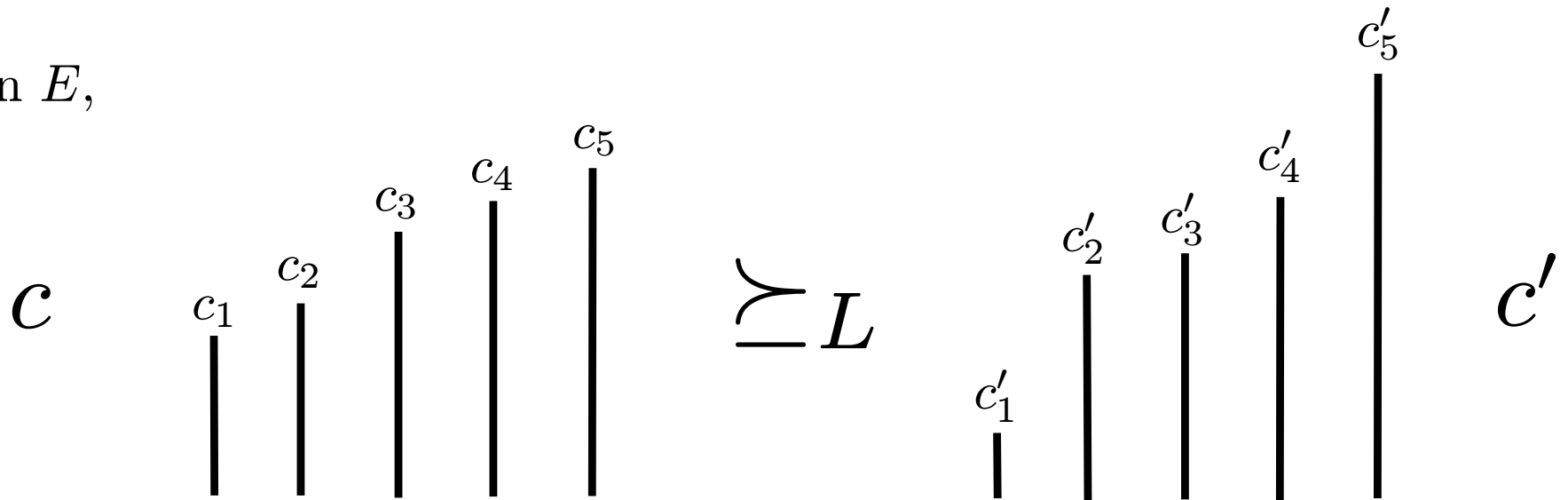
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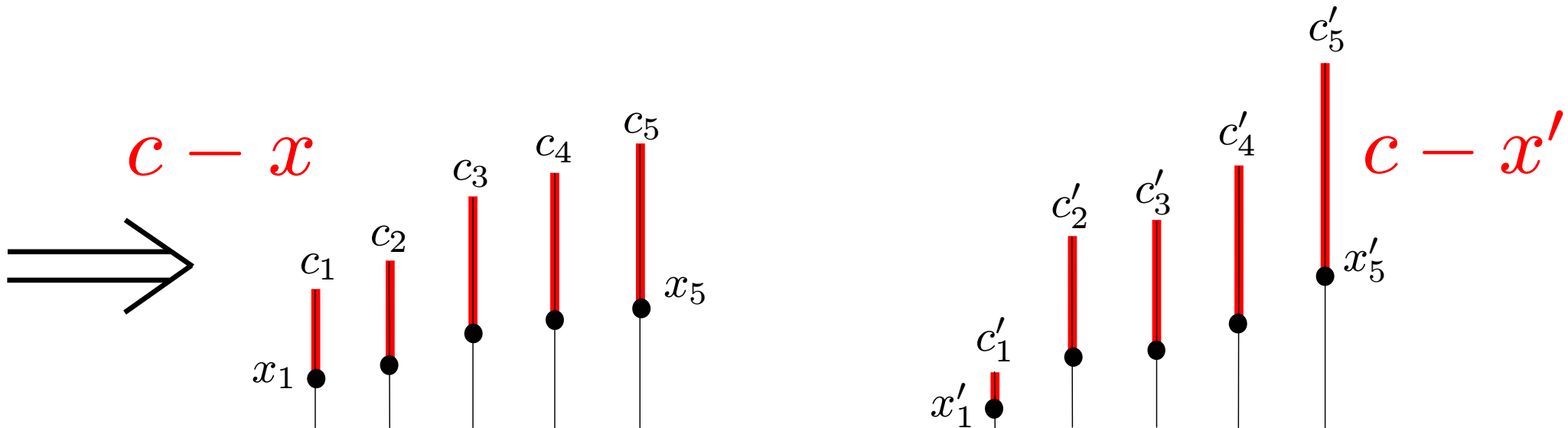
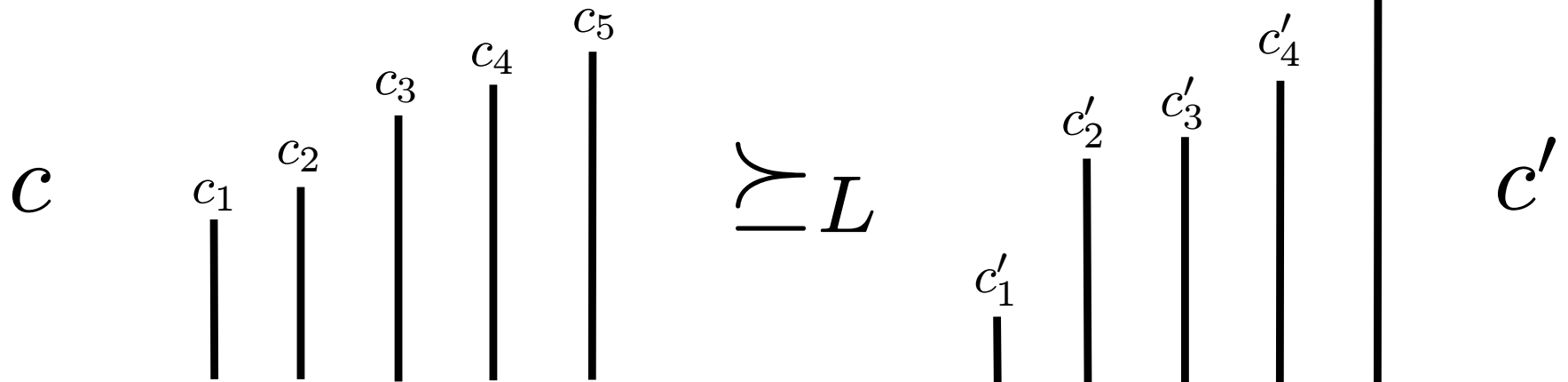
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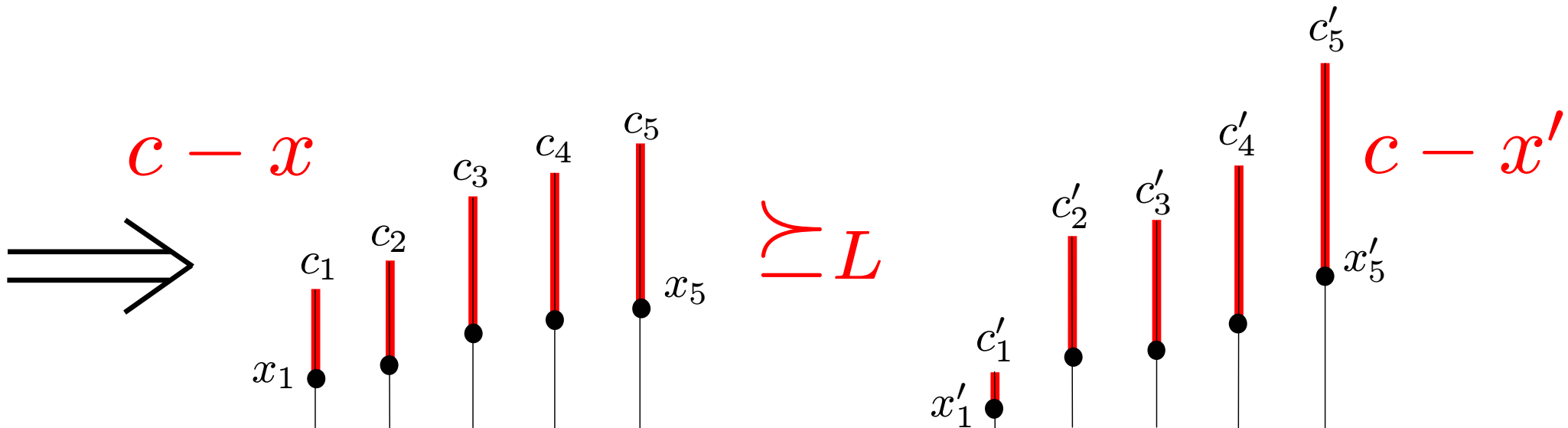
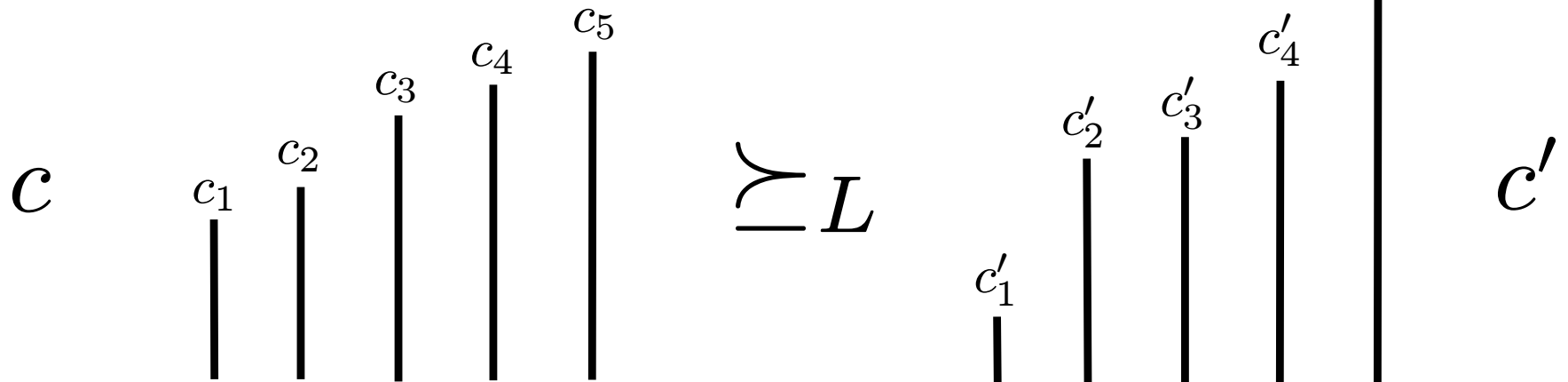
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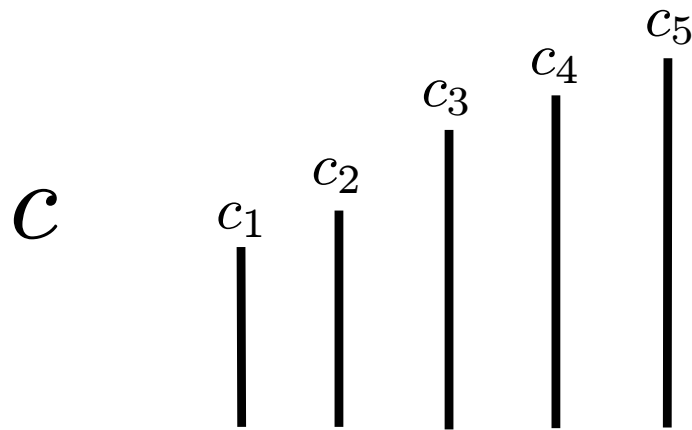
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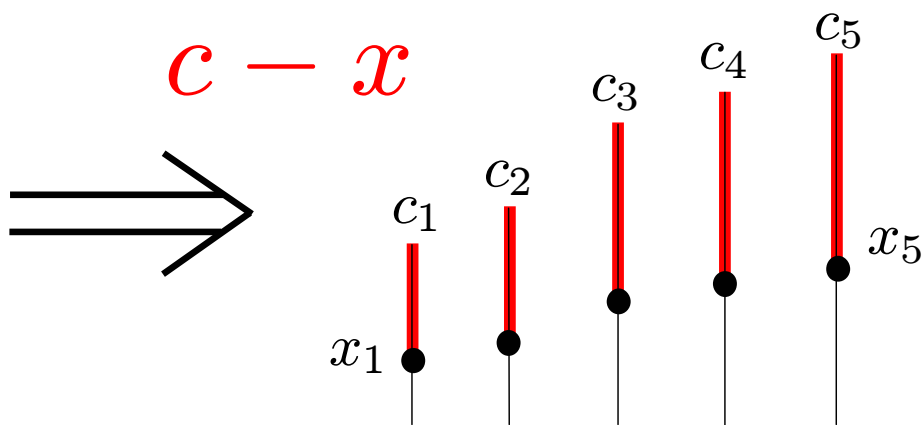
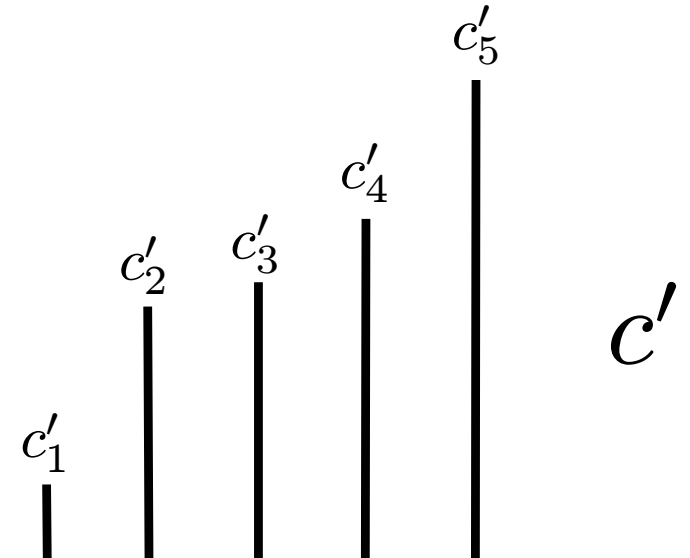
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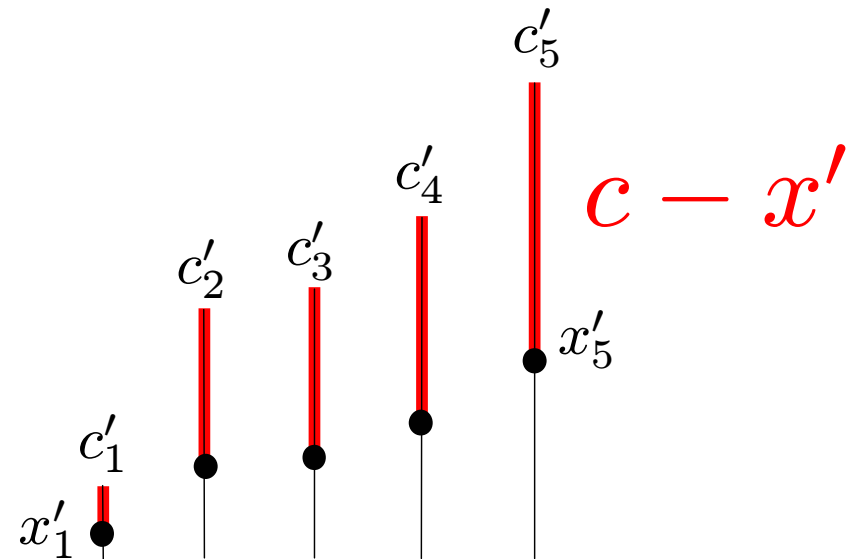
Given  $E$ , ← “ $\sum_{i \in N} c_i - T$ ”



$\succsim_L$



$\succsim_L$



# Axioms

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- Continuity (*cont*):

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- **Continuity (*cont*):**

For each convergent sequence of claim problems  $\{(c^k, E^k)\}_{k \in \mathbb{N}}$ ,

$$S(c^k, E^k) \xrightarrow[k \rightarrow \infty]{} S(\lim_{k \rightarrow \infty} (c^k, E^k)).$$

- **Order preservation in gains (*ord-pres-g*):**

For each  $(c, E) \in \mathcal{C}$  and each  $\{i, j\} \subseteq N$ ,

$$c_i \geq c_j \implies S_i(c, E) \geq S_j(c, E).$$

- **Order preservation in losses (*ord-pres-l*):**

For each  $(c, E) \in \mathcal{C}$  and each  $\{i, j\} \subseteq N$ ,

$$c_i \geq c_j \implies c_i - S_i(c, E) \geq c_j - S_j(c, E).$$

# Overview of results

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# Overview of results

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$S$  satisfying *cont*, *ord-pres-g*, *ord-pres-l*, *ineq-pres-g*, *ineq-pres-l* ?

# Overview of results

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$S$  satisfying  $cont$ ,  $ord-pres-g$ ,  $ord-pres-l$ ,  $ineq-pres-g$ ,  $ineq-pres-l$  ?

- $n = 2$ :

- $n = 3$ :

- $n > 3$ :

# Overview of results

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- $n = 2$ :

# Overview of results

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- $n = 2$ :

$$S: \text{cont}, \begin{array}{cc} \text{ord-pres-g} & \text{ineq-pres-g} \\ \text{ord-pres-l} & \text{ineq-pres-l} \end{array} \iff$$

$S \in$  “Minimal award family (for the two-agent case)”.

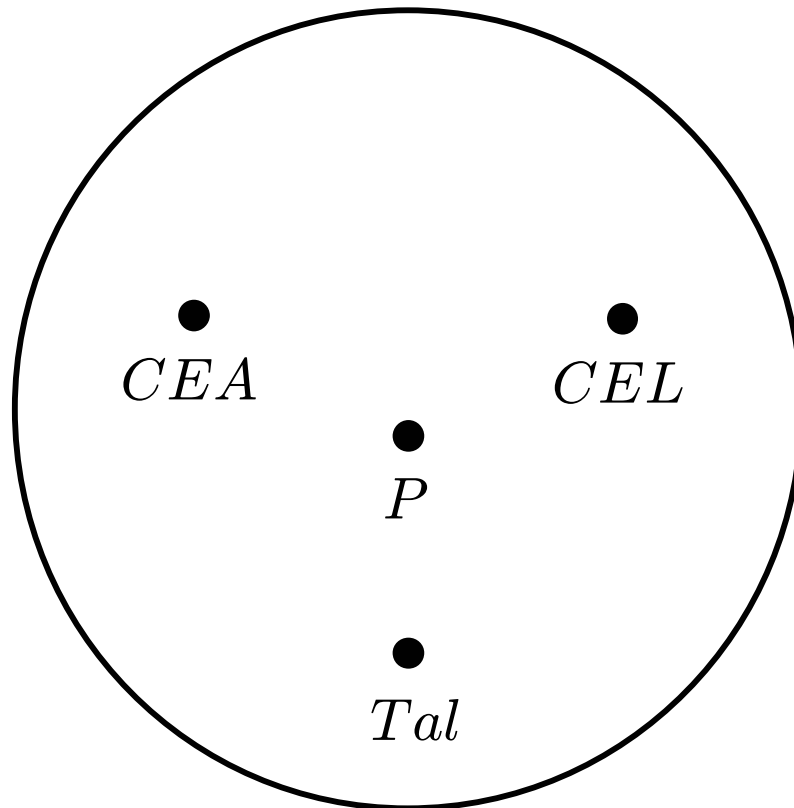
# Overview of results

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- $n = 2$ :

$$S: \text{cont}, \begin{array}{cc} \text{ord-pres-g} & \text{ineq-pres-g} \\ \text{ord-pres-l} & \text{ineq-pres-l} \end{array} \iff$$

$S \in$  “Minimal award family (for the two-agent case)”.



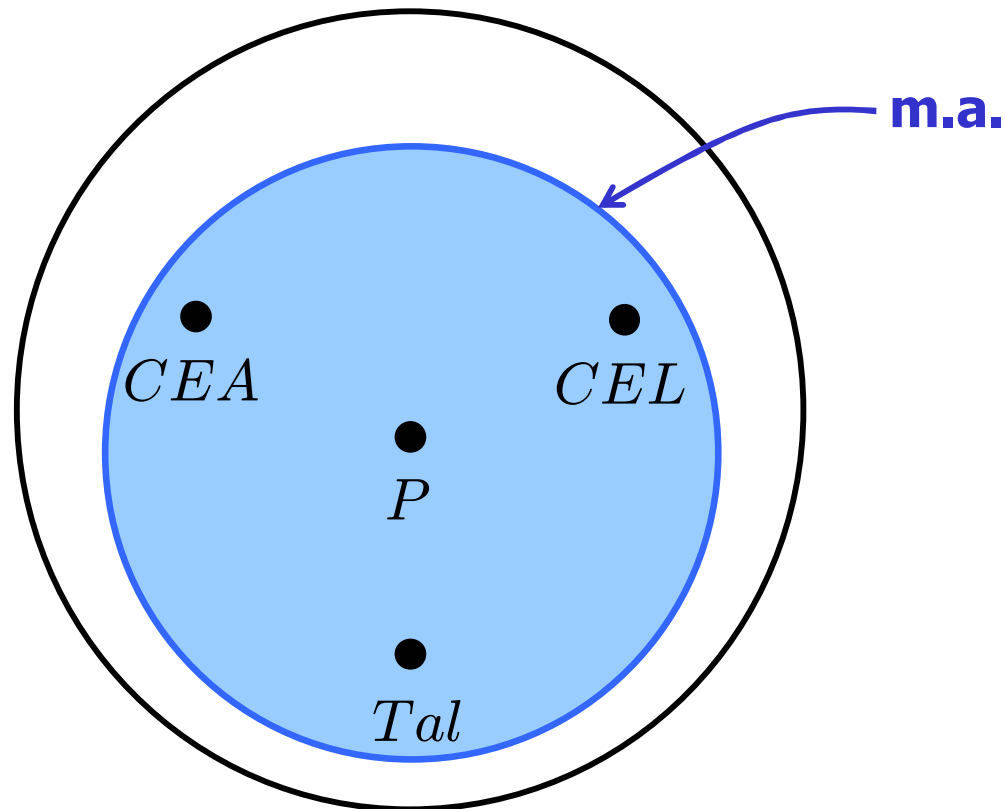
# Overview of results

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- $n = 2$ :

$$S: \text{cont}, \begin{array}{cc} \text{ord-pres-g} & \text{ineq-pres-g} \\ \text{ord-pres-l} & \text{ineq-pres-l} \end{array} \iff$$

$S \in$  “Minimal award family (for the two-agent case)”.

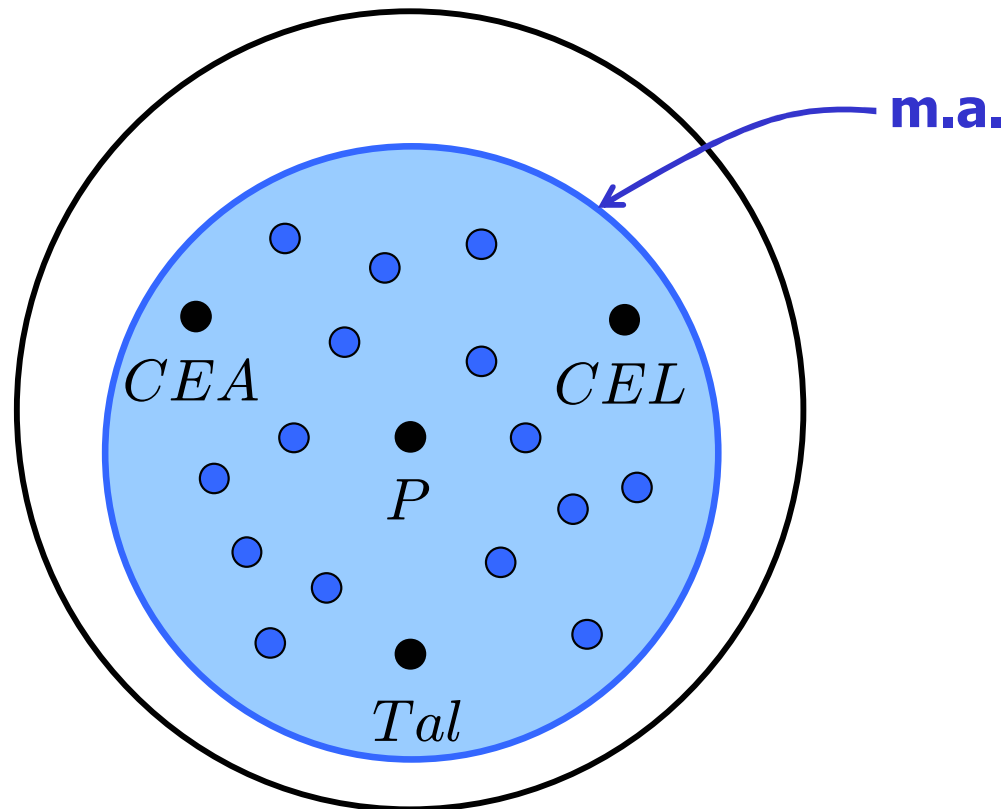


# Overview of results

- $n = 2$ :

$$S: \text{cont}, \begin{matrix} \text{ord-pres-g} & \text{ineq-pres-g} \\ \text{ord-pres-l} & \text{ineq-pres-l} \end{matrix} \iff$$

$S \in$  “Minimal award family (for the two-agent case)”.



# Overview of results

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- $n = 3$ :

# Overview of results

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- $n = 3$ :

$$S: \text{cont}, \begin{array}{cc} \text{ord-pres-g} & \text{ineq-pres-g} \\ \text{ord-pres-l} & \text{ineq-pres-l} \end{array} \iff$$

$S \in$  “Minimal award family (for the three-agent case)”.

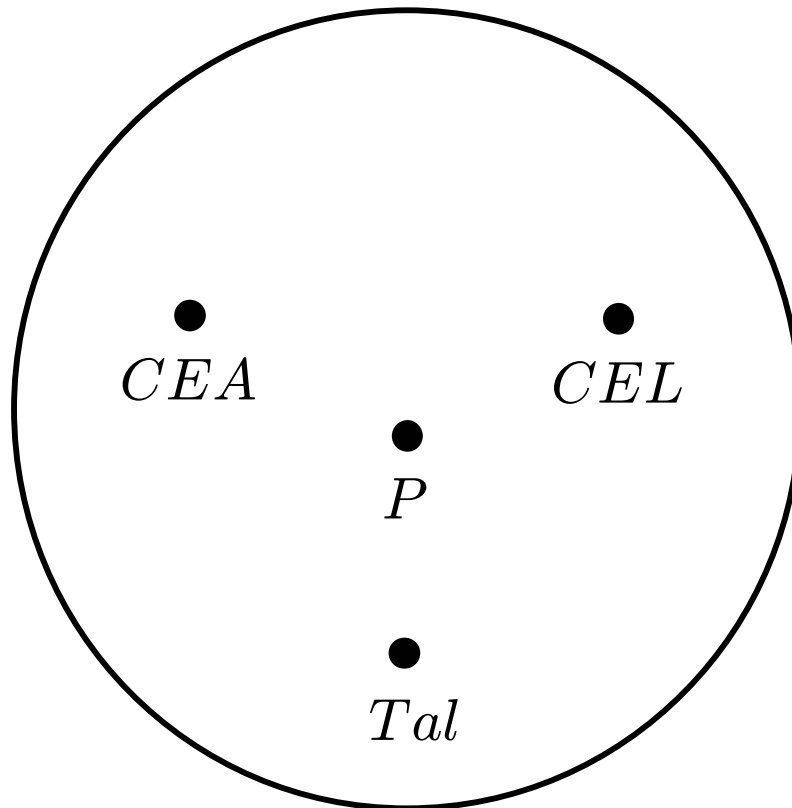
# Overview of results

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- $n = 3$ :

$S: cont, \begin{matrix} ord-pres-g & ineq-pres-g \\ ord-pres-l & ineq-pres-l \end{matrix} \iff$

$S \in$  “Minimal award family (for the three-agent case)”.



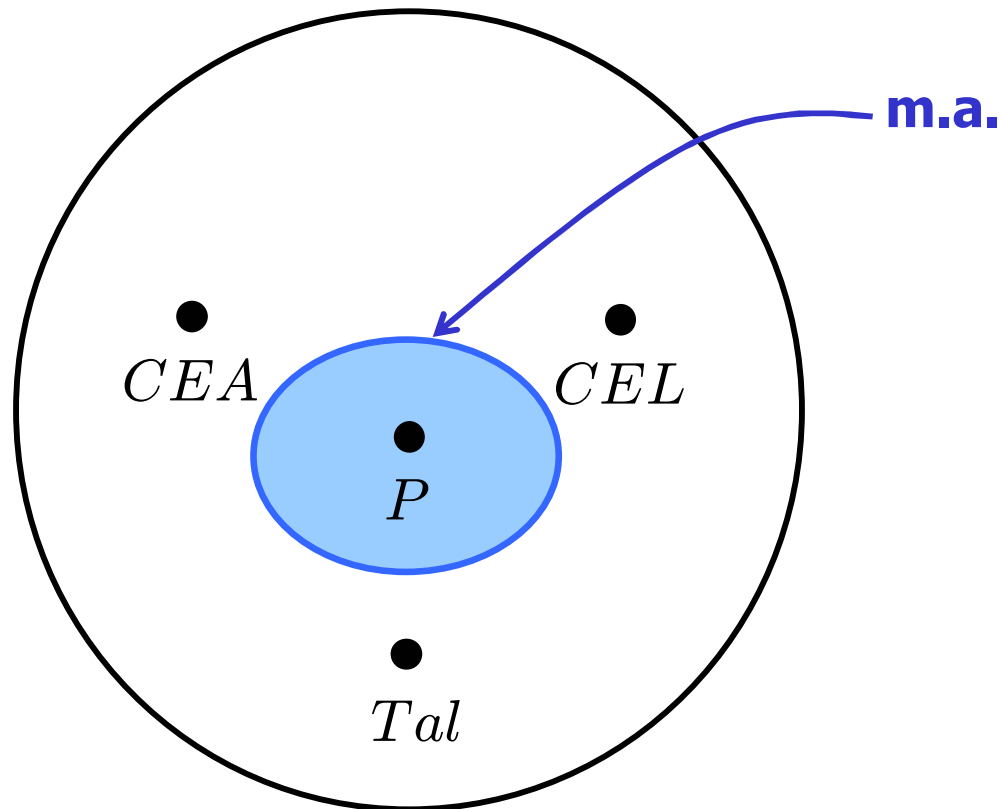
# Overview of results

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- $n = 3$ :

$$S: \text{cont}, \begin{array}{cc} \text{ord-pres-g} & \text{ineq-pres-g} \\ \text{ord-pres-l} & \text{ineq-pres-l} \end{array} \iff$$

$S \in$  “Minimal award family (for the three-agent case)”.

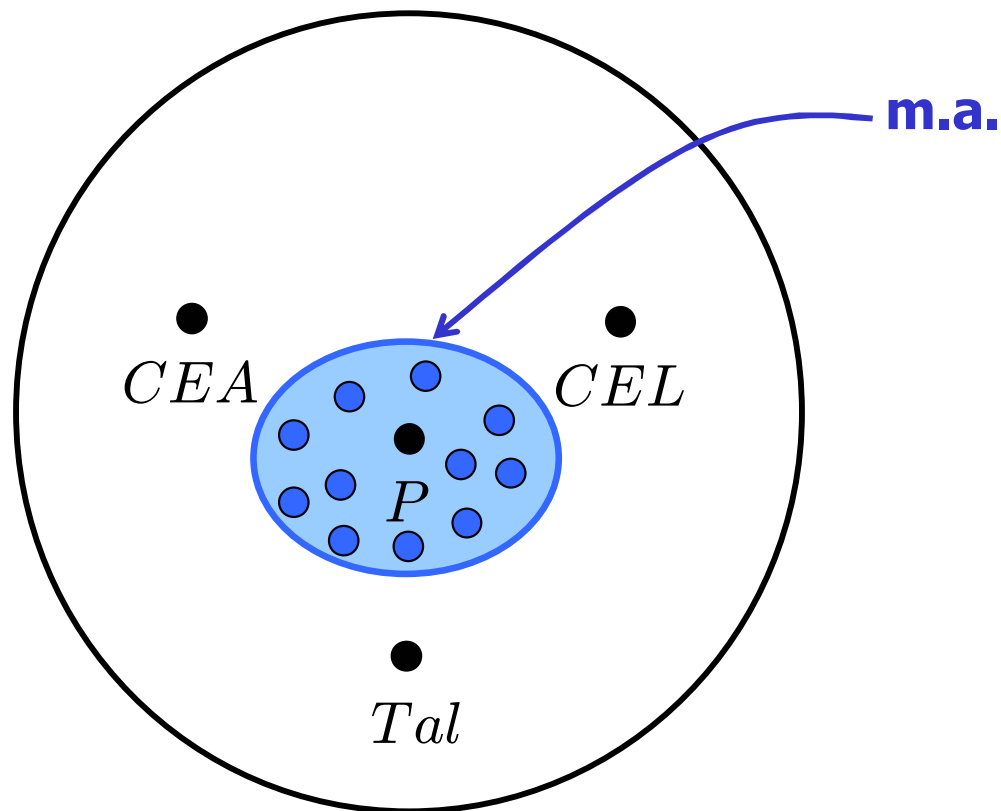


# Overview of results

- $n = 3$ :

$$S: \text{cont}, \quad \begin{array}{cc} \text{ord-pres-g} & \text{ineq-pres-g} \\ \text{ord-pres-l} & \text{ineq-pres-l} \end{array} \iff$$

$S \in$  “Minimal award family (for the three-agent case)”.



# Overview of results

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- $n > 3$ :

# Overview of results

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- $n > 3$ :

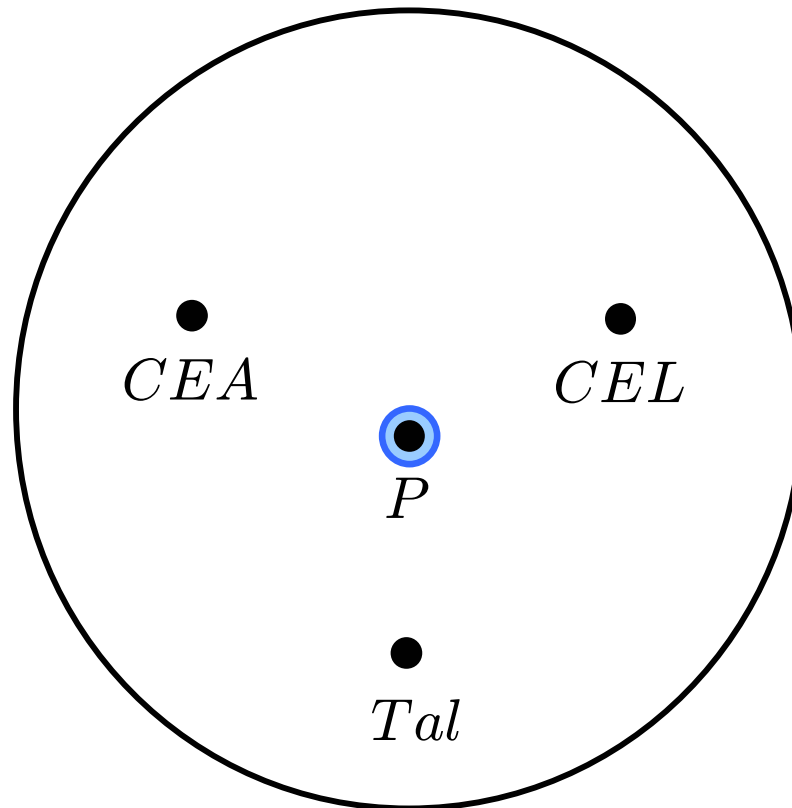
$S: \text{cont}, \begin{matrix} \text{ord-pres-g} & \text{ineq-pres-g} \\ \text{ord-pres-l} & \text{ineq-pres-l} \end{matrix} \iff S = \text{Proportional rule.}$

# Overview of results

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- $n > 3$ :

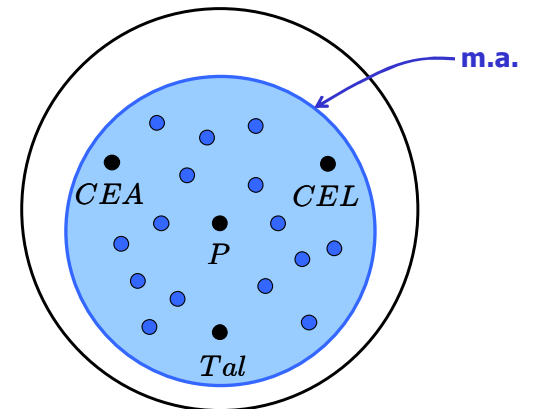
$S: \text{cont}, \begin{matrix} \text{ord-pres-g} \\ \text{ord-pres-l} \end{matrix}, \begin{matrix} \text{ineq-pres-g} \\ \text{ineq-pres-l} \end{matrix} \iff S = \text{Proportional rule.}$



# Overview of results

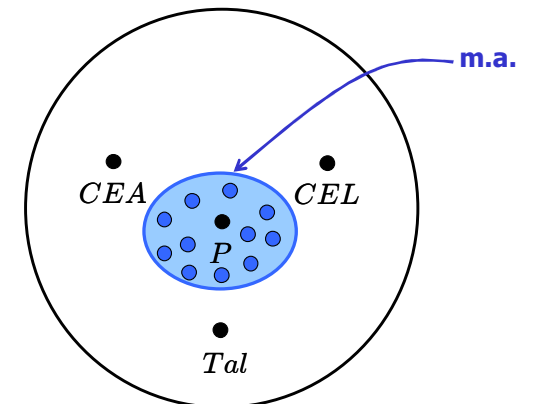
- $n = 2$ :

$S$ :  $cont$ ,  $ord-pres-g$ ,  $ineq-pres-g$   
 $ord-pres-l$ ,  $ineq-pres-l$   $\iff$



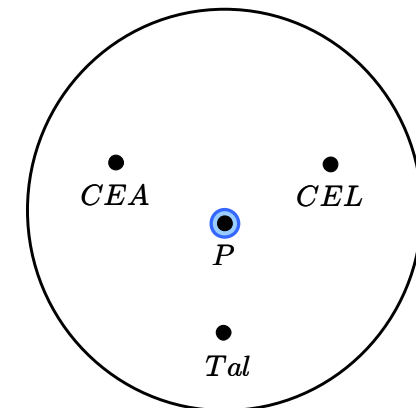
- $n = 3$ :

$S$ :  $\text{---}$ ,  $\text{---}$ ,  $\text{---}$   
 $\text{---}$ ,  $\text{---}$   $\iff$



- $n > 3$ :

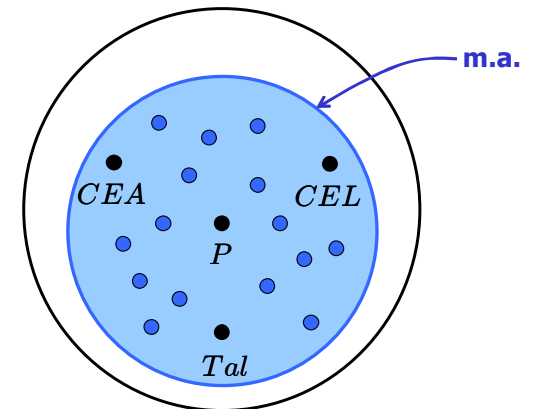
$S$ :  $\text{---}$ ,  $\text{---}$ ,  $\text{---}$   
 $\text{---}$ ,  $\text{---}$   $\iff$



# Overview of results

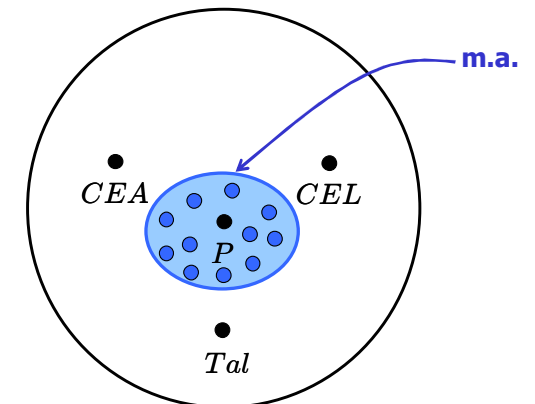
- $n = 2$ :

$S$ :  $cont$ ,  $ord-pres-g$ ,  $ineq-pres-g$   
 $ord-pres-l$ ,  $ineq-pres-l$   $\iff$



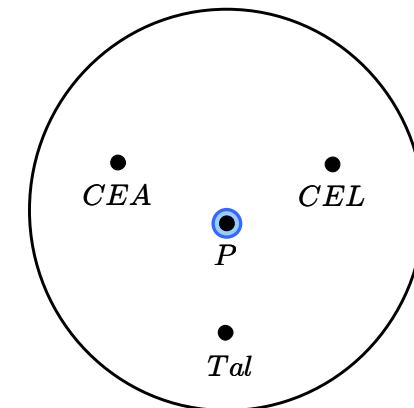
- $n = 3$ :

$S$ :  $\text{---}$ ,  $\text{---}$ ,  $\text{---}$   
 $\text{---}$ ,  $\text{---}$   $\iff$



- $n > 3$ :

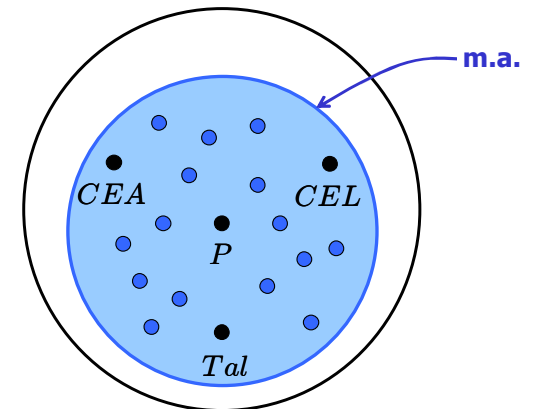
$S$ :  $\text{---}$ ,  $\text{---}$   
 $\text{---}$ ,  $\text{---}$   $\iff$



# Overview of results

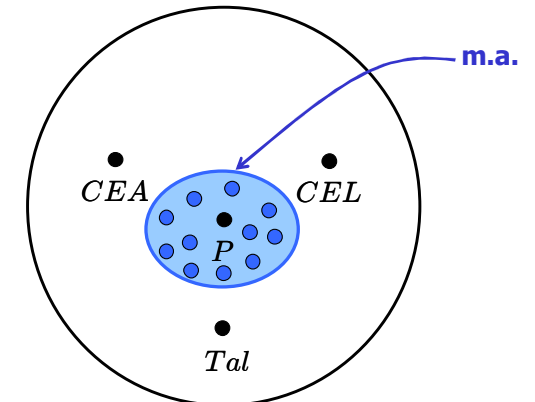
- $n = 2$ :

$S$ :  $cont$ ,  $ord-pres-g$ ,  $ineq-pres-g$   
 $ord-pres-l$ ,  $ineq-pres-l$   $\iff$



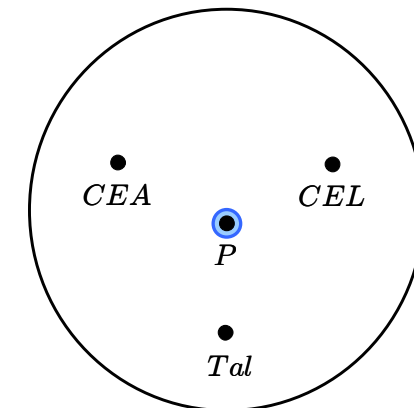
- $n = 3$ :

$S$ :  $\text{---}$ ,  $\text{---}$ ,  $\text{---}$   
 $\text{---}$ ,  $\text{---}$   $\iff$



- $n > 3$ :

$S$ :  $\text{---}$ ,  $\text{---}$   
 $\text{---}$   $\iff$



# Initial steps

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# Initial steps

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Let  $S$ :  $cont, \begin{matrix} ord-pres-g & ineq-pres-g \\ ord-pres-l & ineq-pres-l \end{matrix} .$

# Initial steps

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Let  $S: \begin{matrix} cont, & ord-pres-g & ineq-pres-g \\ & ord-pres-l & ineq-pres-l \end{matrix} .$

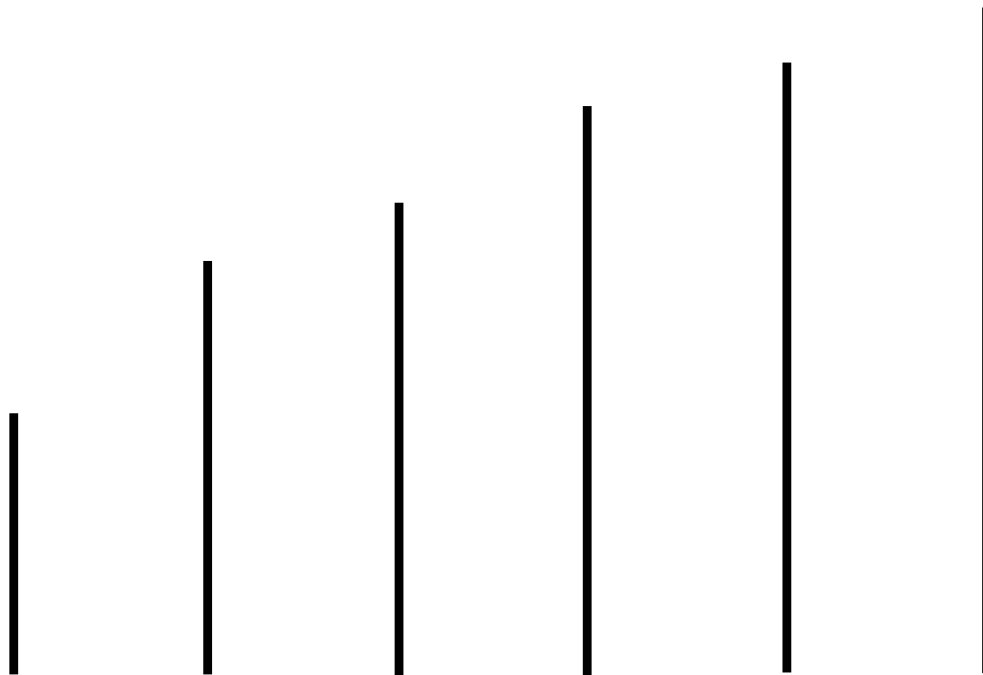
For each  $(c, E) \in \mathcal{C}$

# Initial steps

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Let  $S: \begin{matrix} cont, & ord-pres-g & ineq-pres-g \\ & ord-pres-l & ineq-pres-l \end{matrix} .$

For each  $(c, E) \in \mathcal{C}$

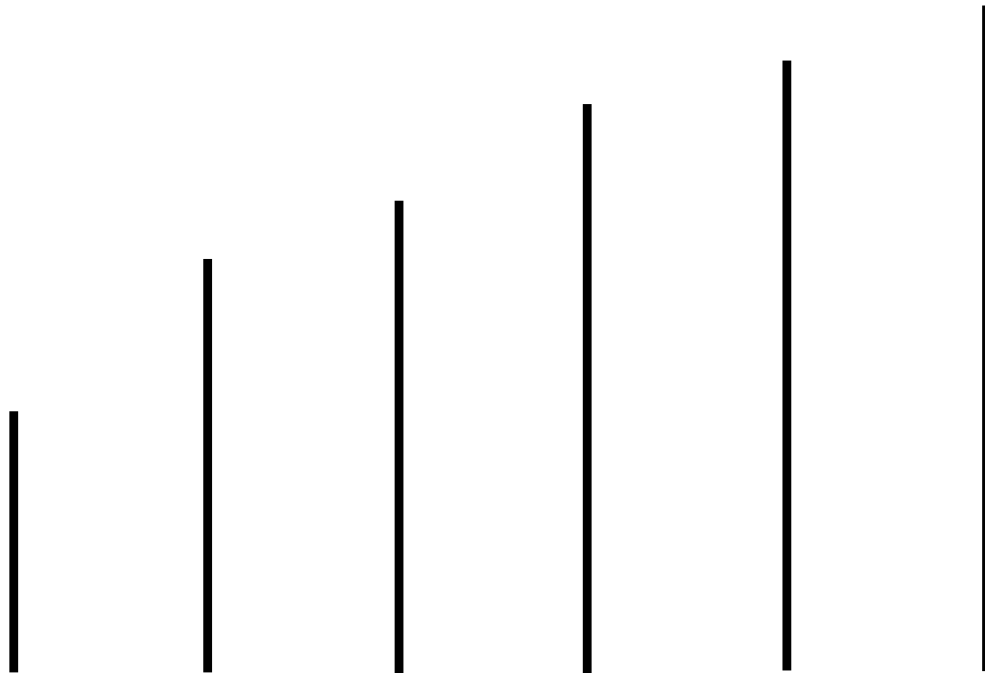


# Initial steps

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Let  $S: \begin{matrix} cont, & ord-pres-g & ineq-pres-g \\ & ord-pres-l & ineq-pres-l \end{matrix}$ .

For each  $(c, E) \in \mathcal{C}$  and each  $i \in \arg \min_{k \in N} c_k$ ,

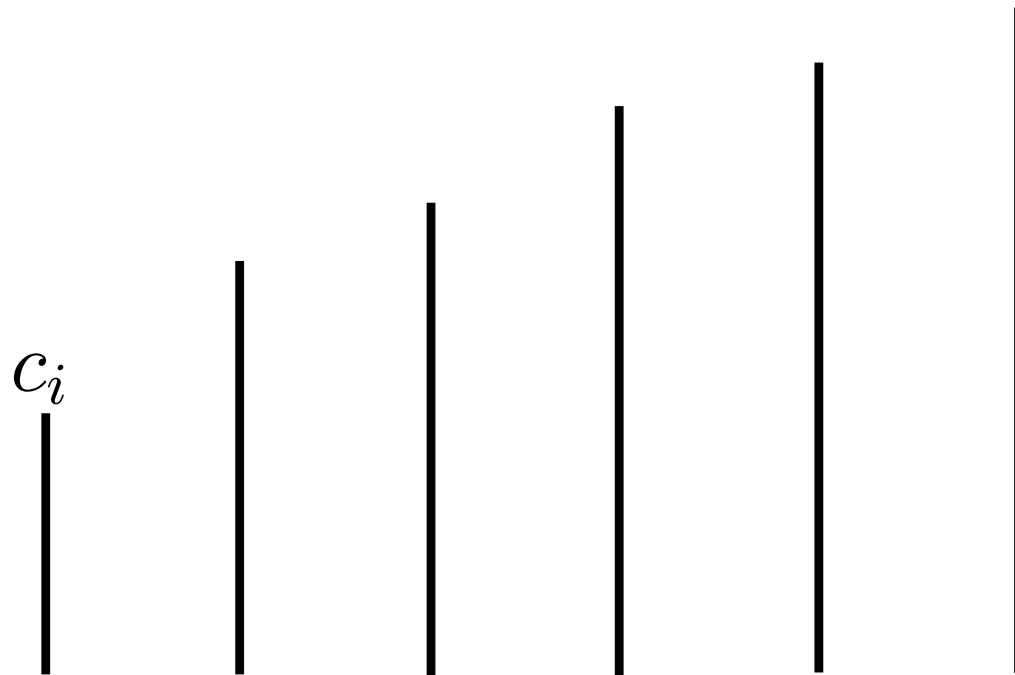


# Initial steps

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Let  $S: \begin{matrix} cont, & ord-pres-g & ineq-pres-g \\ & ord-pres-l & ineq-pres-l \end{matrix}$ .

For each  $(c, E) \in \mathcal{C}$  and each  $i \in \arg \min_{k \in N} c_k$ ,

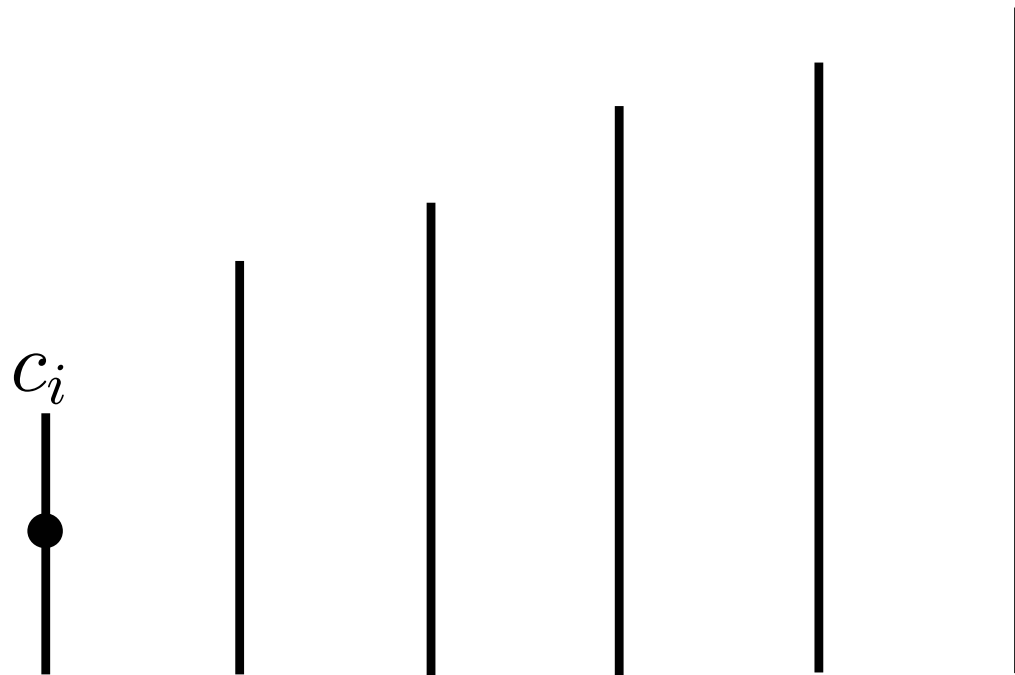


# Initial steps

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Let  $S: \begin{matrix} cont, & ord-pres-g & ineq-pres-g \\ & ord-pres-l & ineq-pres-l \end{matrix}$ .

For each  $(c, E) \in \mathcal{C}$  and each  $i \in \arg \min_{k \in N} c_k$ ,

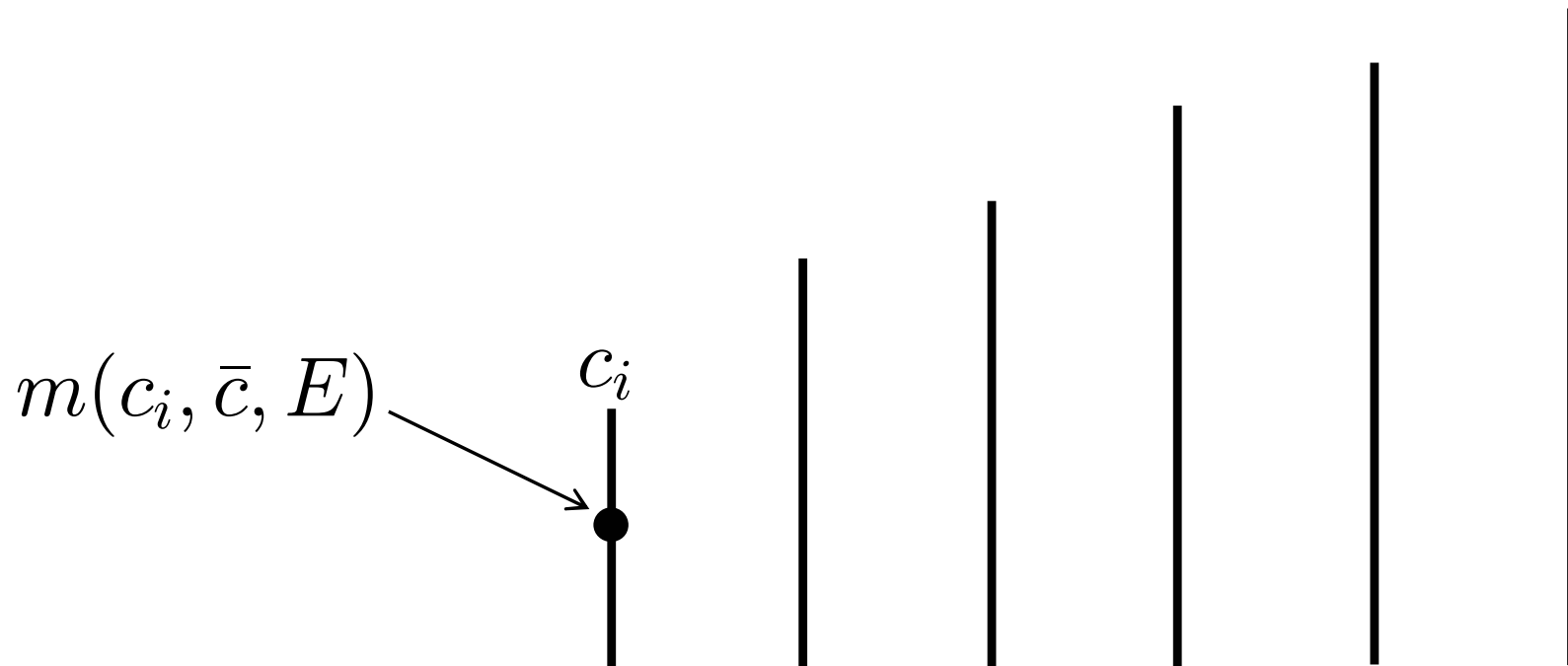


# Initial steps

---

Let  $S: \begin{matrix} cont, & ord-pres-g & ineq-pres-g \\ & ord-pres-l & ineq-pres-l \end{matrix}$ .

For each  $(c, E) \in \mathcal{C}$  and each  $i \in \arg \min_{k \in N} c_k$ ,

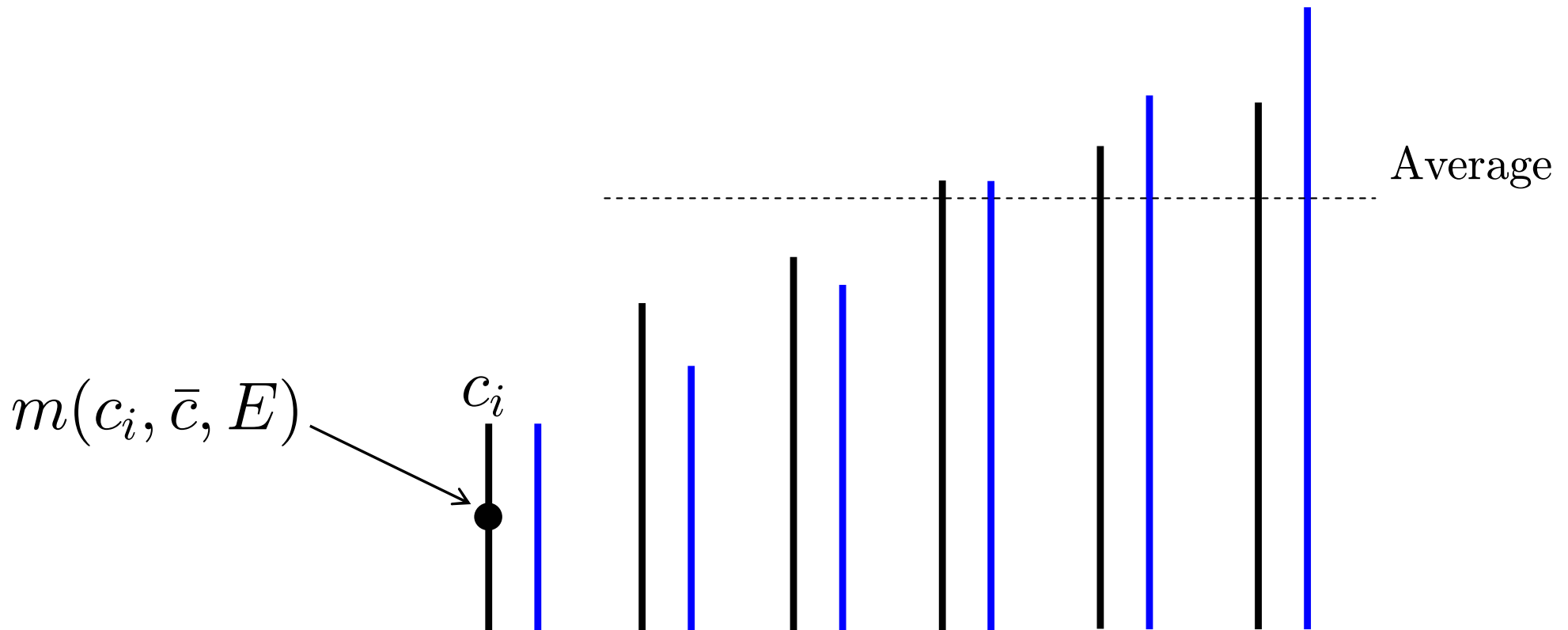


# Initial steps

---

Let  $S: \begin{matrix} cont, & ord-pres-g & ineq-pres-g \\ & ord-pres-l & ineq-pres-l \end{matrix}$ .

For each  $(c, E) \in \mathcal{C}$  and each  $i \in \arg \min_{k \in N} c_k$ ,

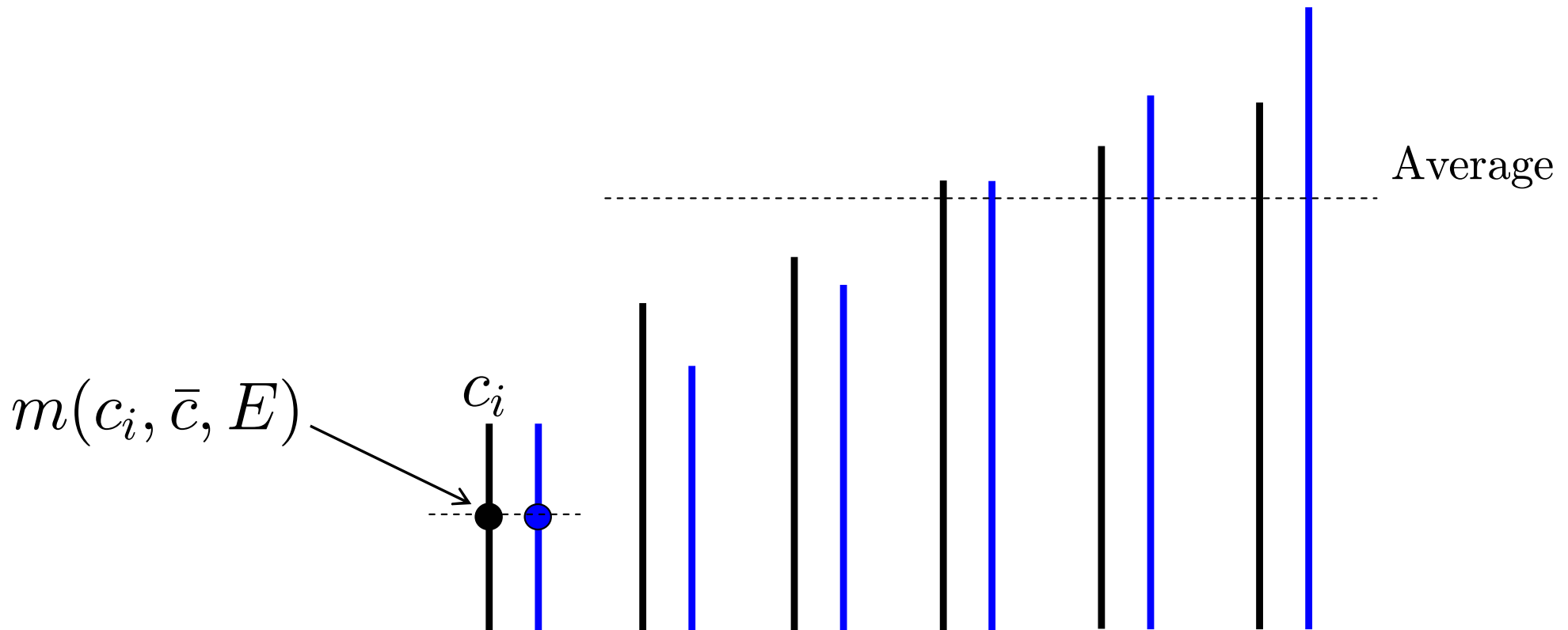


# Initial steps

---

Let  $S: \begin{matrix} cont, & ord-pres-g & ineq-pres-g \\ & ord-pres-l, & ineq-pres-l \end{matrix}$ .

For each  $(c, E) \in \mathcal{C}$  and each  $i \in \arg \min_{k \in N} c_k$ ,

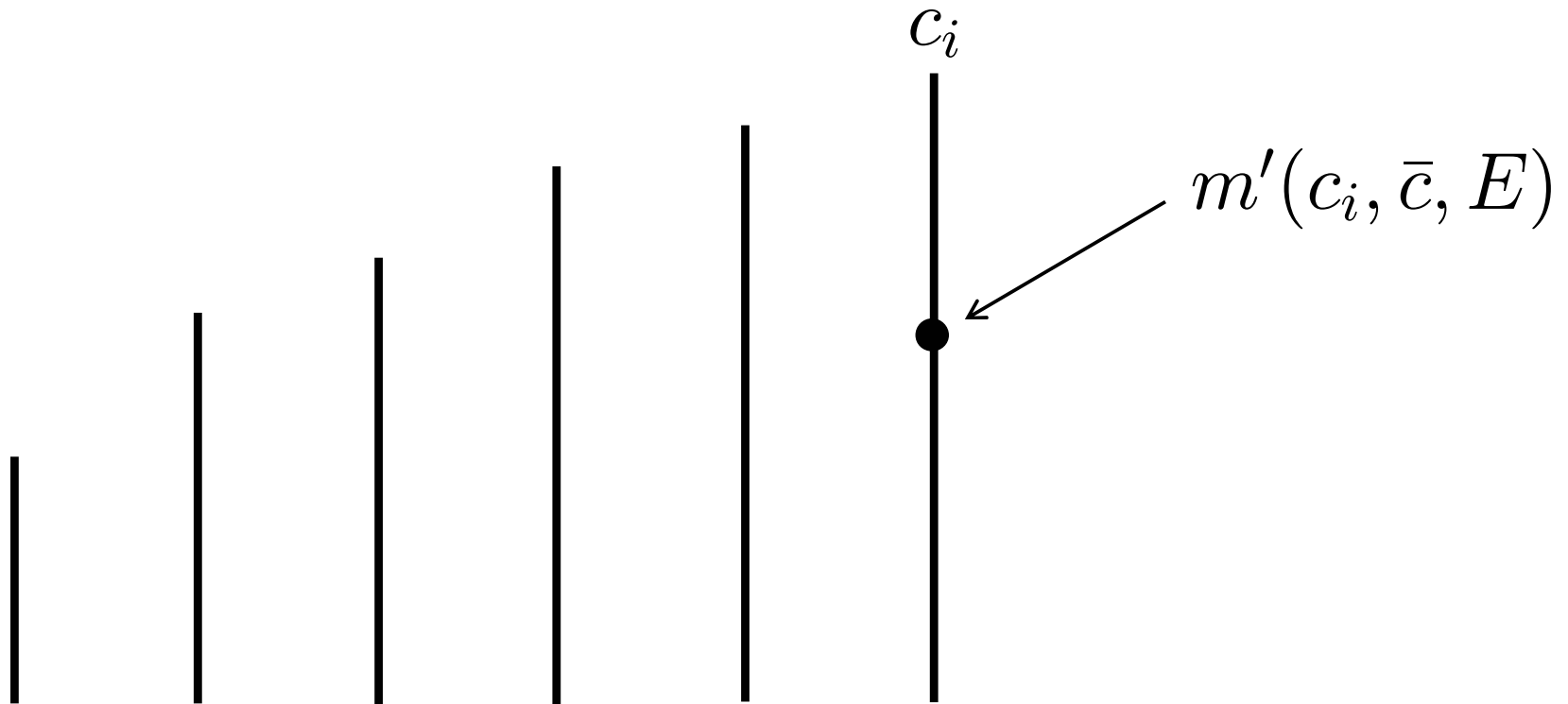


# Initial steps

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Let  $S: \begin{matrix} cont, & ord-pres-g & ineq-pres-g \\ & ord-pres-l & ineq-pres-l \end{matrix}$ .

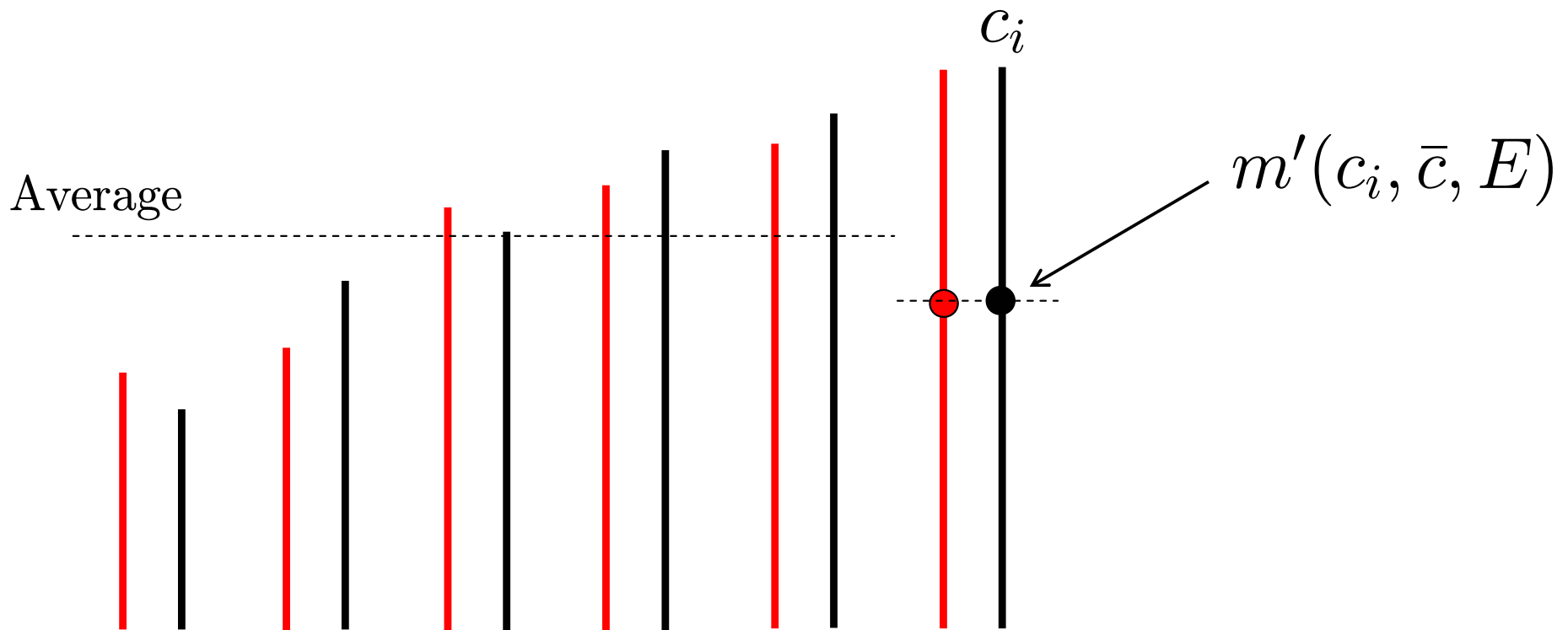
For each  $(c, E) \in \mathcal{C}$  and each  $i \in \arg \max_{k \in N} c_k$ ,



# Initial steps

Let  $S: \begin{matrix} cont, & ord-pres-g & ineq-pres-g \\ & ord-pres-l, & ineq-pres-l \end{matrix}$ .

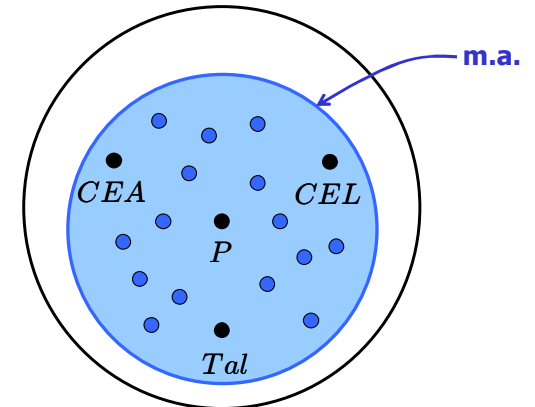
For each  $(c, E) \in \mathcal{C}$  and each  $i \in \arg \max_{k \in N} c_k$ ,



# Overview of results

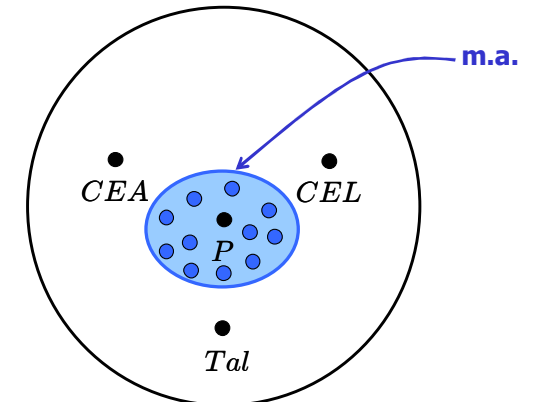
- $n = 2$ :

$S$ :  $cont$ ,  $ord-pres-g$ ,  $ineq-pres-g$   
 $ord-pres-l$ ,  $ineq-pres-l$   $\iff$



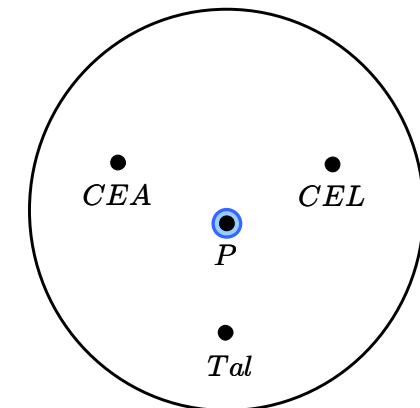
- $n = 3$ :

$S$ :  $\text{---}$ ,  $\text{---}$ ,  $\text{---}$   
 $\text{---}$ ,  $\text{---}$   $\iff$



- $n > 3$ :

$S$ :  $\text{---}$ ,  $\text{---}$   
 $\text{---}$ ,  $\text{---}$   $\iff$

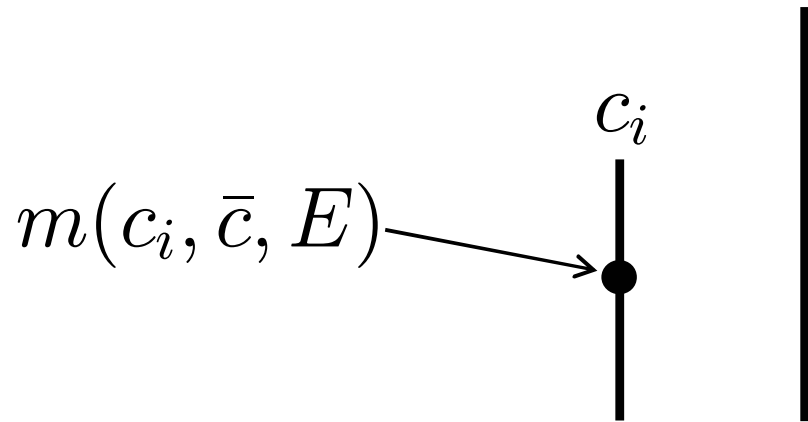


# Results (n=2)

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Let  $S: \begin{matrix} cont, & ord-pres-g & ineq-pres-g \\ & ord-pres-l & ineq-pres-l \end{matrix}$ .

For each  $(c, E) \in \mathcal{C}$  and each  $i \in \arg \min_{k \in N} c_k$ ,



# Results (n=2)

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$$m(c_i, \bar{c}, E)$$

# Results (n=2)

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$$m(c_i, \bar{c}, E)$$

- Continuous

# Results (n=2)

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$$m(c_i, \bar{c}, E)$$

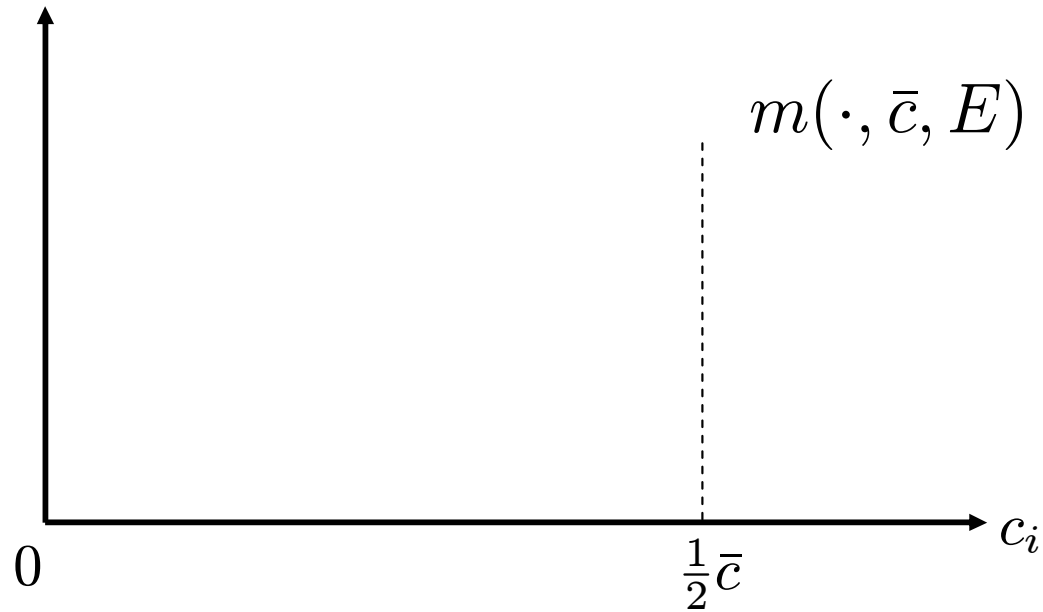
- Continuous
- Given  $\bar{c}$  and  $E$ ,

# Results (n=2)

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$$m(c_i, \bar{c}, E)$$

- Continuous
- Given  $\bar{c}$  and  $E$ ,

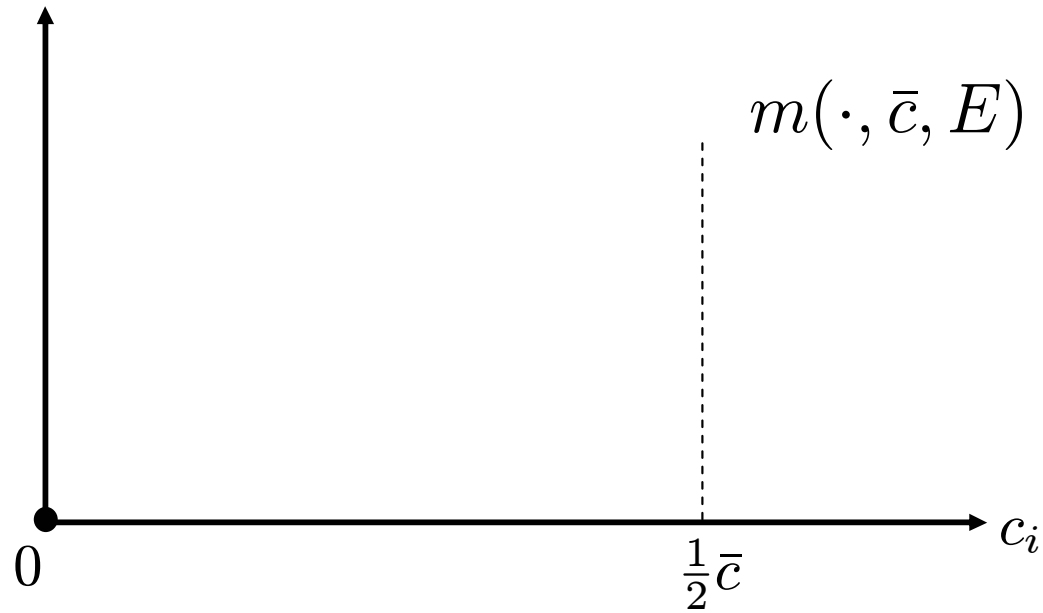


# Results (n=2)

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$$m(c_i, \bar{c}, E)$$

- Continuous
- Given  $\bar{c}$  and  $E$ ,

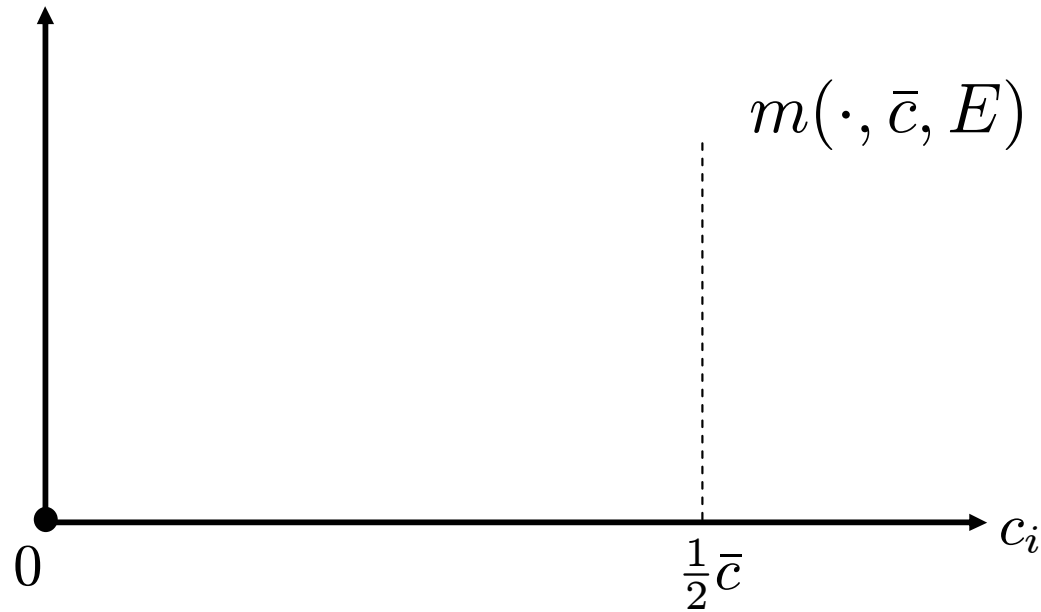


# Results (n=2)

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$$m(c_i, \bar{c}, E)$$

- Continuous
- Given  $\bar{c}$  and  $E$ ,



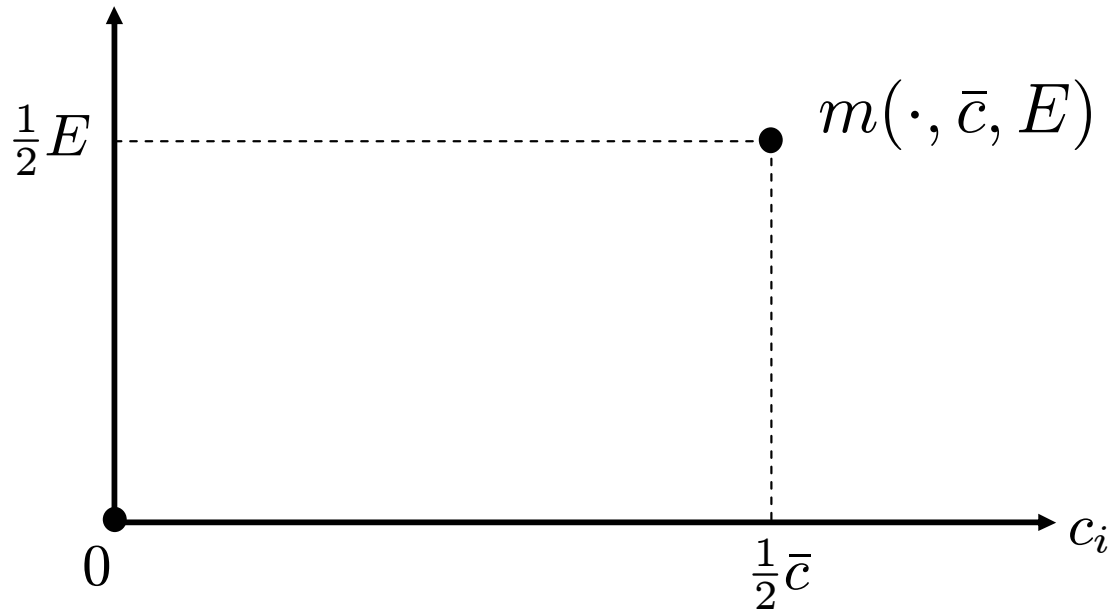
$$m(0, \bar{c}, E) = 0$$

# Results (n=2)

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$$m(c_i, \bar{c}, E)$$

- Continuous
- Given  $\bar{c}$  and  $E$ ,



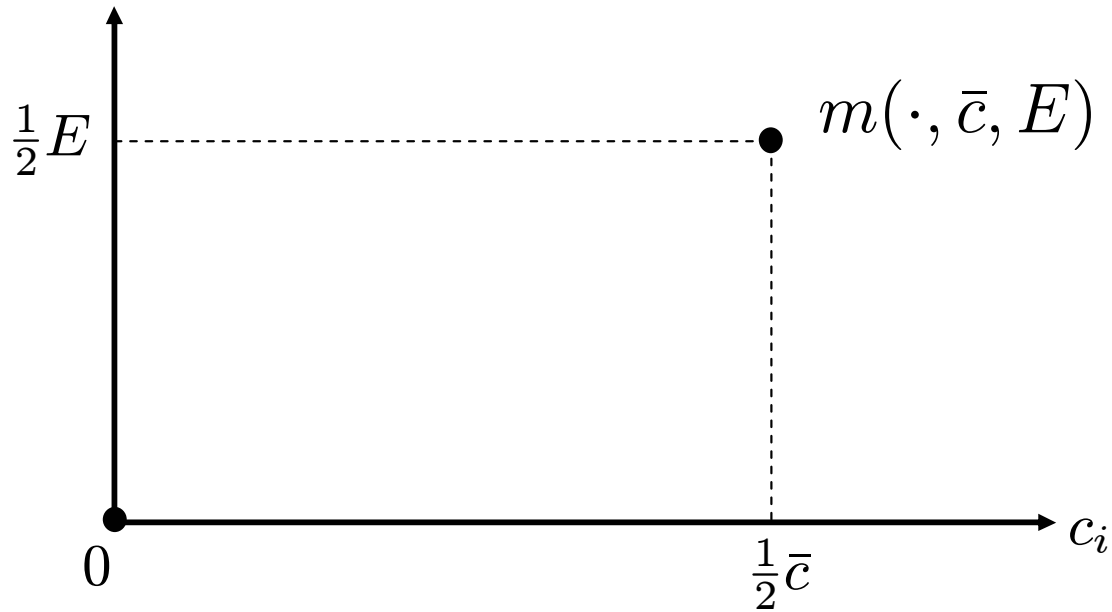
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# Results (n=2)

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$$m(c_i, \bar{c}, E)$$

- Continuous
- Given  $\bar{c}$  and  $E$ ,



$$m(0, \bar{c}, E) = 0$$

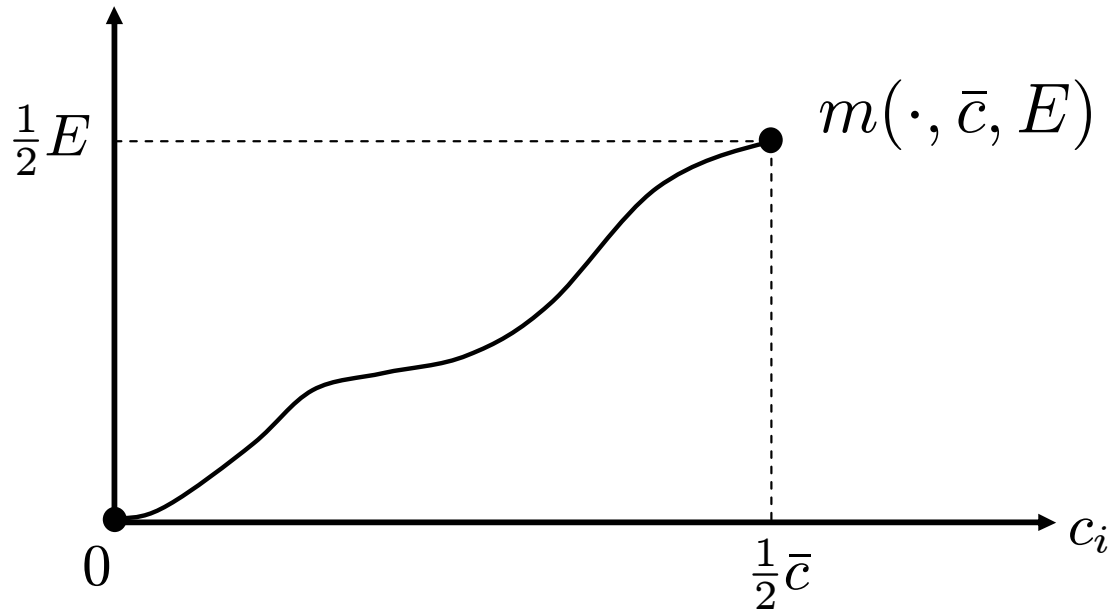
$$m(\frac{1}{2}\bar{c}, \bar{c}, E) = \frac{1}{2}E$$

# Results (n=2)

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$$m(c_i, \bar{c}, E)$$

- Continuous
- Given  $\bar{c}$  and  $E$ ,



$$m(0, \bar{c}, E) = 0$$

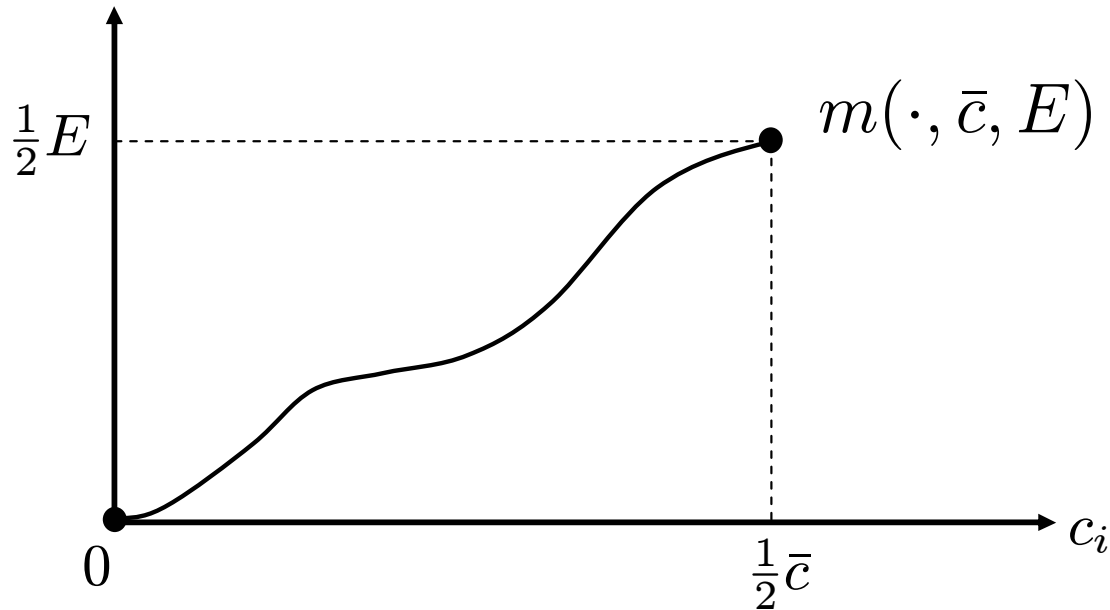
$$m(\frac{1}{2}\bar{c}, \bar{c}, E) = \frac{1}{2}E$$

# Results (n=2)

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$$m(c_i, \bar{c}, E)$$

- Continuous
- Given  $\bar{c}$  and  $E$ ,



$$m(0, \bar{c}, E) = 0$$

$$m(\frac{1}{2}\bar{c}, \bar{c}, E) = \frac{1}{2}E$$

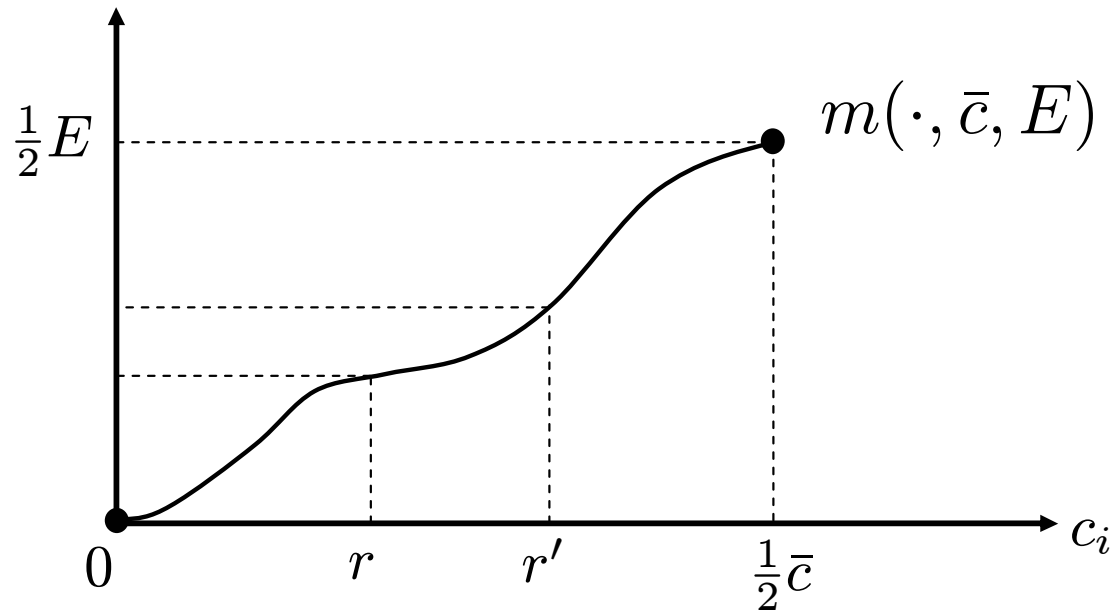
non-decreasing

# Results (n=2)

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$$m(c_i, \bar{c}, E)$$

- Continuous
- Given  $\bar{c}$  and  $E$ ,



$$m(0, \bar{c}, E) = 0$$

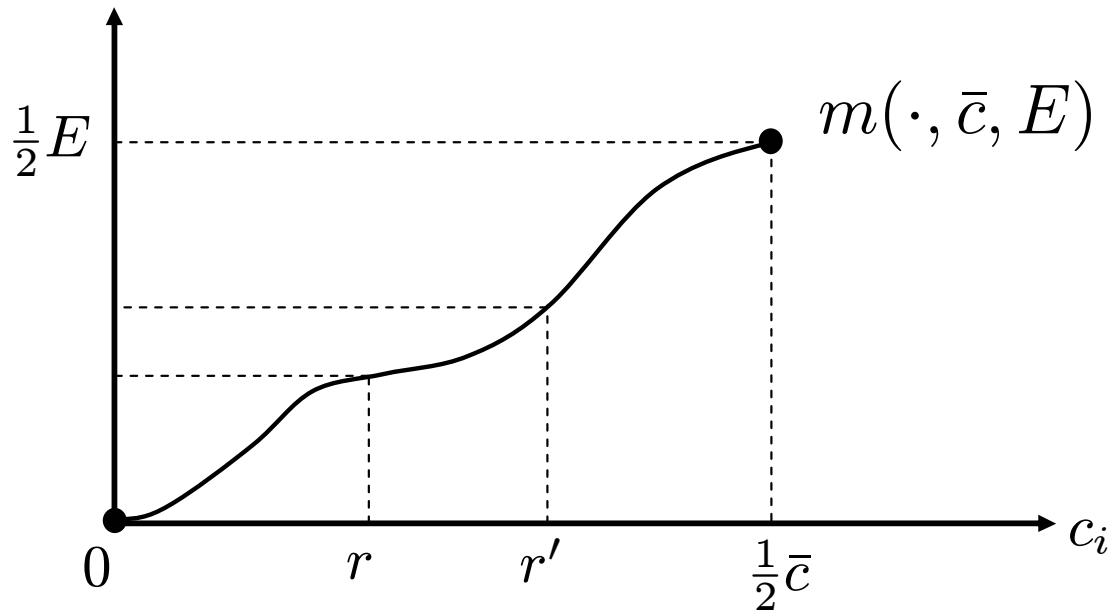
$$m(\frac{1}{2}\bar{c}, \bar{c}, E) = \frac{1}{2}E$$

non-decreasing

# Results (n=2)

$$m(c_i, \bar{c}, E)$$

- Continuous
- Given  $\bar{c}$  and  $E$ ,



$$m(0, \bar{c}, E) = 0$$

$$m(\frac{1}{2}\bar{c}, \bar{c}, E) = \frac{1}{2}E$$

non-decreasing

$$r' \geq r, m(r', \bar{c}, E) - m(r, \bar{c}, E) \leq r' - r$$

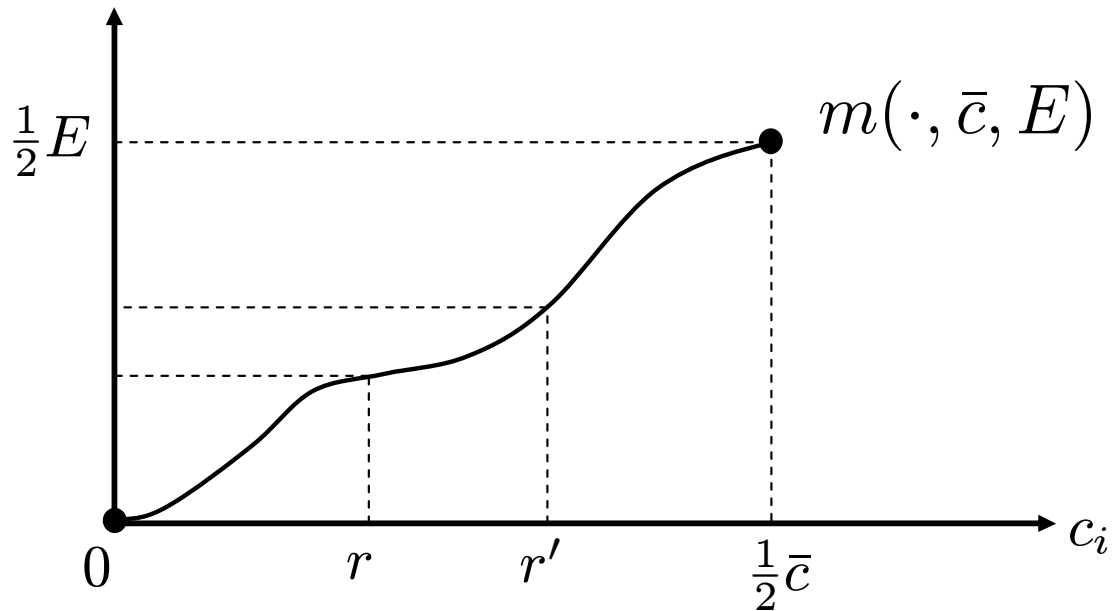
# Results (n=2)

- Minimal award function,  $m$ :

$$m(c_i, \bar{c}, E)$$

- Continuous

- Given  $\bar{c}$  and  $E$ ,



$$m(0, \bar{c}, E) = 0$$

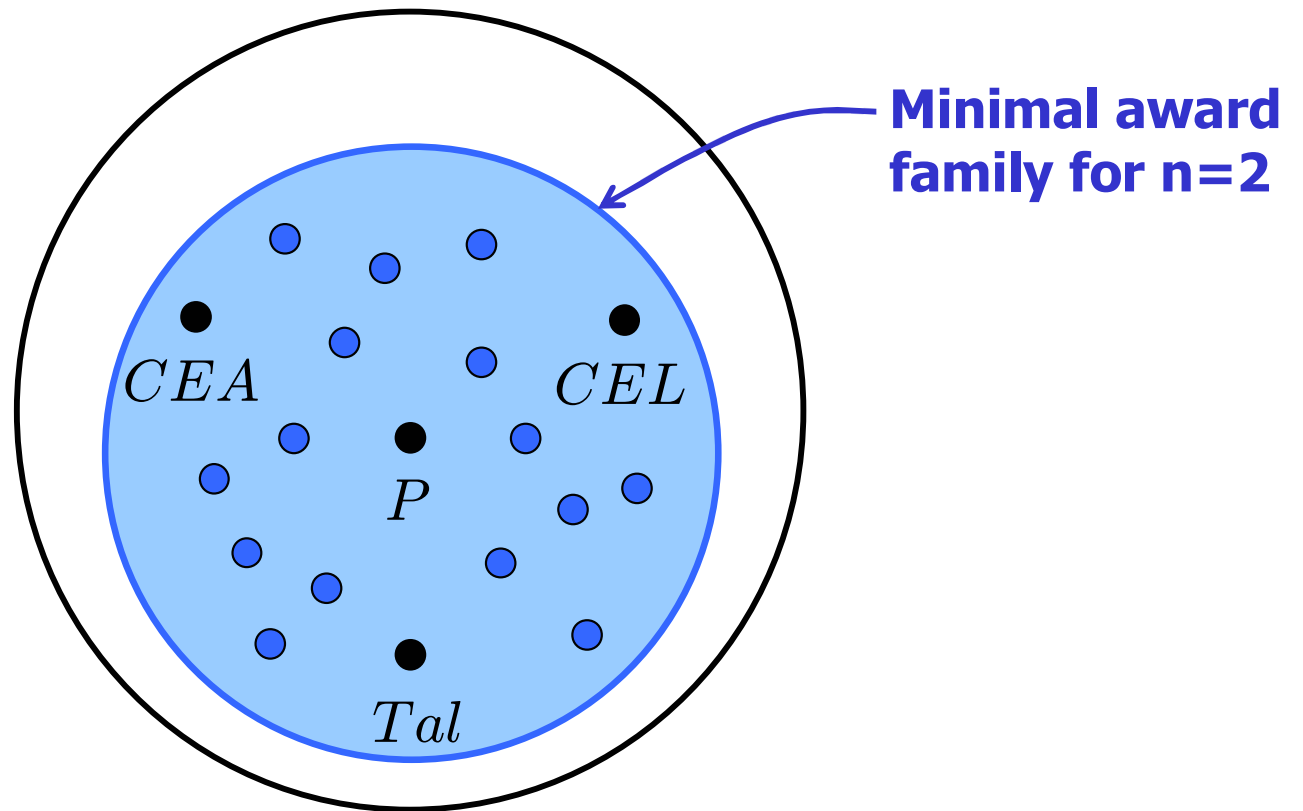
$$m(\frac{1}{2}\bar{c}, \bar{c}, E) = \frac{1}{2}E$$

non-decreasing

$$r' \geq r, m(r', \bar{c}, E) - m(r, \bar{c}, E) \leq r' - r$$

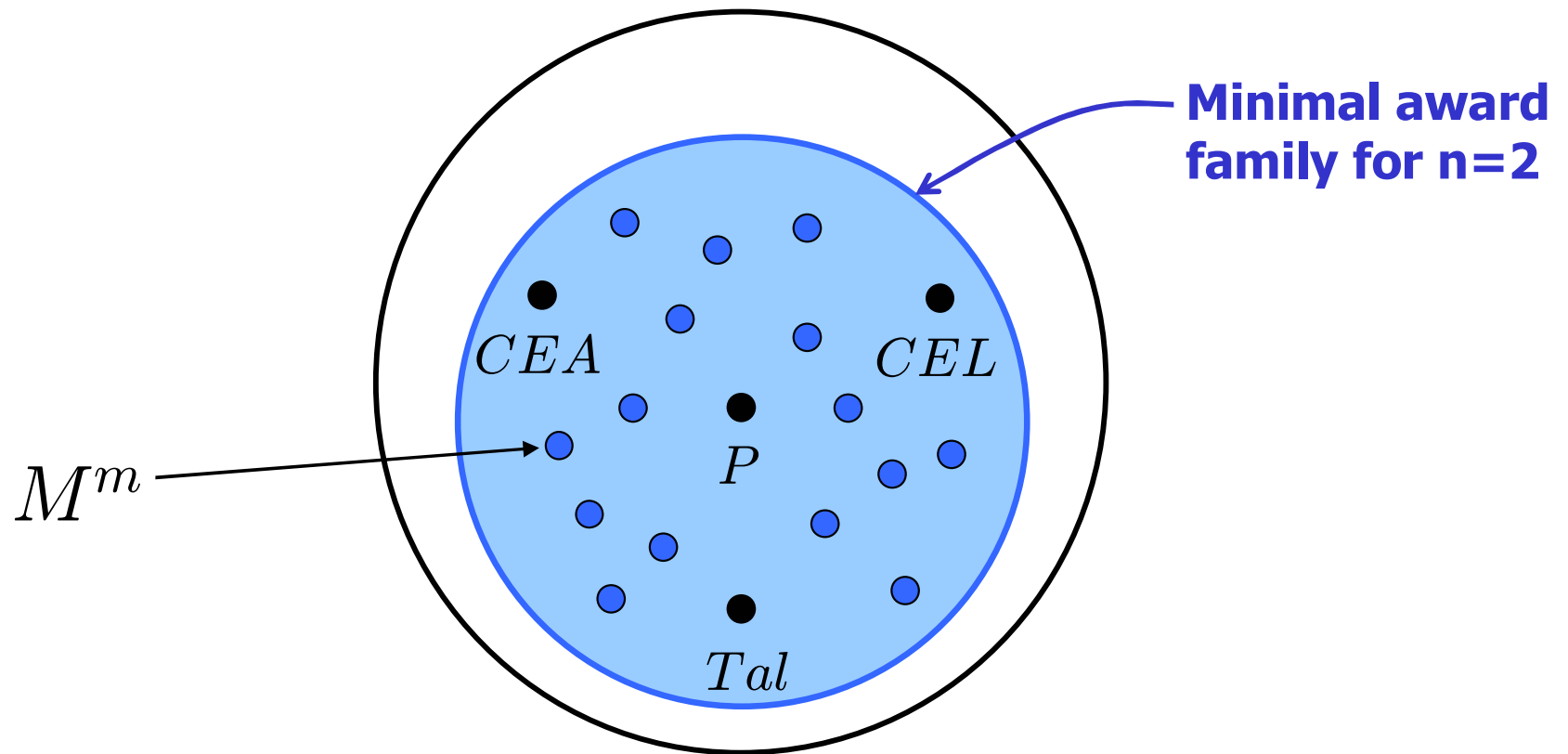
# Results (n=2)

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# Results (n=2)

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# Results ( $n=2$ )

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- Minimal award rule associated to  $m$ ,  $M^m$ :

# Results ( $n=2$ )

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- Minimal award rule associated to  $m$ ,  $M^m$ :

For each  $(c, E) \in \mathcal{C}$ , and each  $i \in N$ ,

# Results ( $n=2$ )

---

- Minimal award rule associated to  $m$ ,  $M^m$ :

For each  $(c, E) \in \mathcal{C}$ , and each  $i \in N$ ,

$$M_i^m(c, E) \equiv \left\{ \right.$$

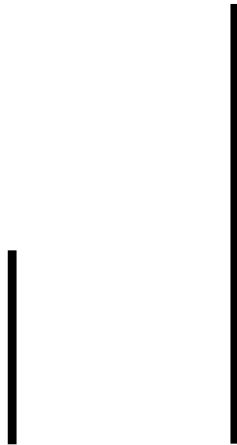
# Results (n=2)

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- Minimal award rule associated to  $m$ ,  $M^m$ :

For each  $(c, E) \in \mathcal{C}$ , and each  $i \in N$ ,

$$M_i^m(c, E) \equiv \left\{ \begin{array}{l} \end{array} \right.$$



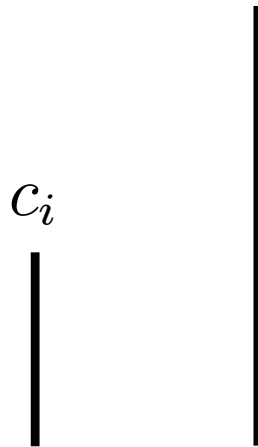
# Results ( $n=2$ )

---

- Minimal award rule associated to  $m$ ,  $M^m$ :

For each  $(c, E) \in \mathcal{C}$ , and each  $i \in N$ ,

$$M_i^m(c, E) \equiv \left\{ \right.$$



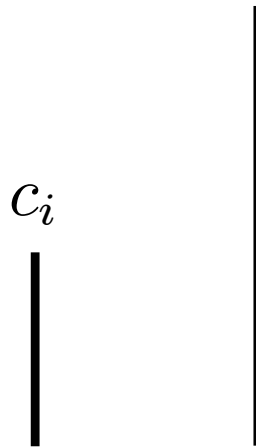
# Results ( $n=2$ )

---

- Minimal award rule associated to  $m$ ,  $M^m$ :

For each  $(c, E) \in \mathcal{C}$ , and each  $i \in N$ ,

$$M_i^m(c, E) \equiv \begin{cases} & \text{if } c_i = \min c < \max c \end{cases}$$



# Results ( $n=2$ )

---

- Minimal award rule associated to  $m$ ,  $M^m$ :

For each  $(c, E) \in \mathcal{C}$ , and each  $i \in N$ ,

$$M_i^m(c, E) \equiv \begin{cases} m(c_i, \bar{c}, E) & \text{if } c_i = \min c < \max c \end{cases}$$



$c_i$

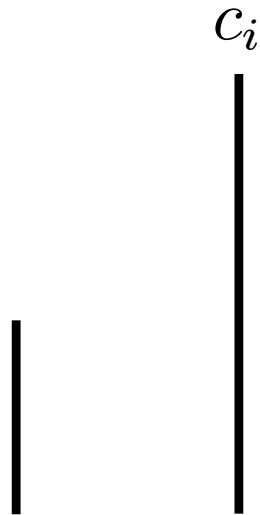
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---

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For each  $(c, E) \in \mathcal{C}$ , and each  $i \in N$ ,

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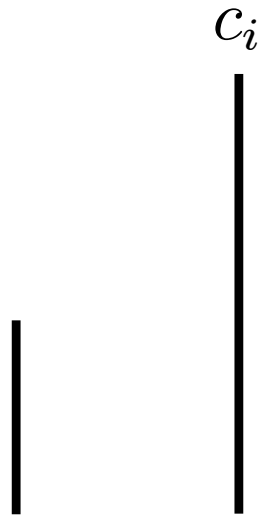
# Results ( $n=2$ )

---

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For each  $(c, E) \in \mathcal{C}$ , and each  $i \in N$ ,

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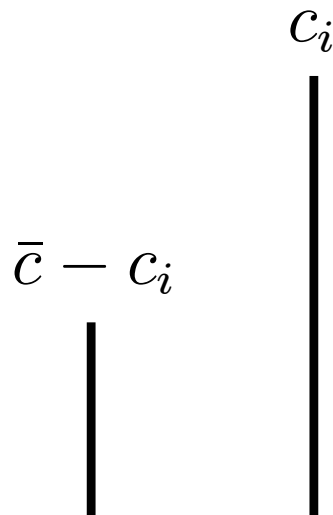
# Results ( $n=2$ )

---

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For each  $(c, E) \in \mathcal{C}$ , and each  $i \in N$ ,

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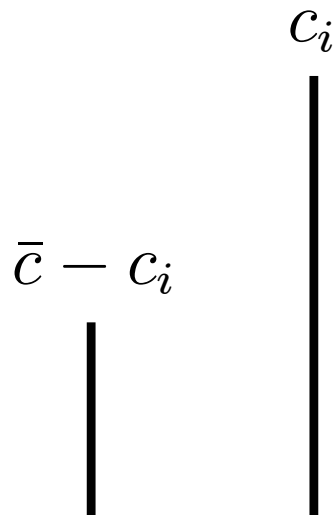
# Results ( $n=2$ )

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- Minimal award rule associated to  $m$ ,  $M^m$ :

For each  $(c, E) \in \mathcal{C}$ , and each  $i \in N$ ,

$$M_i^m(c, E) \equiv \begin{cases} m(c_i, \bar{c}, E) & \text{if } c_i = \min c < \max c \\ E - m(\bar{c} - c_i, \bar{c}, E) & \text{if } c_i = \max c. \end{cases}$$



# Results ( $n=2$ )

---

**Theorem 1:** Assume  $n = 2$ .

$S: \text{cont}, \begin{matrix} \text{ord-pres-g} & \text{ineq-pres-g} \\ \text{ord-pres-l} & \text{ineq-pres-l} \end{matrix} \iff$

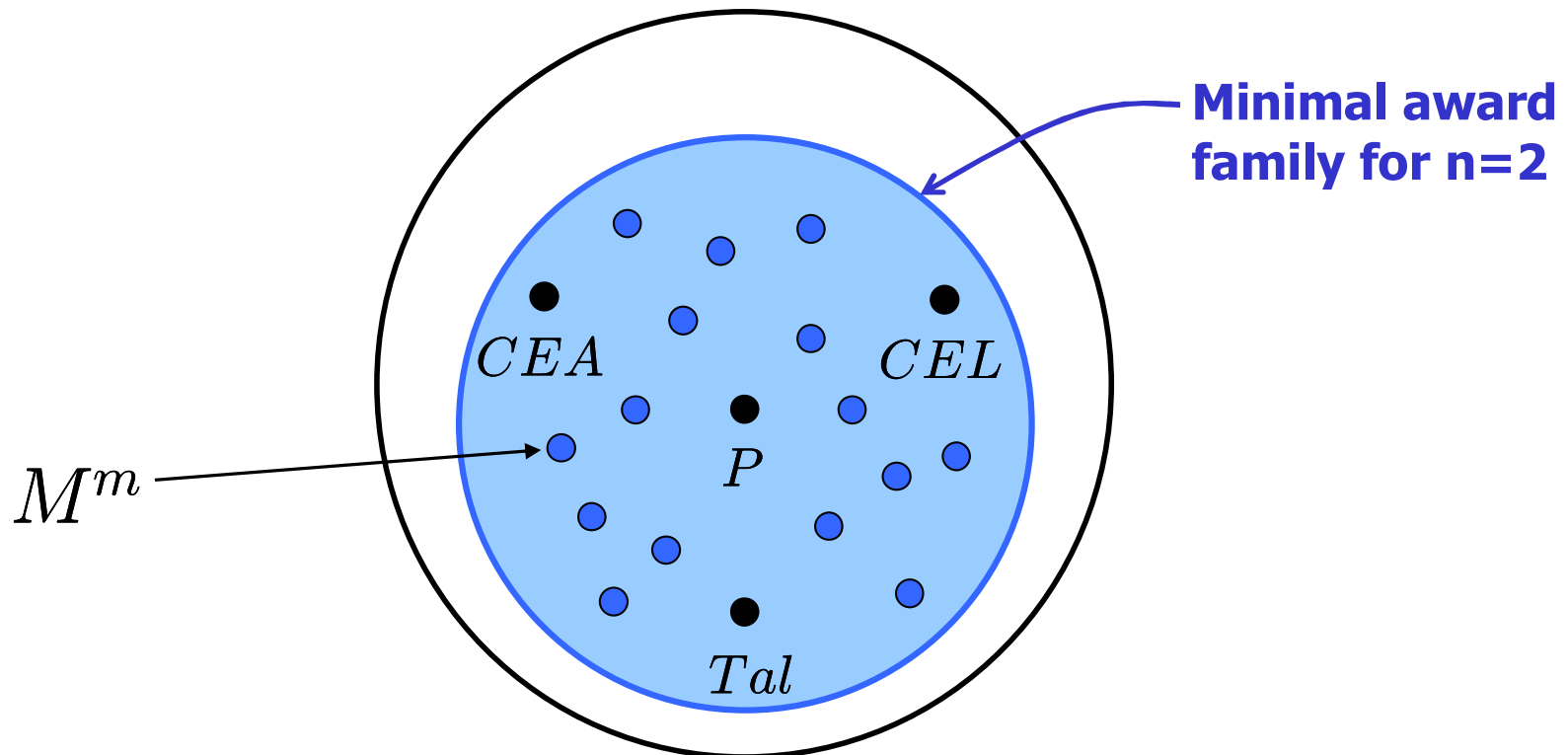
$S \in \text{“Minimal award family”}.$

# Results ( $n=2$ )

**Theorem 1:** Assume  $n = 2$ .

$$S: \begin{matrix} \text{cont,} & \text{ord-pres-g} & \text{ineq-pres-g} \\ & \text{ord-pres-l} & \text{ineq-pres-l} \end{matrix} \iff$$

$S \in$  “Minimal award family”.

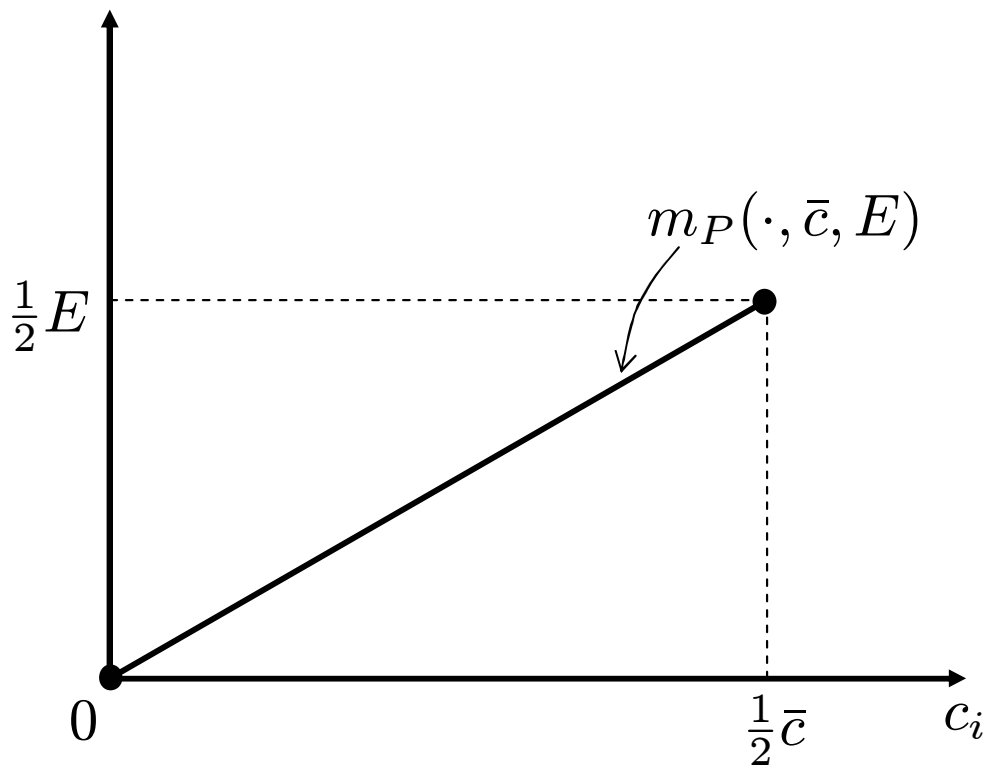


# Results ( $n=2$ )

**Theorem 1:** Assume  $n = 2$ .

$$S: \begin{array}{ll} \text{cont, } & \text{ord-pres-g} \\ & \text{ord-pres-l} \end{array} \text{ , } \begin{array}{l} \text{ineq-pres-g} \\ \text{ineq-pres-l} \end{array} \iff$$

$S \in$  “Minimal award family”.

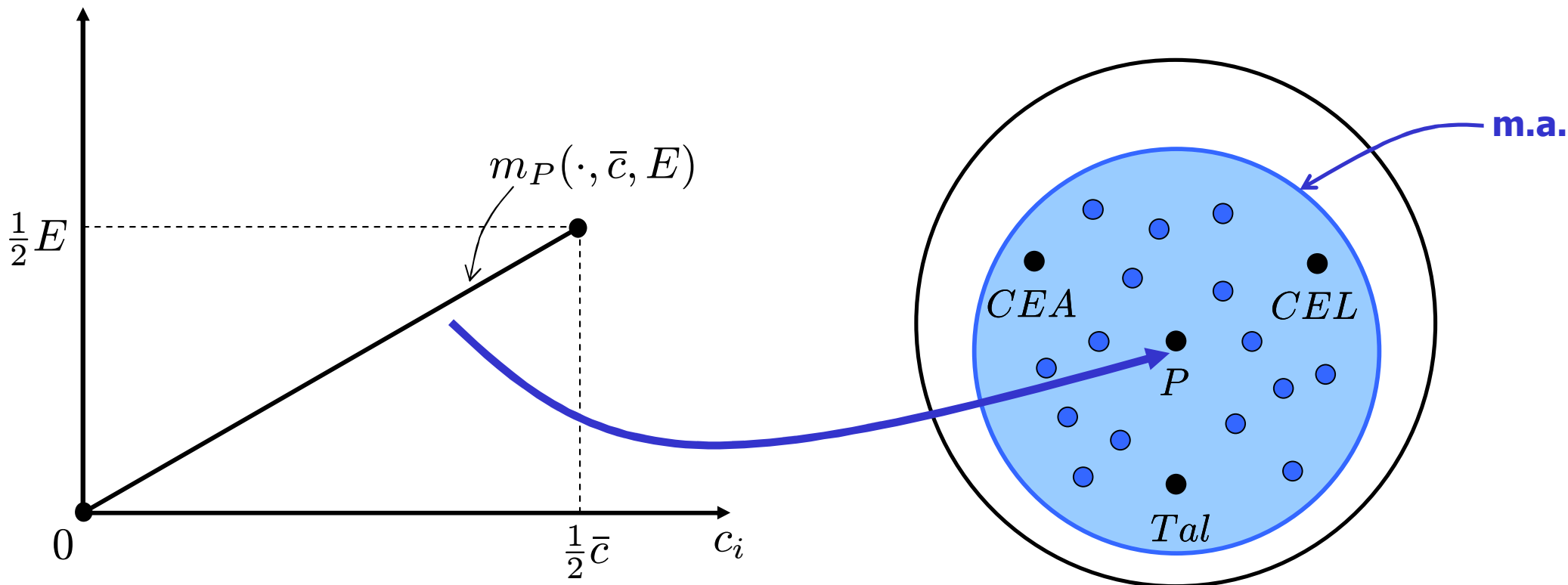


# Results ( $n=2$ )

**Theorem 1:** Assume  $n = 2$ .

$$S: \begin{matrix} \text{cont,} & \text{ord-pres-g} & \text{ineq-pres-g} \\ & \text{ord-pres-l} & \text{ineq-pres-l} \end{matrix} \iff$$

$S \in$  “Minimal award family”.

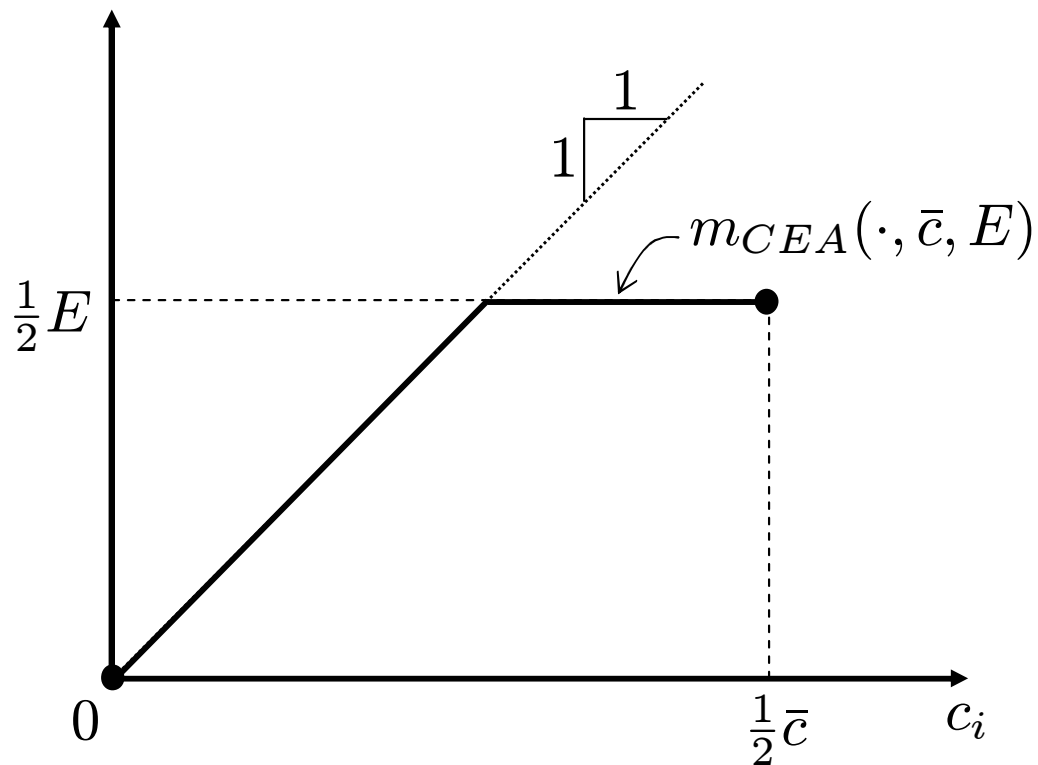


# Results ( $n=2$ )

**Theorem 1:** Assume  $n = 2$ .

$$S: \begin{array}{ll} \text{cont,} & \text{ord-pres-g} & \text{ineq-pres-g} \\ & \text{ord-pres-l} & \text{ineq-pres-l} \end{array} \iff$$

$S \in$  “Minimal award family”.

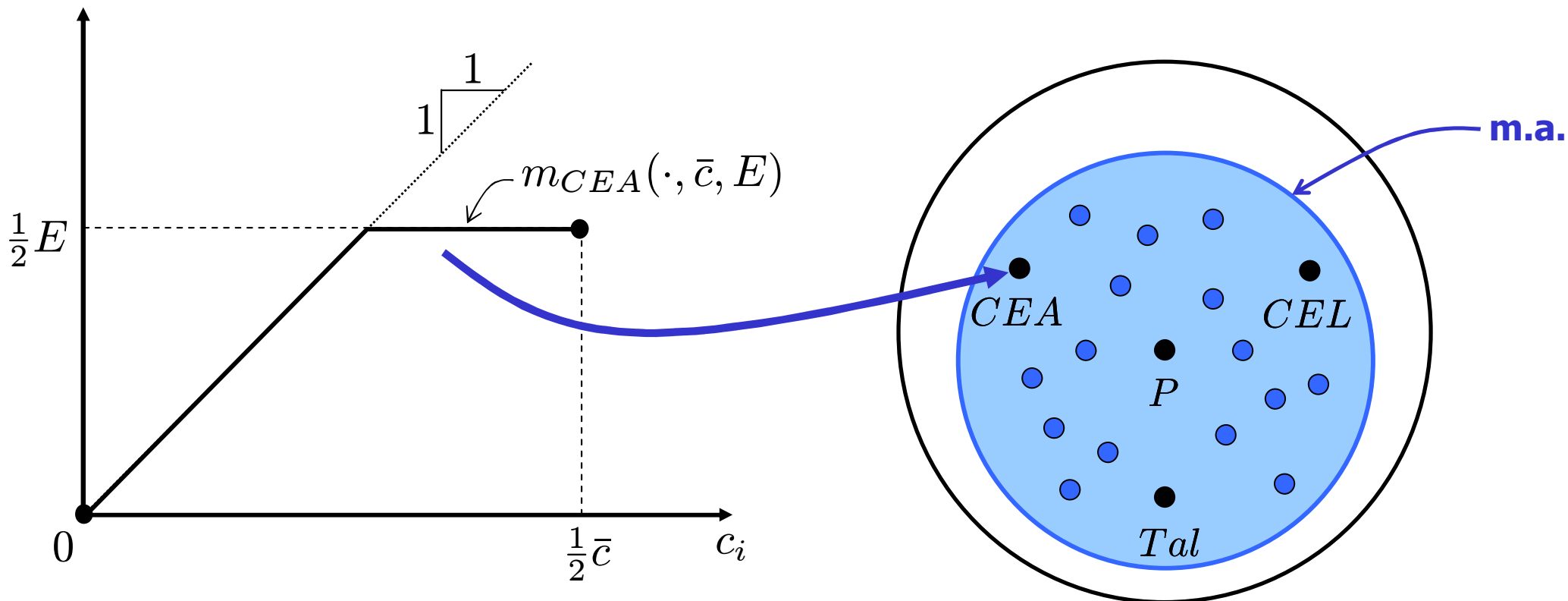


# Results ( $n=2$ )

**Theorem 1:** Assume  $n = 2$ .

$$S: \begin{matrix} \text{cont,} & \text{ord-pres-g} & \text{ineq-pres-g} \\ & \text{ord-pres-l} & \text{ineq-pres-l} \end{matrix} \iff$$

$S \in$  “Minimal award family”.

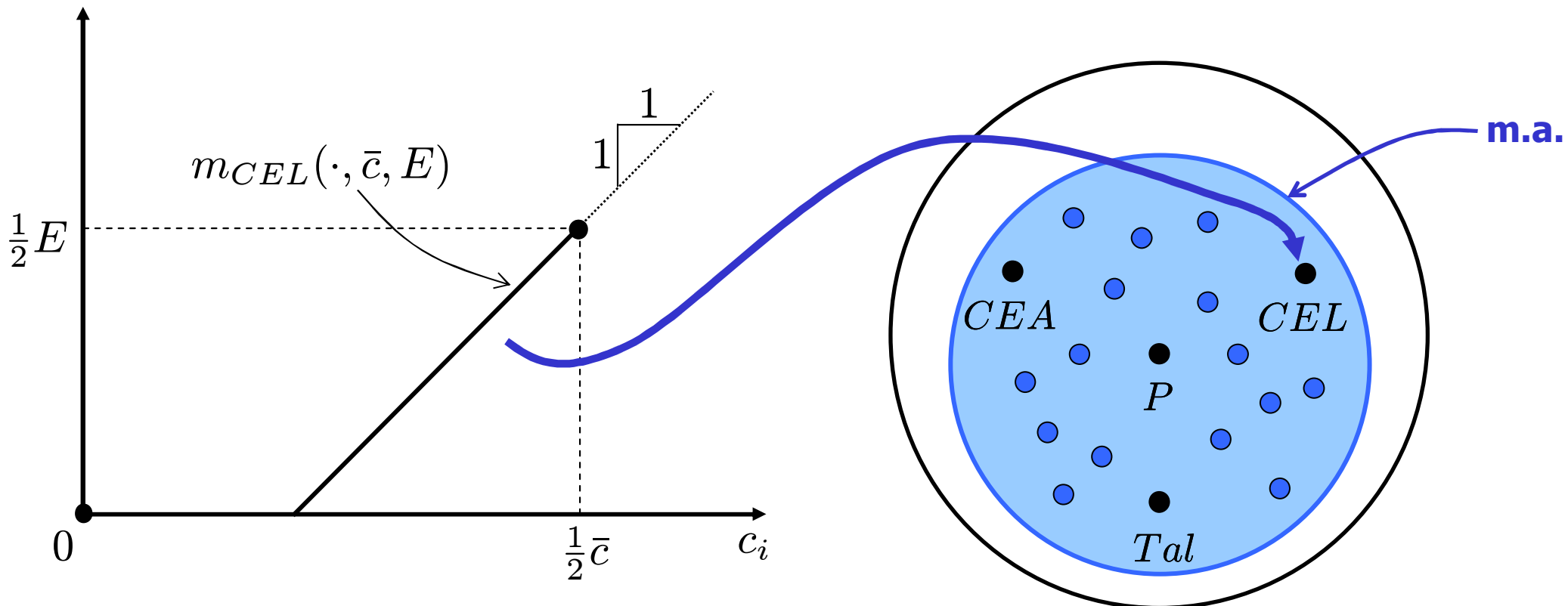


# Results ( $n=2$ )

**Theorem 1:** Assume  $n = 2$ .

$$S: \begin{matrix} cont, & ord-pres-g & ineq-pres-g \\ & ord-pres-l & ineq-pres-l \end{matrix} \iff$$

$S \in$  “Minimal award family”.

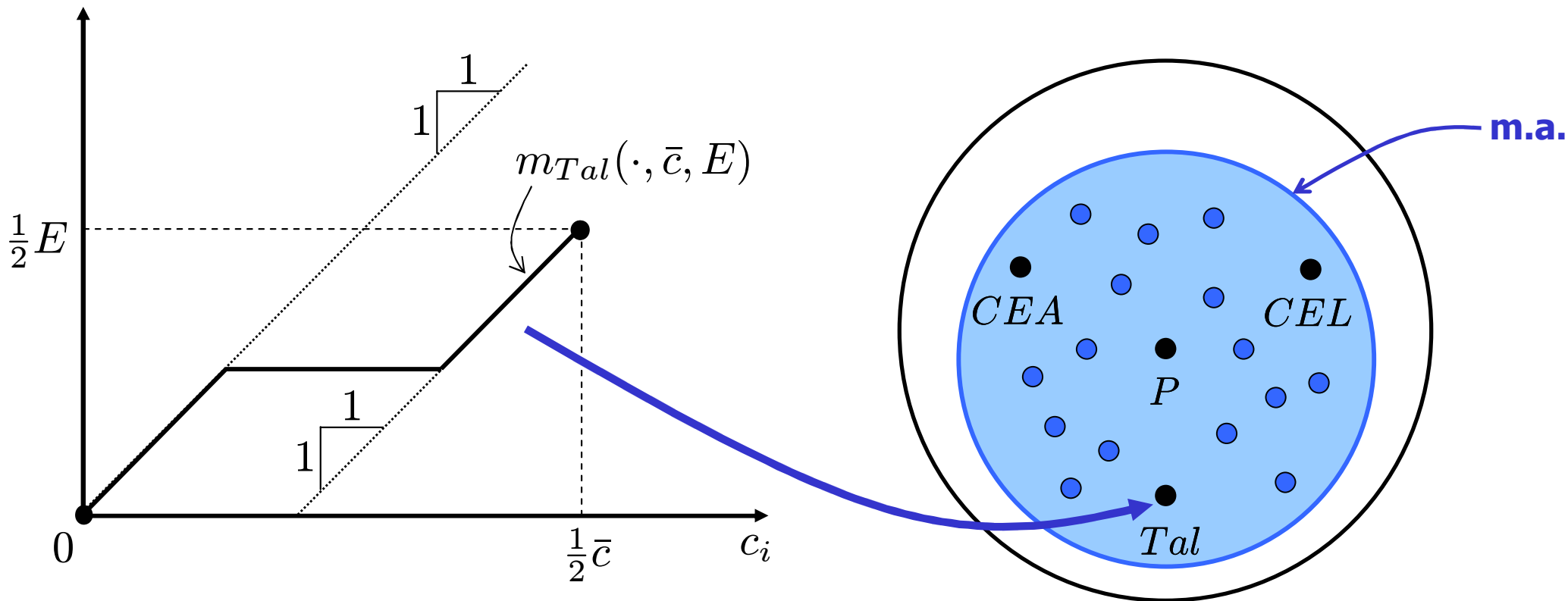


# Results ( $n=2$ )

**Theorem 1:** Assume  $n = 2$ .

$$S: \begin{matrix} \text{cont}, & \text{ord-pres-g} & \text{ineq-pres-g} \\ & \text{ord-pres-l} & \text{ineq-pres-l} \end{matrix} \iff$$

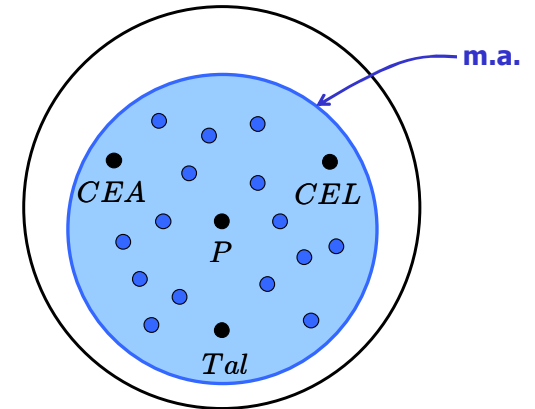
$S \in$  “Minimal award family”.



# Overview of results

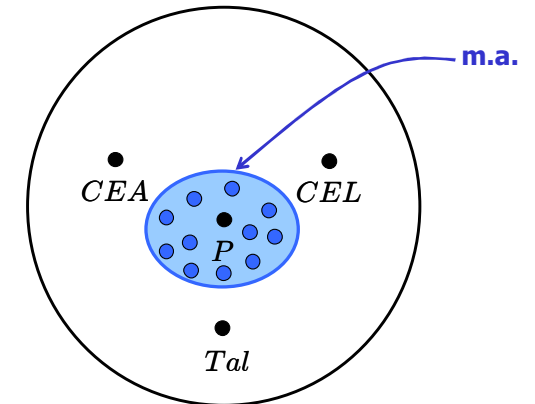
- $n = 2$ :

$S$ :  $cont$ ,  $ord-pres-g$ ,  $ineq-pres-g$   
 $ord-pres-l$ ,  $ineq-pres-l$   $\iff$



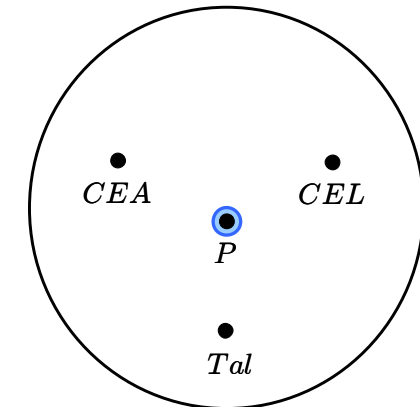
- $n = 3$ :

$S$ :  $\text{---}$ ,  $\text{---}$ ,  $\text{---}$   
 $\text{---}$ ,  $\text{---}$   $\iff$



- $n > 3$ :

$S$ :  $\text{---}$ ,  $\text{---}$   
 $\text{---}$ ,  $\text{---}$   $\iff$

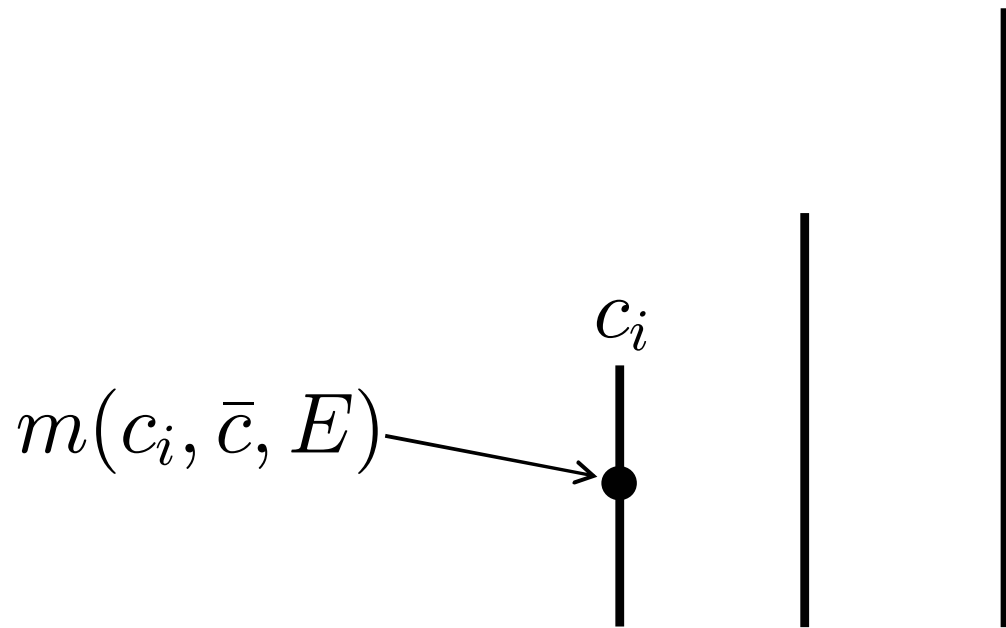


# Results (n=3)

---

Let  $S: \begin{matrix} cont, & ord-pres-g & ineq-pres-g \\ & ord-pres-l, & ineq-pres-l \end{matrix}$ .

For each  $(c, E) \in \mathcal{C}$  and each  $i \in \arg \min_{k \in N} c_k$ ,



# Results (n=3)

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$$m(c_i, \bar{c}, E)$$

# Results (n=3)

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$$m(c_i, \bar{c}, E)$$

- Continuous

# Results ( $n=3$ )

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$$m(c_i, \bar{c}, E)$$

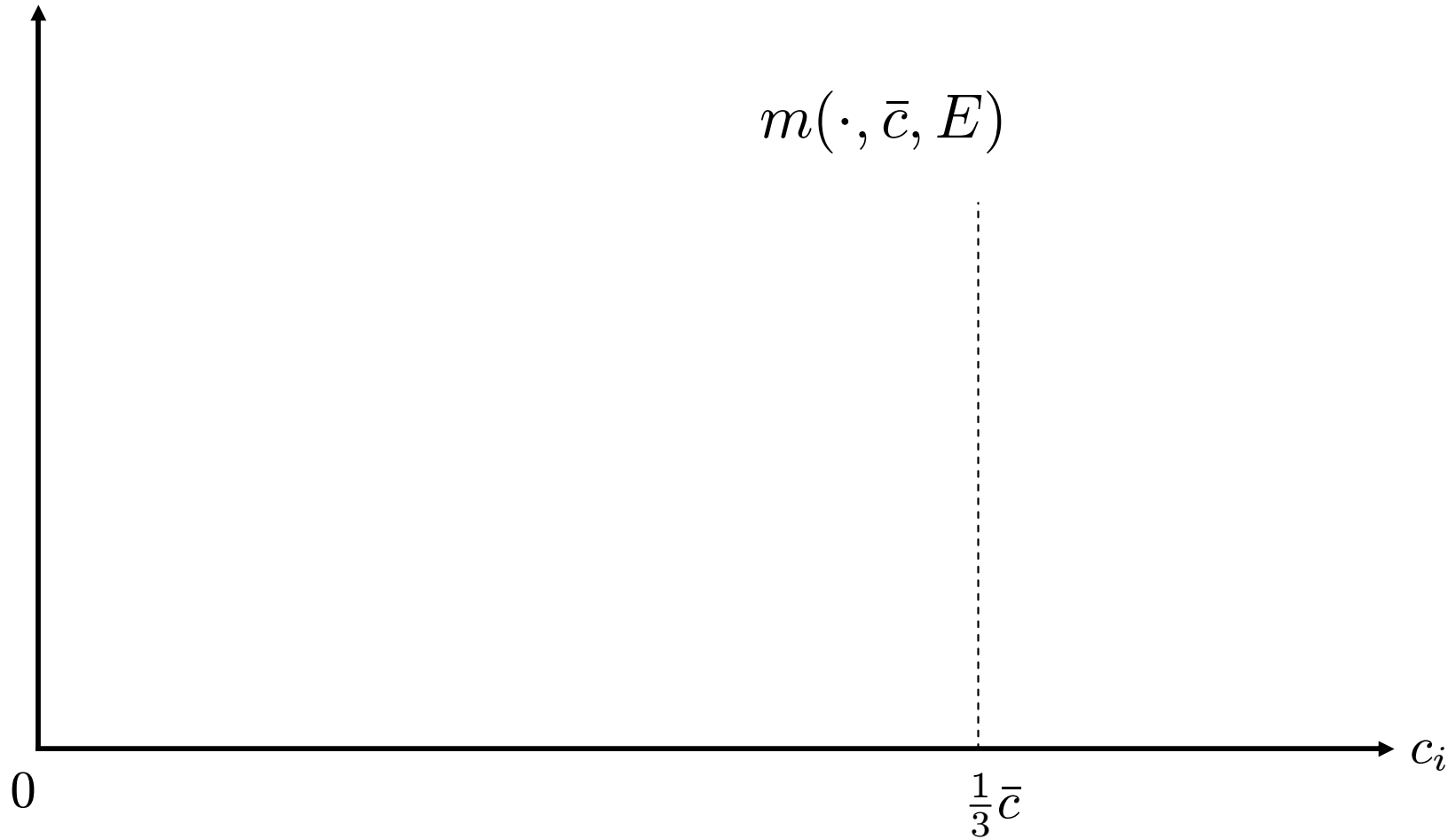
- Continuous
- Given  $\bar{c}$  and  $E$ ,

# Results (n=3)

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$$m(c_i, \bar{c}, E)$$

- Continuous
- Given  $\bar{c}$  and  $E$ ,

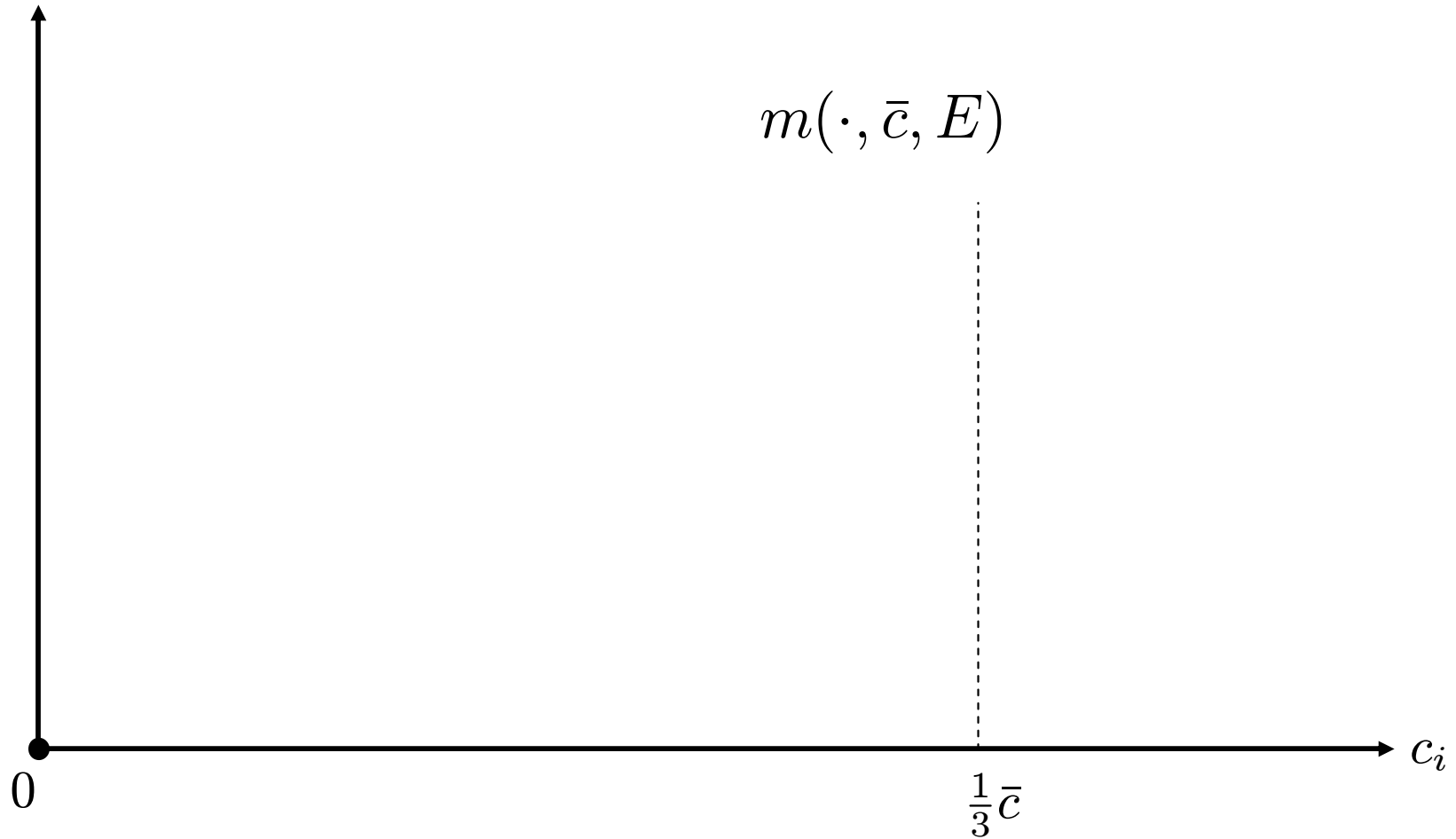


# Results (n=3)

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$$m(c_i, \bar{c}, E)$$

- Continuous
- Given  $\bar{c}$  and  $E$ ,

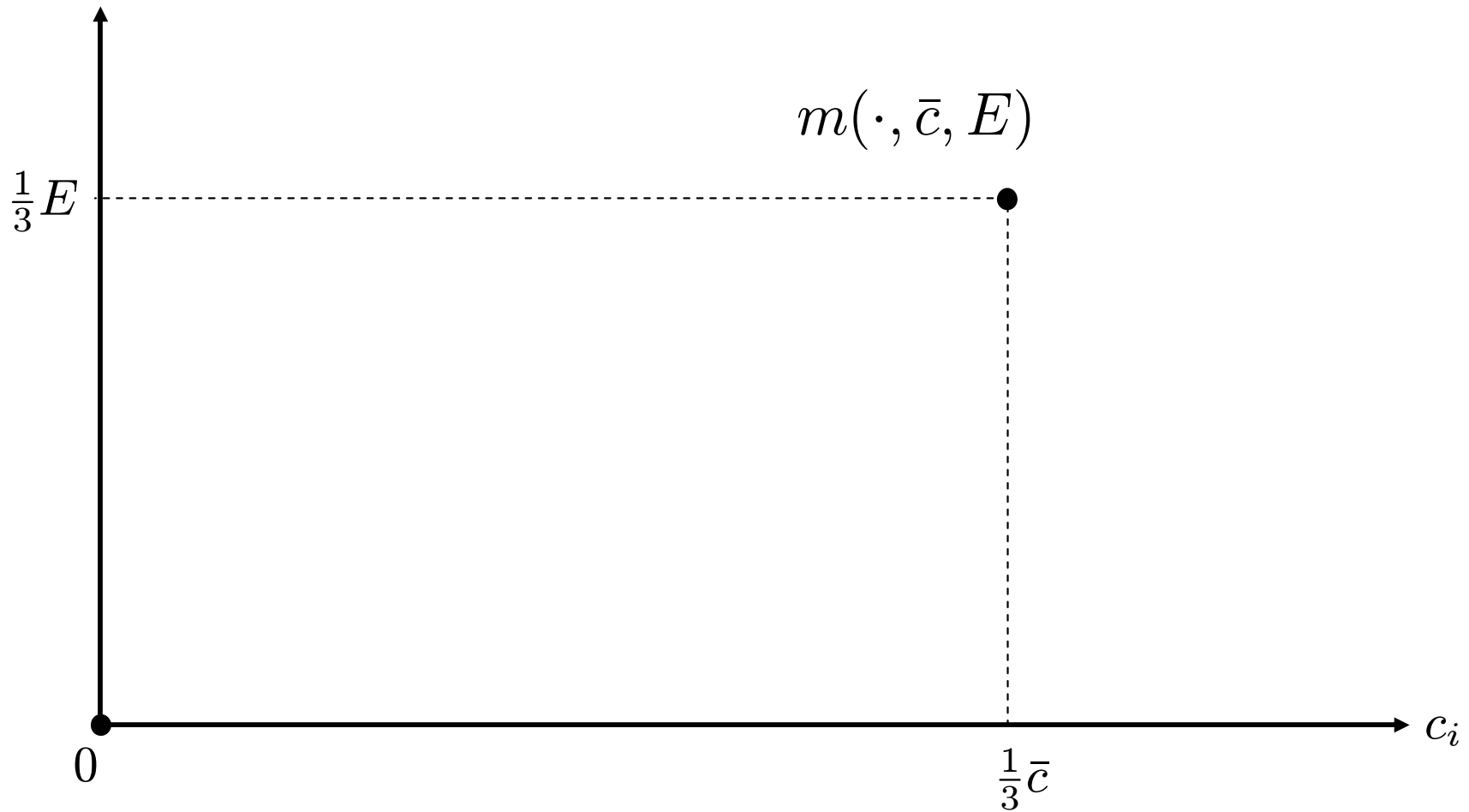


# Results (n=3)

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$$m(c_i, \bar{c}, E)$$

- Continuous
- Given  $\bar{c}$  and  $E$ ,

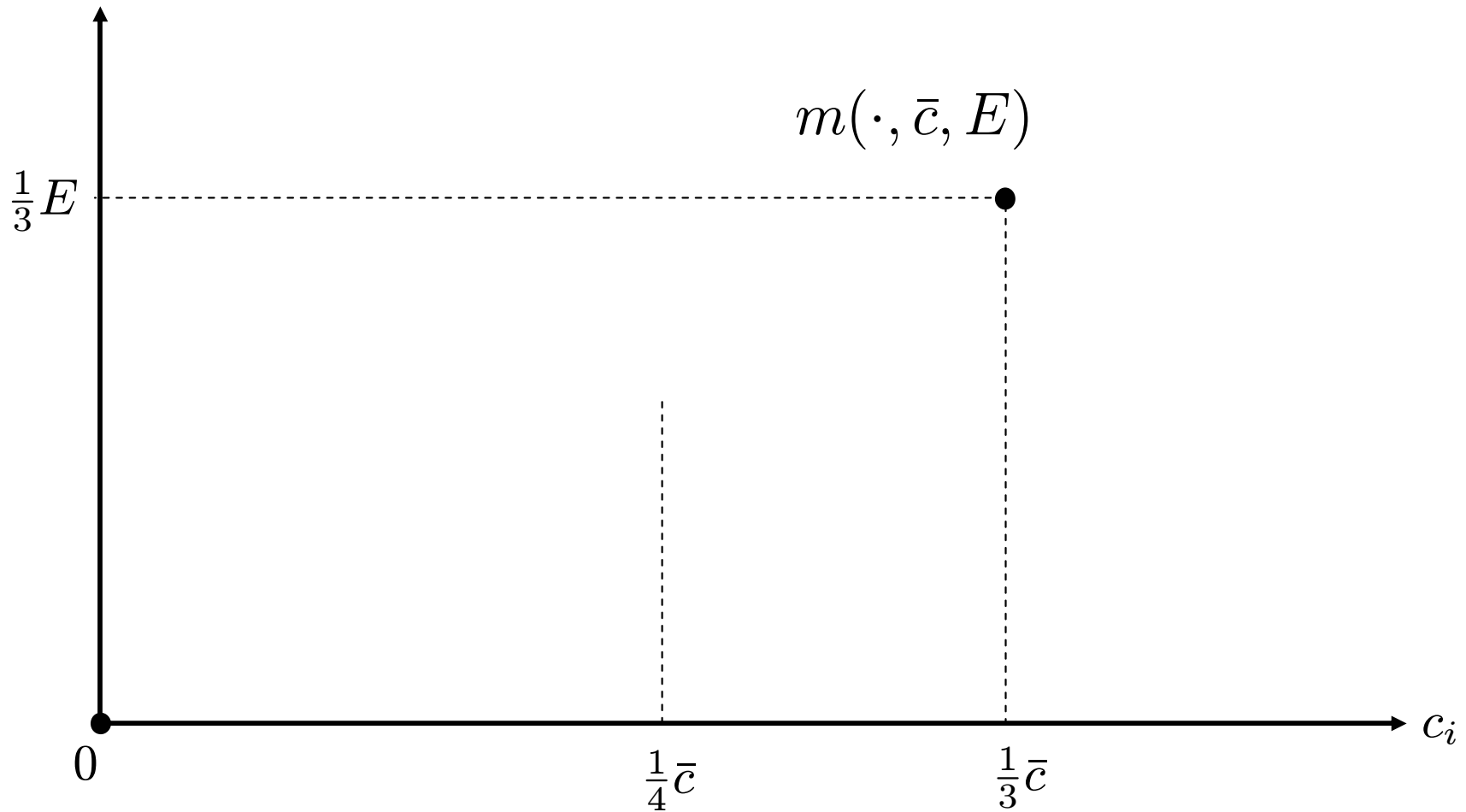


# Results (n=3)

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$$m(c_i, \bar{c}, E)$$

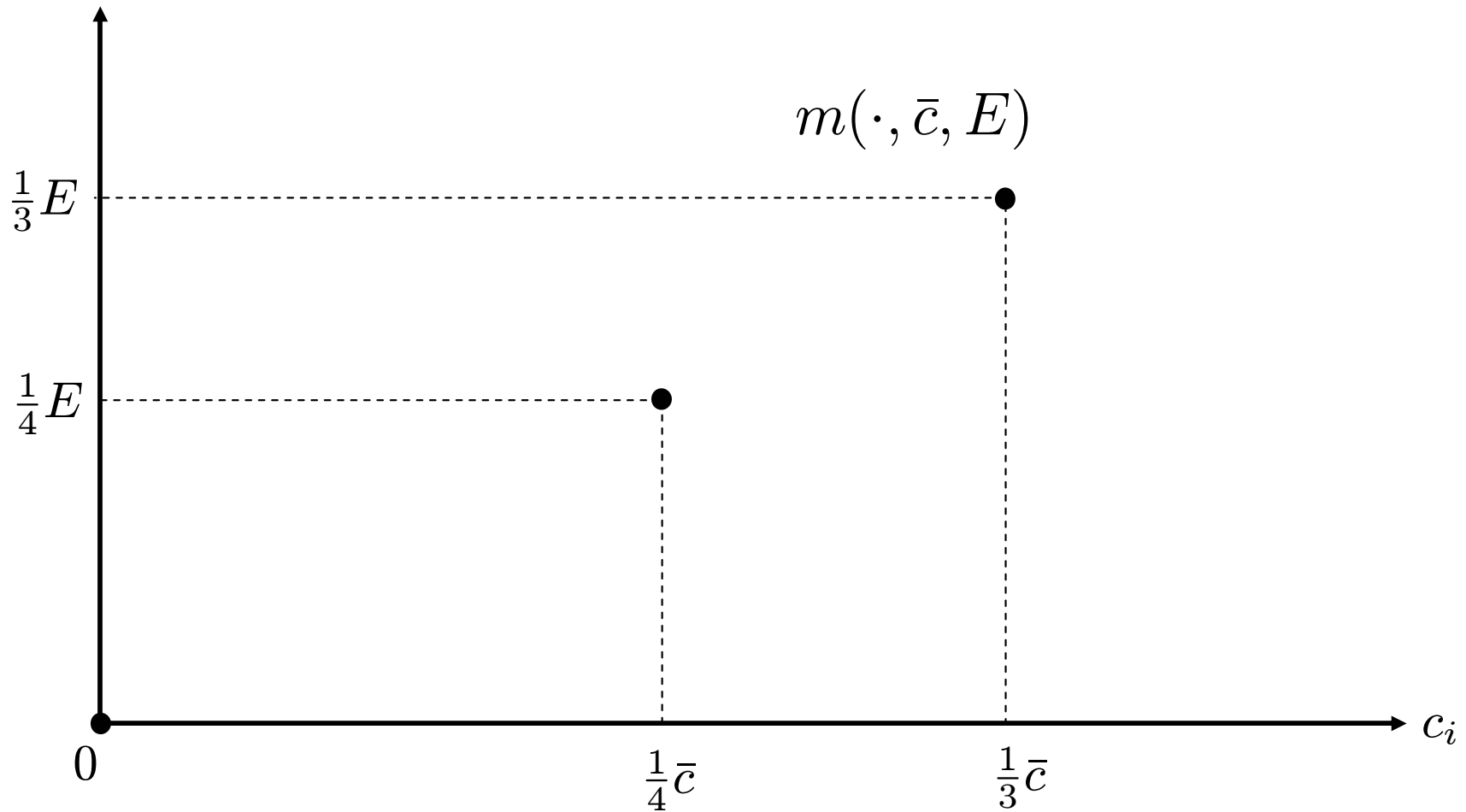
- Continuous
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# Results (n=3)

$$m(c_i, \bar{c}, E)$$

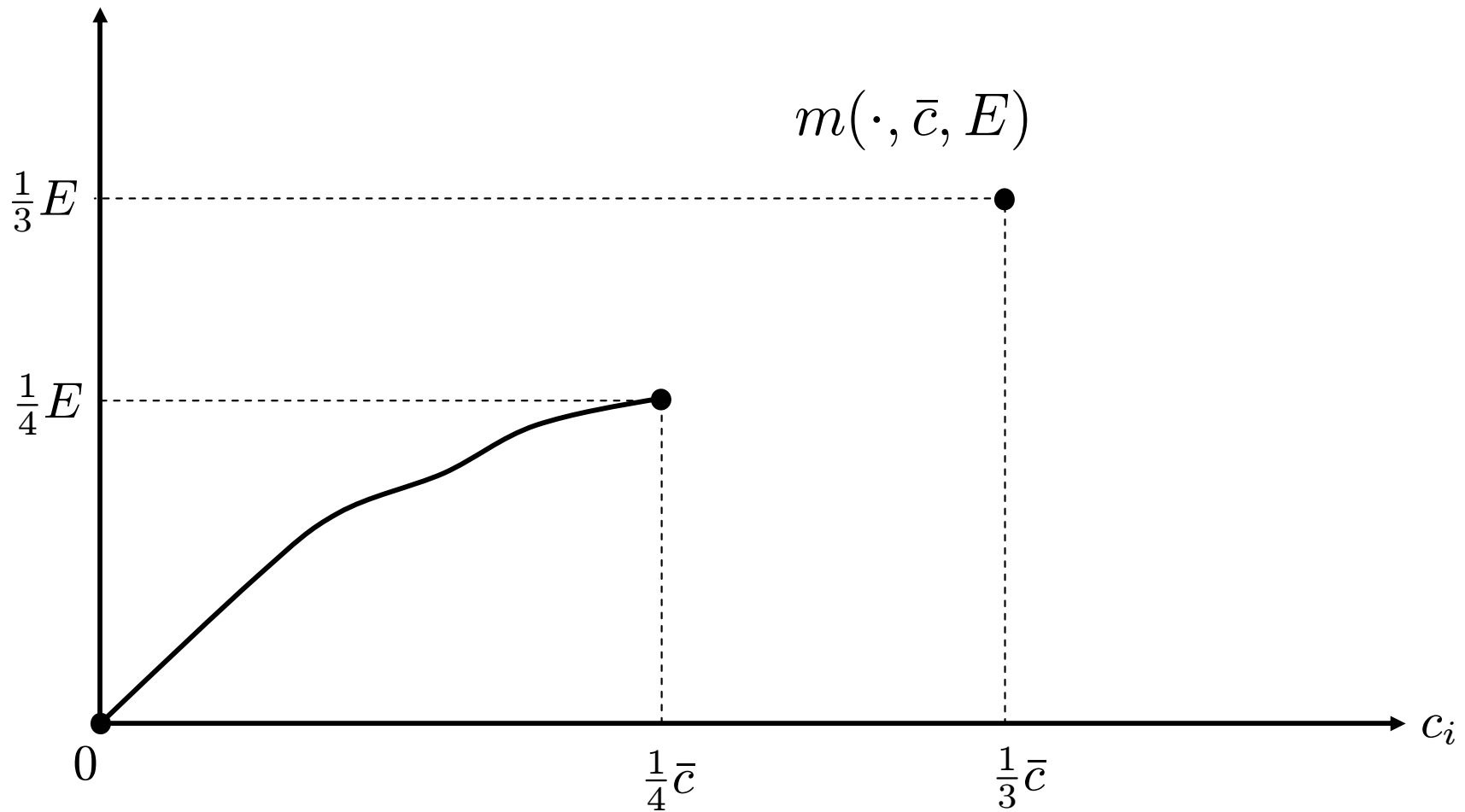
- Continuous
- Given  $\bar{c}$  and  $E$ ,



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$$m(c_i, \bar{c}, E)$$

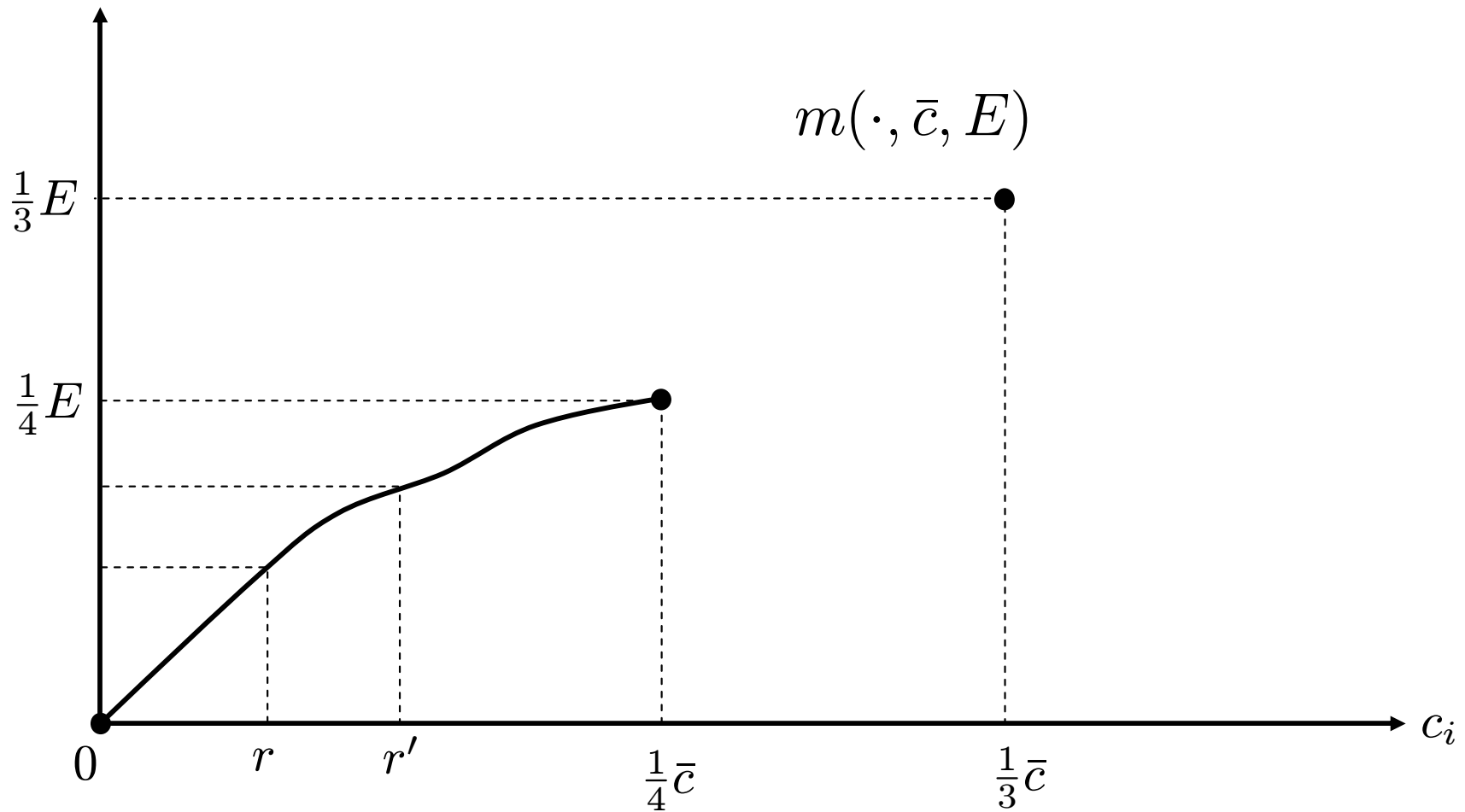
- Continuous
- Given  $\bar{c}$  and  $E$ ,



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$$m(c_i, \bar{c}, E)$$

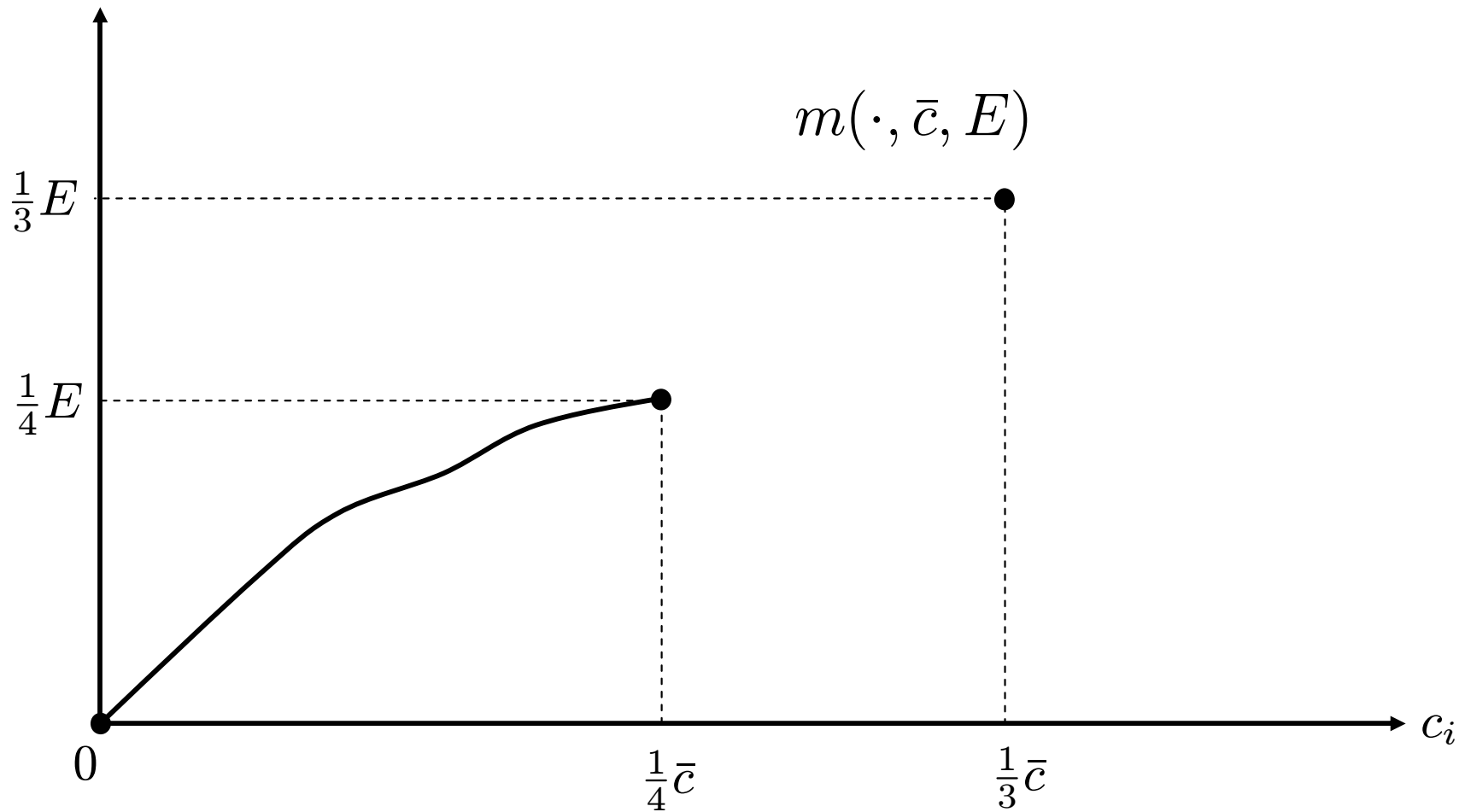
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# Results (n=3)

$$m(c_i, \bar{c}, E)$$

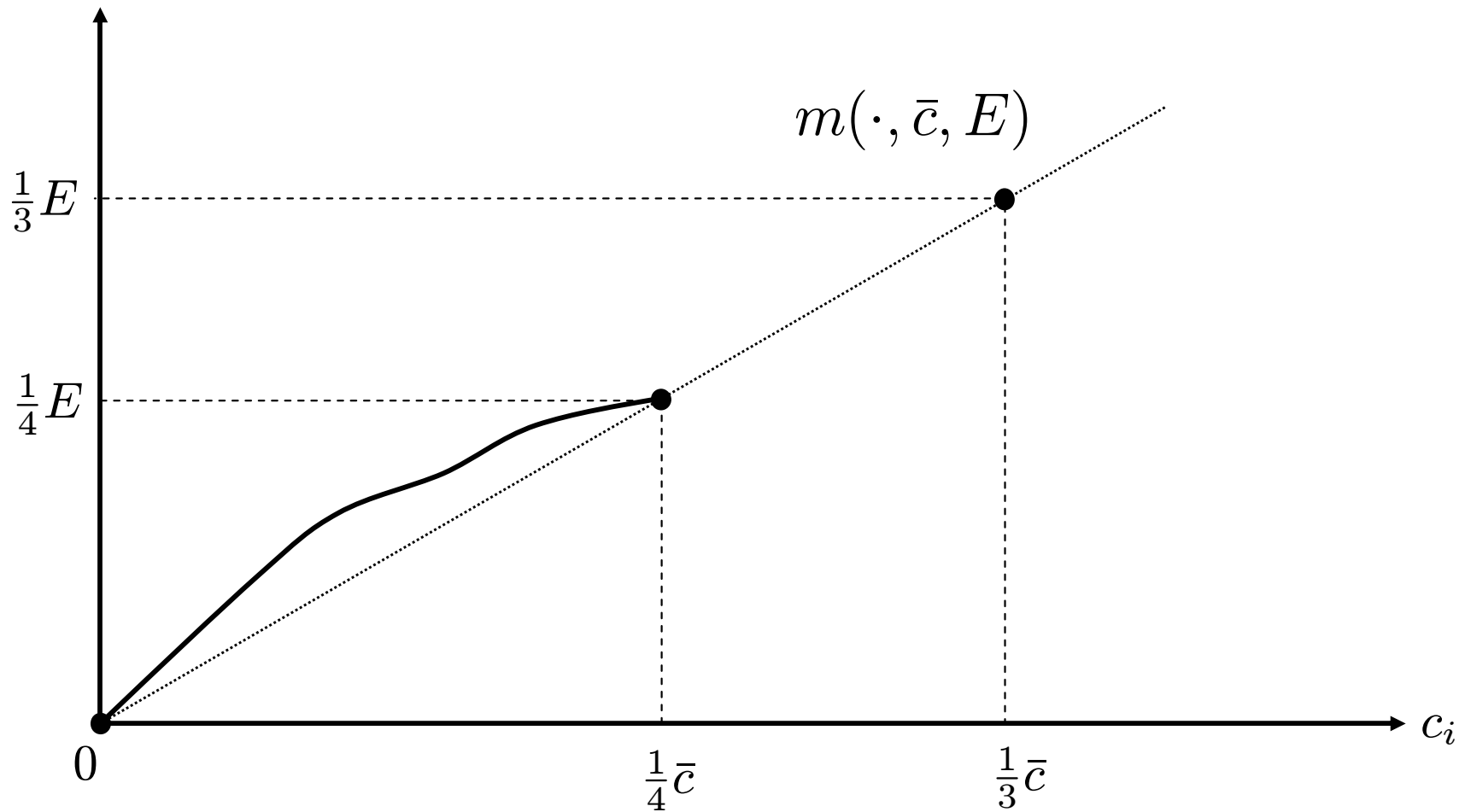
- Continuous
- Given  $\bar{c}$  and  $E$ ,



# Results (n=3)

$$m(c_i, \bar{c}, E)$$

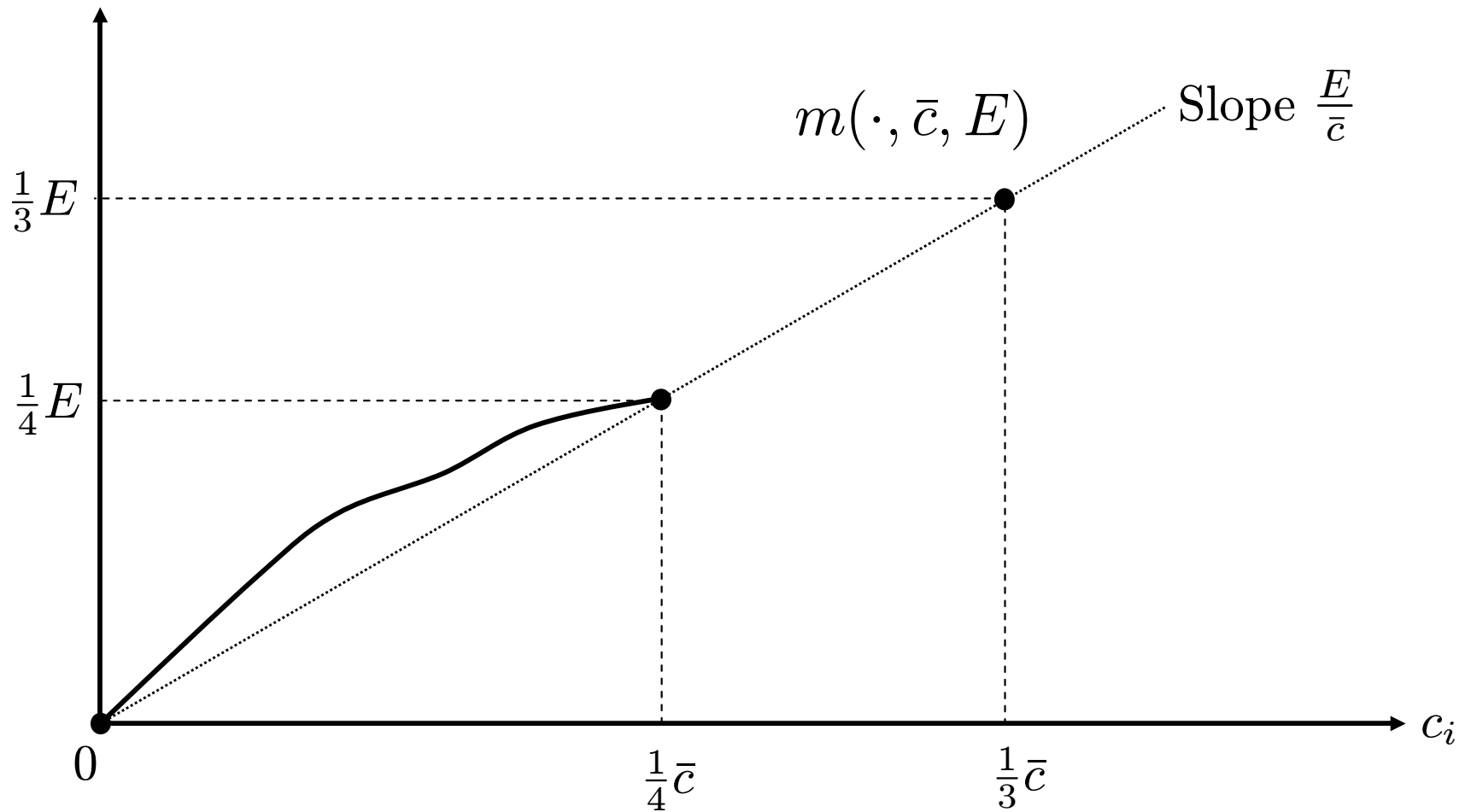
- Continuous
- Given  $\bar{c}$  and  $E$ ,



# Results (n=3)

$$m(c_i, \bar{c}, E)$$

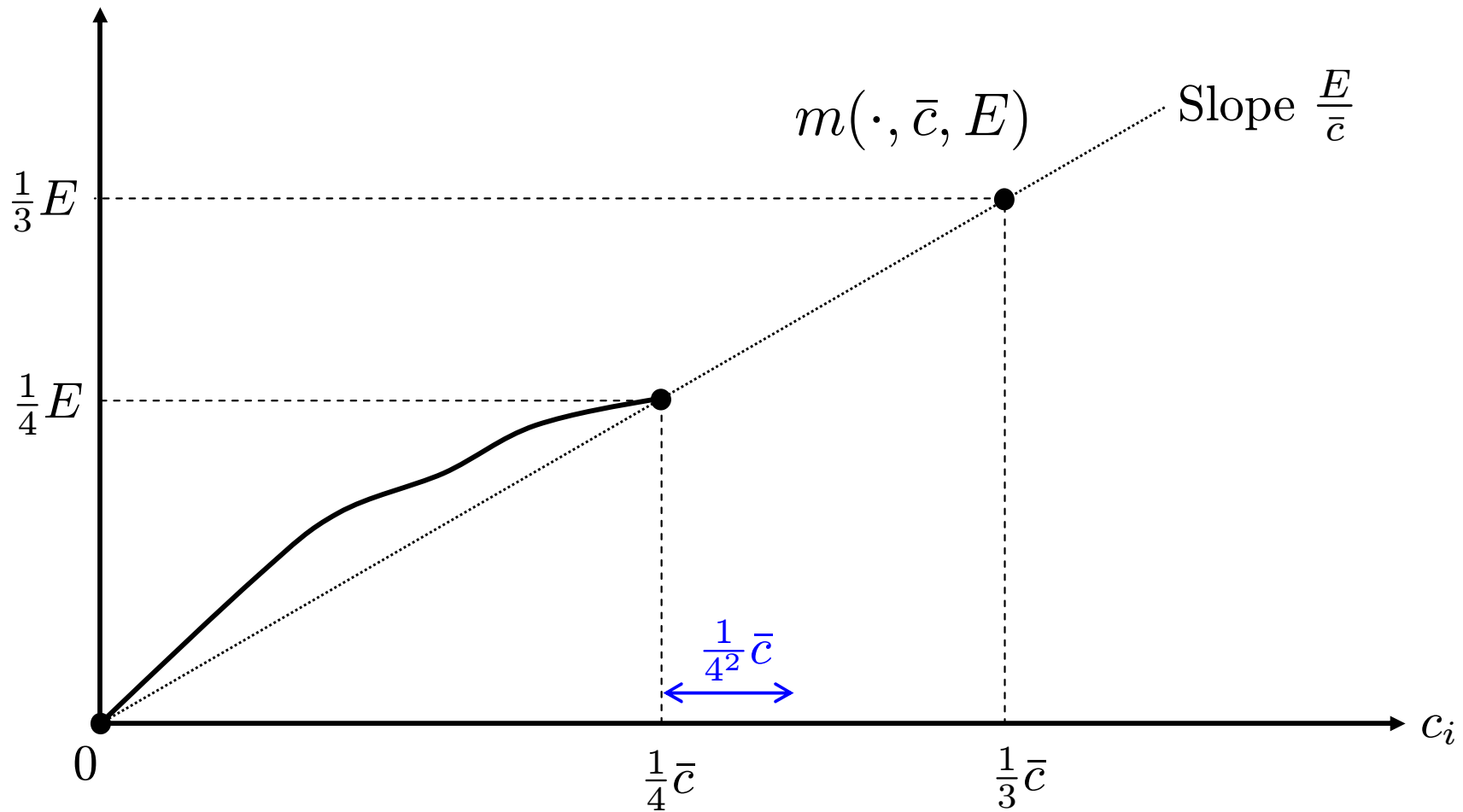
- Continuous
- Given  $\bar{c}$  and  $E$ ,



# Results (n=3)

$$m(c_i, \bar{c}, E)$$

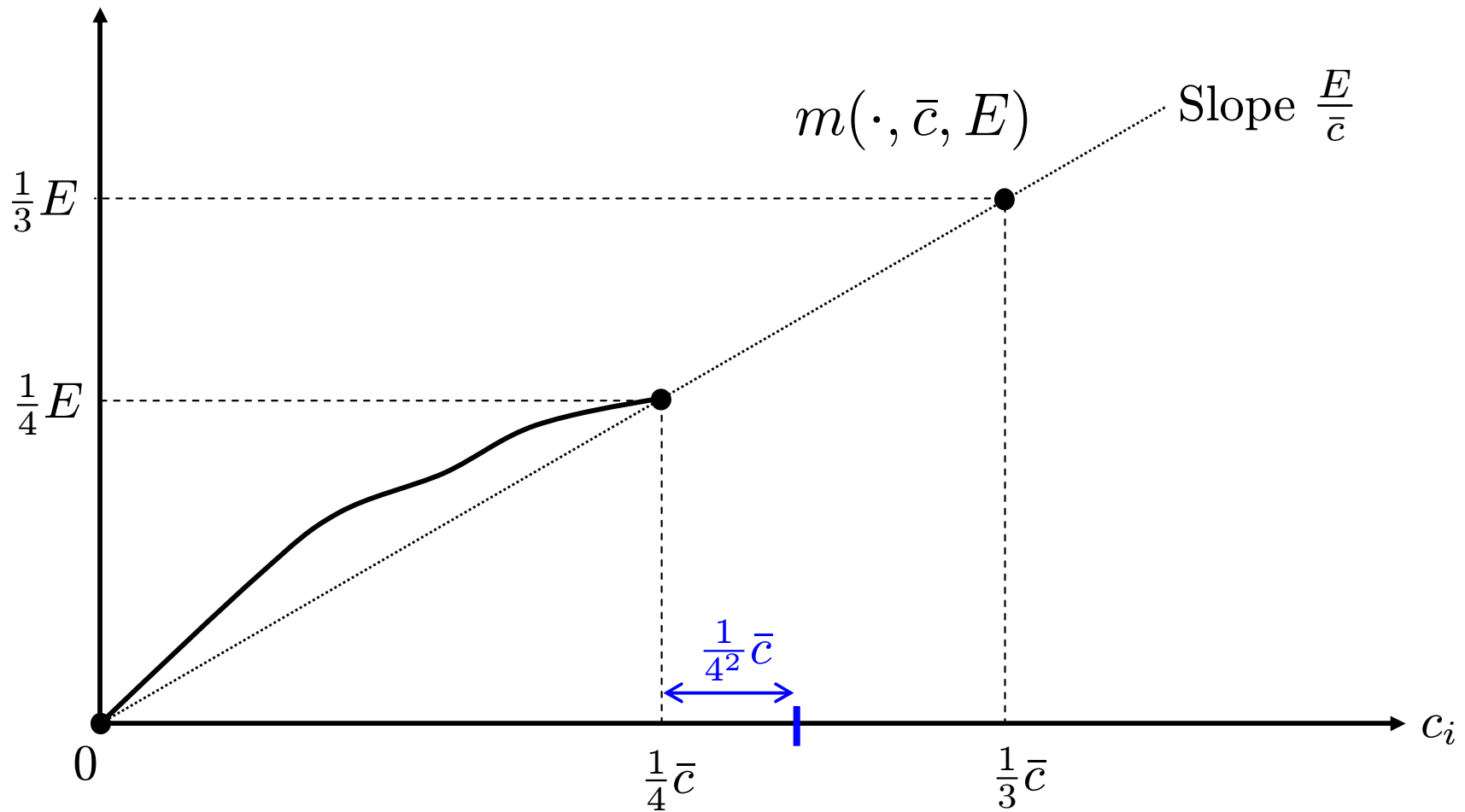
- Continuous
- Given  $\bar{c}$  and  $E$ ,



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$$m(c_i, \bar{c}, E)$$

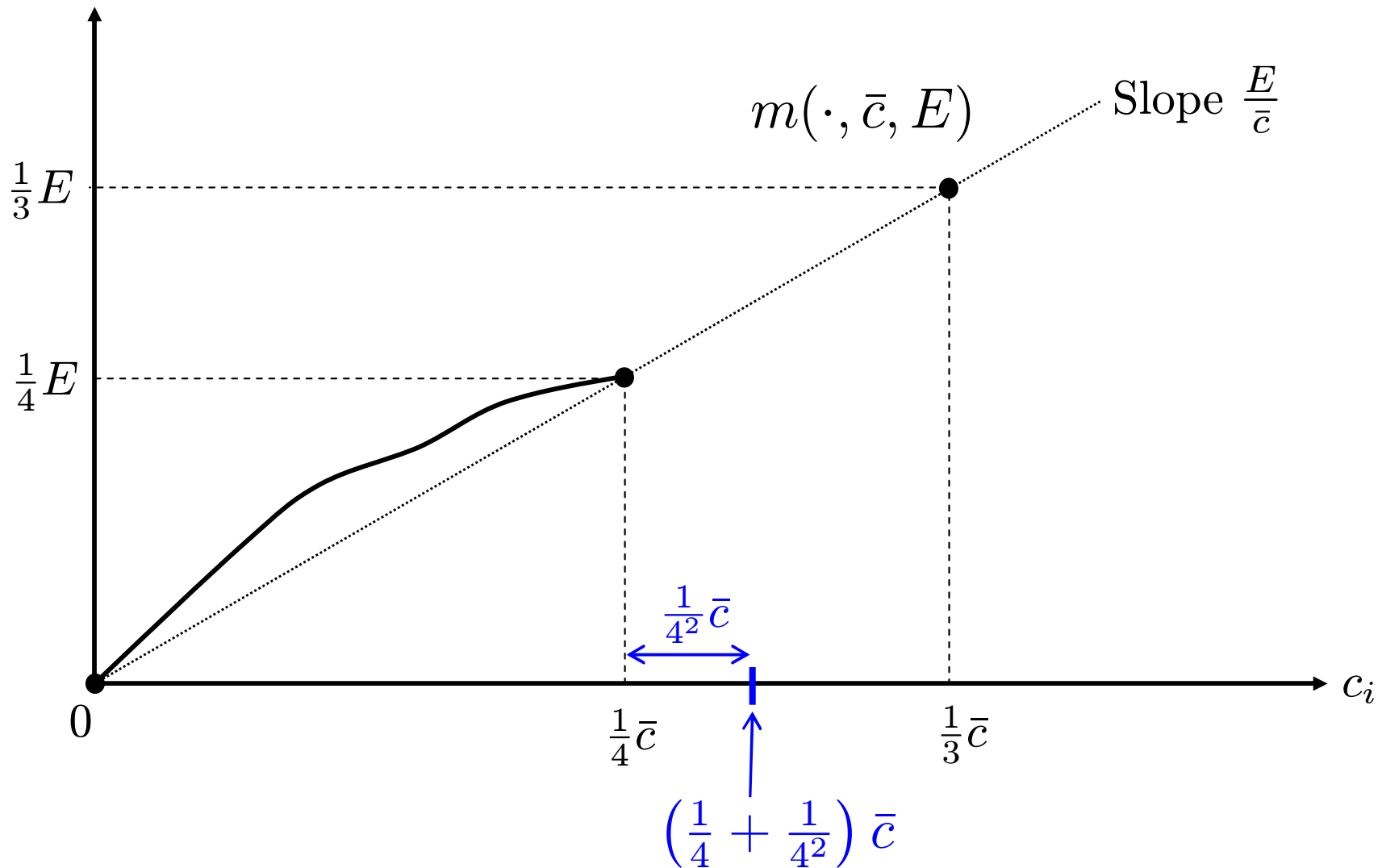
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# Results (n=3)

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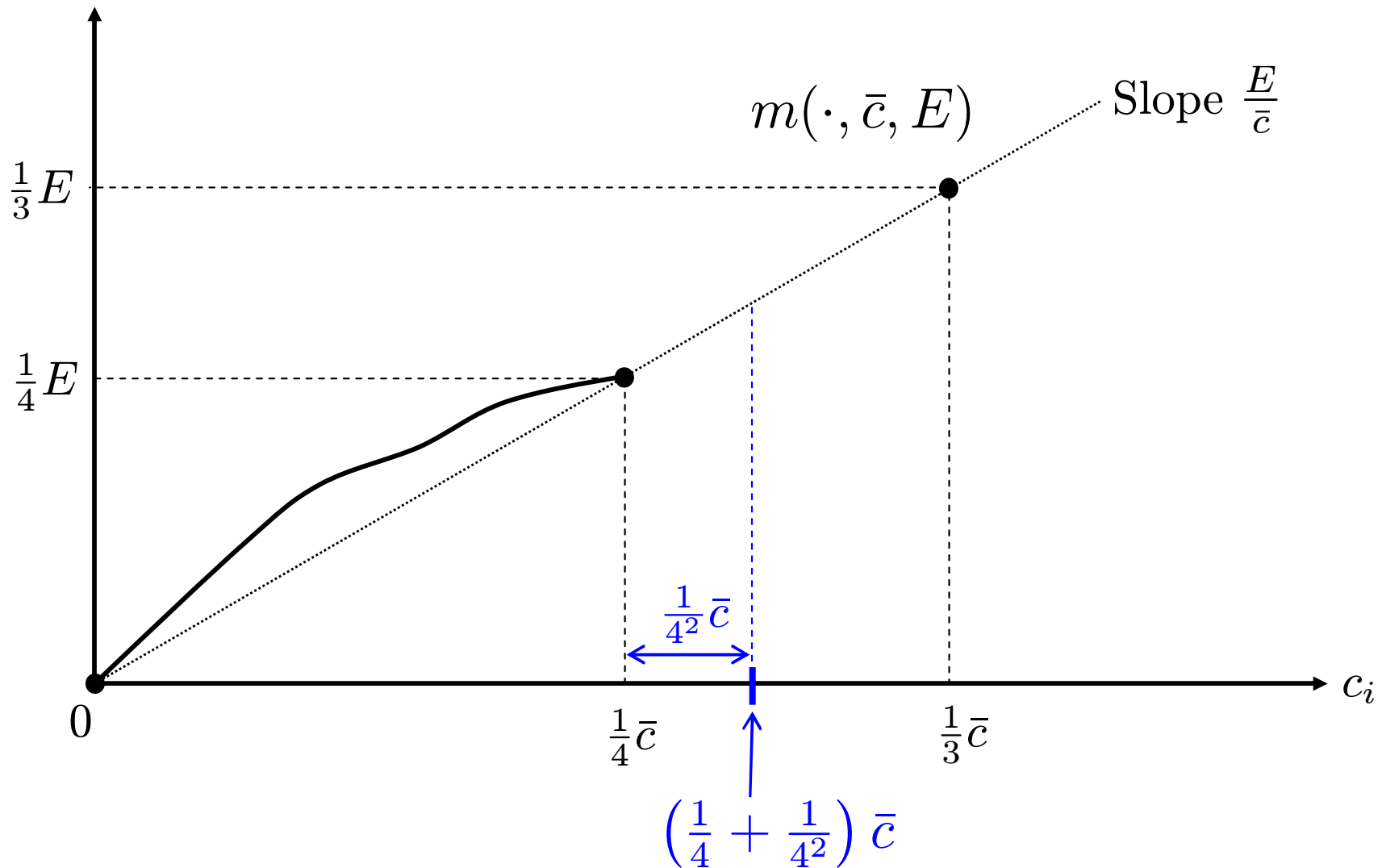
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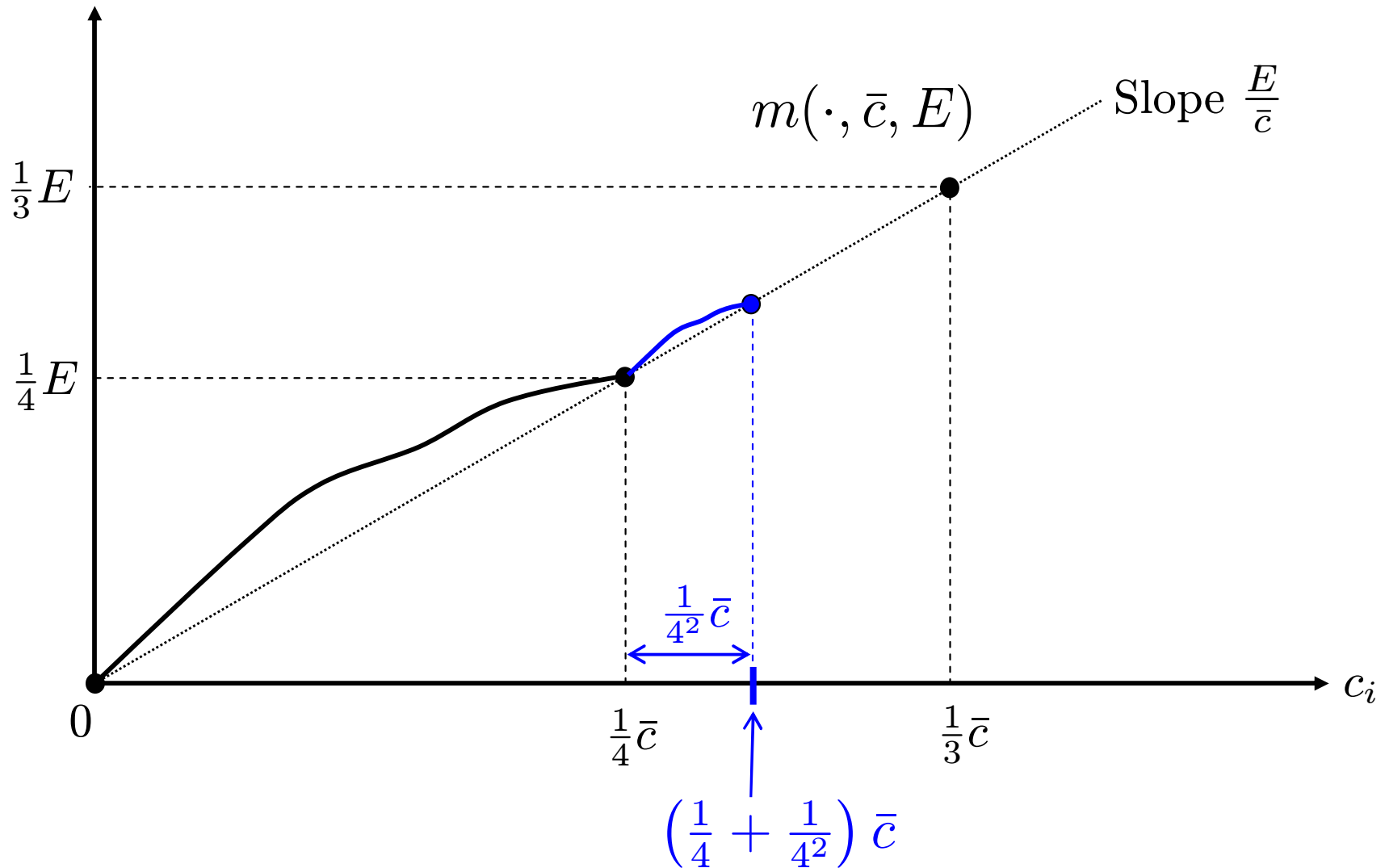
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$$m(c_i, \bar{c}, E)$$

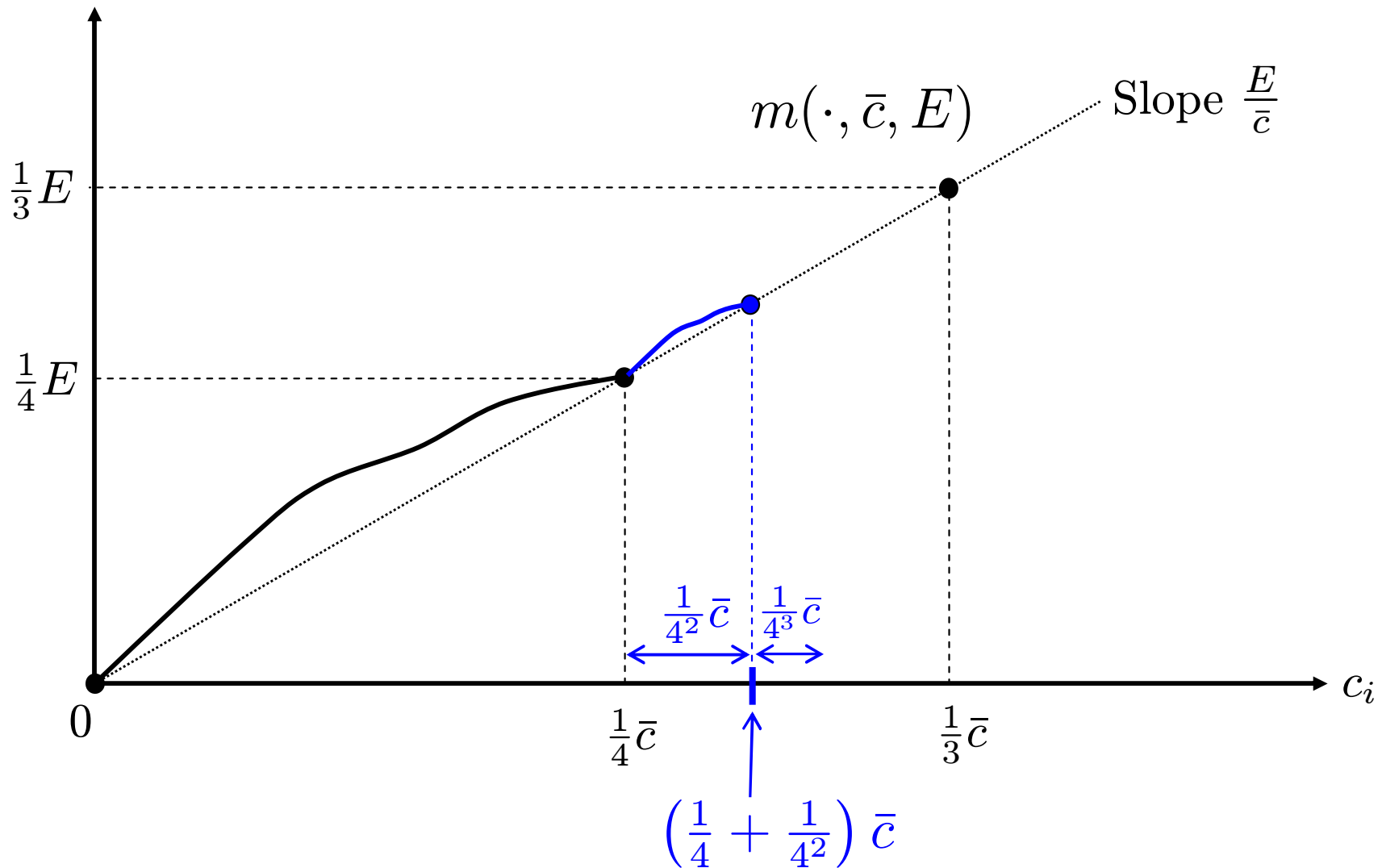
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$$m(c_i, \bar{c}, E)$$

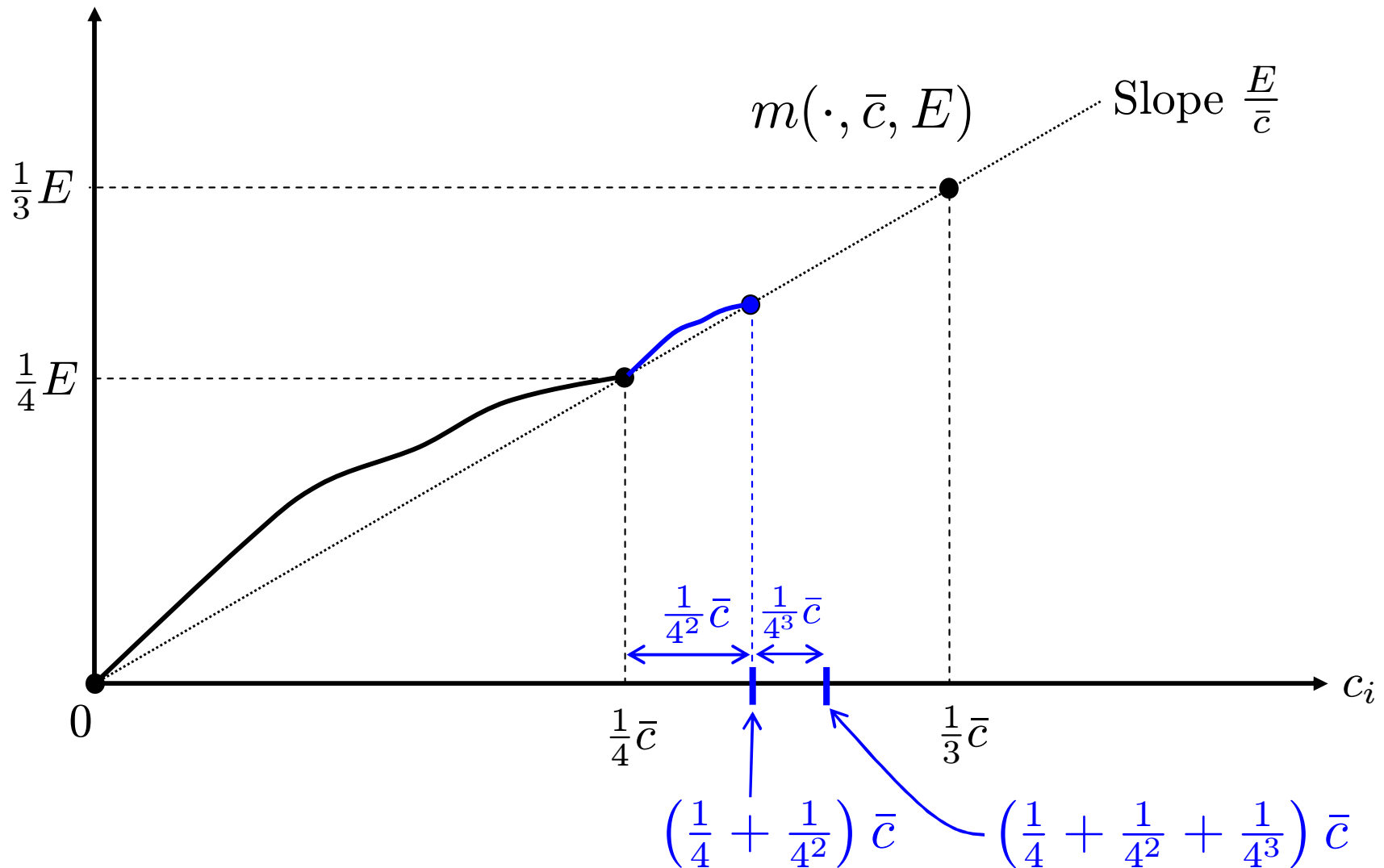
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- Given  $\bar{c}$  and  $E$ ,



# Results (n=3)

$$m(c_i, \bar{c}, E)$$

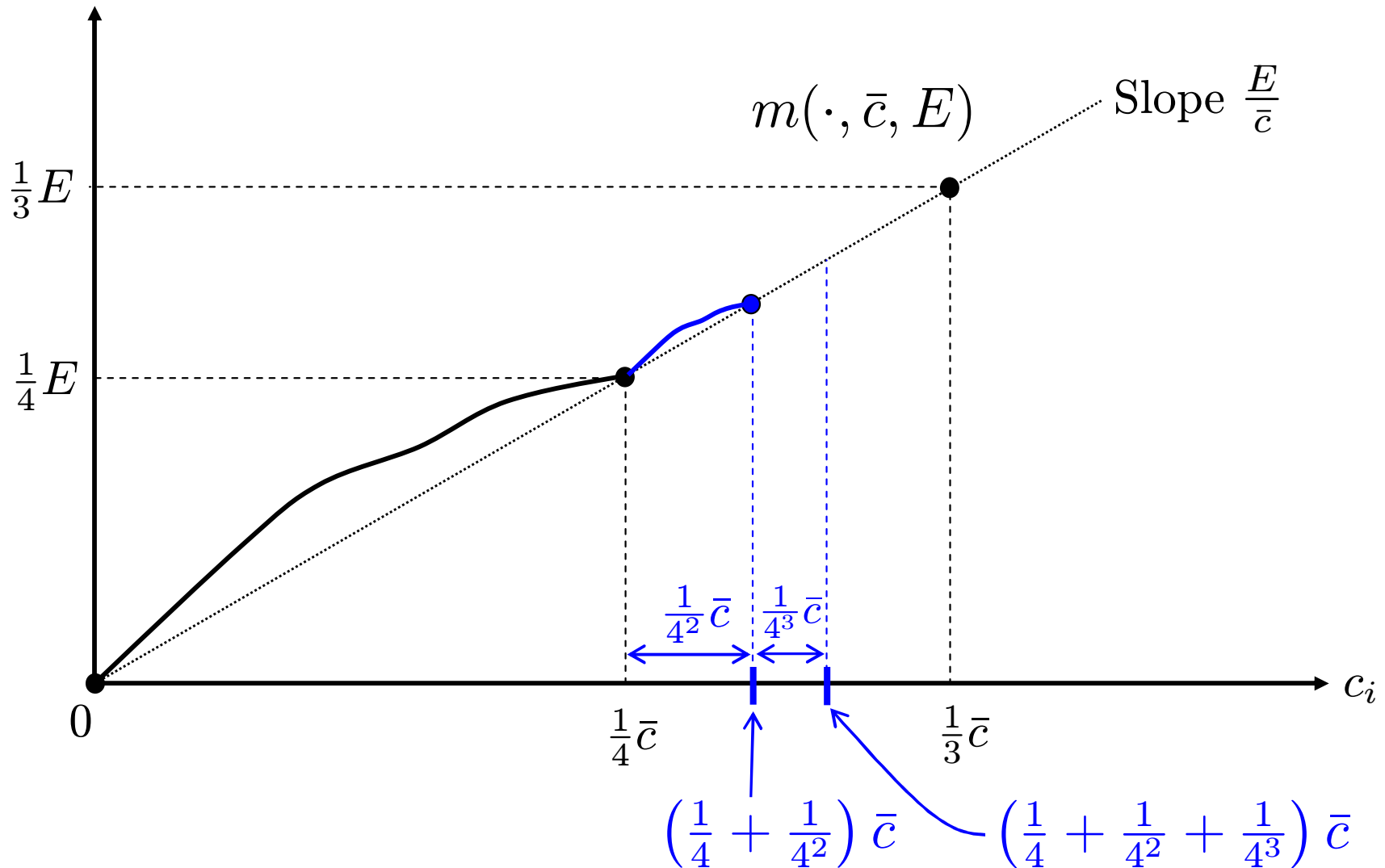
- Continuous
- Given  $\bar{c}$  and  $E$ ,



# Results (n=3)

$$m(c_i, \bar{c}, E)$$

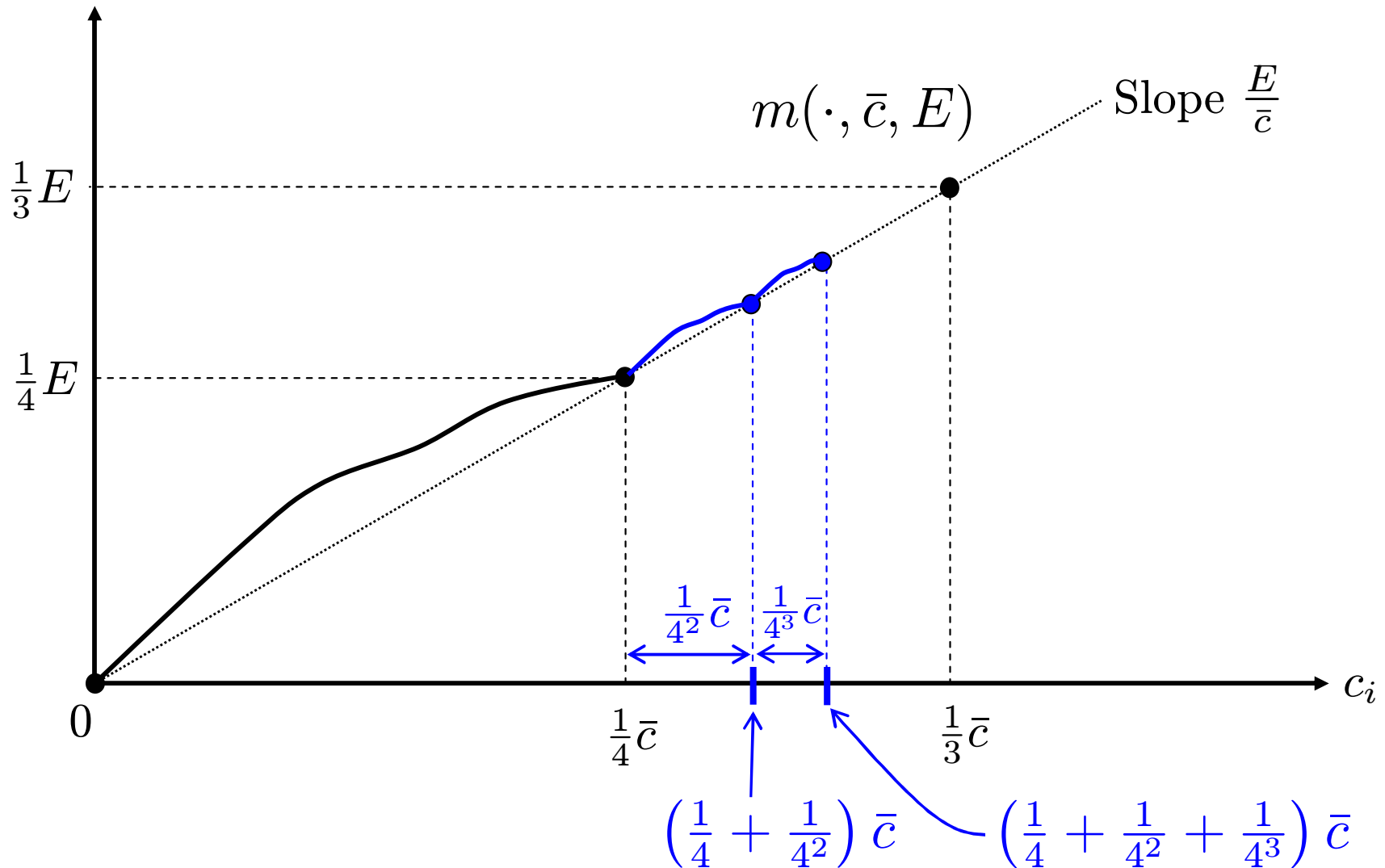
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- Given  $\bar{c}$  and  $E$ ,



# Results (n=3)

$$m(c_i, \bar{c}, E)$$

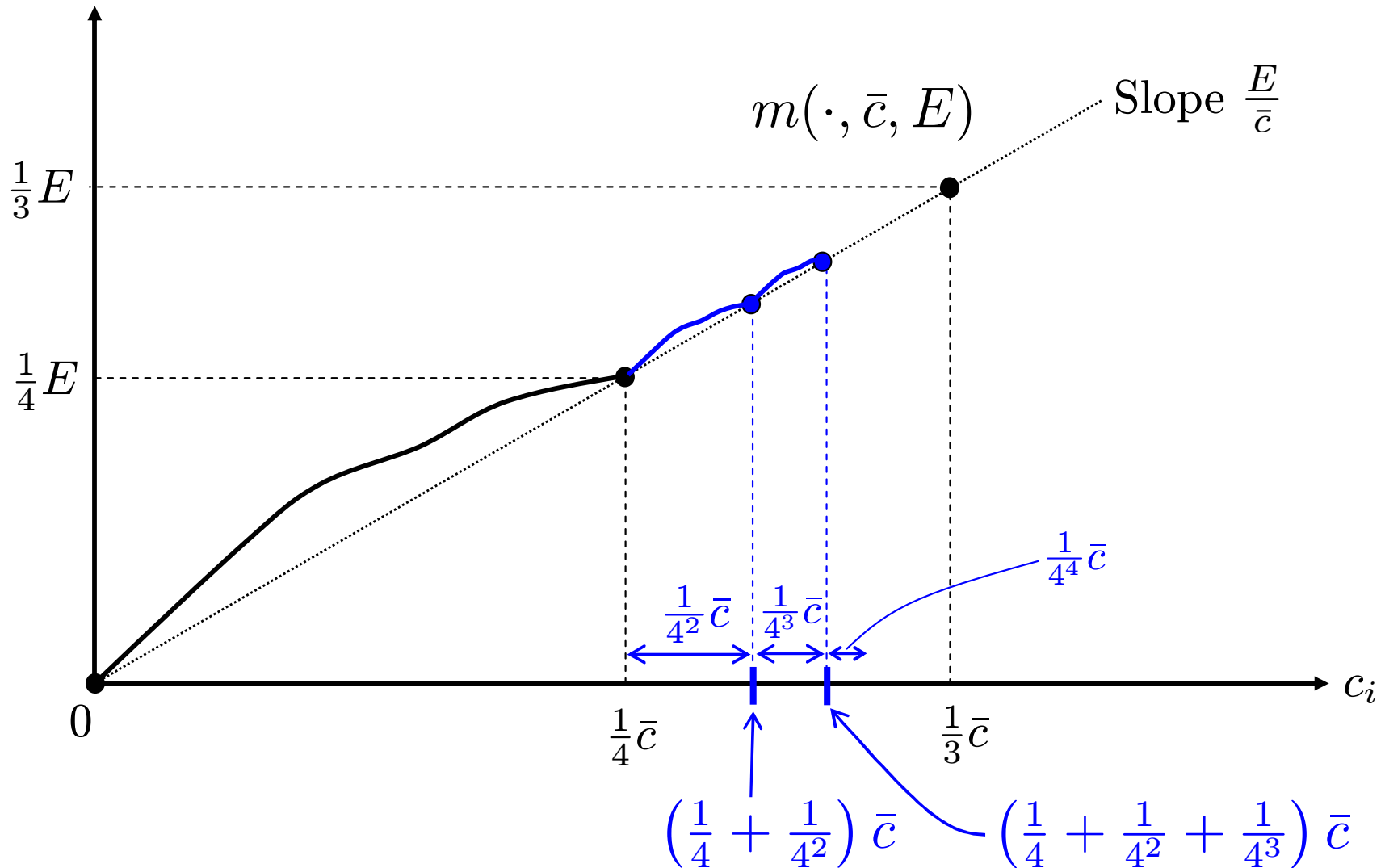
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- Given  $\bar{c}$  and  $E$ ,



# Results (n=3)

$$m(c_i, \bar{c}, E)$$

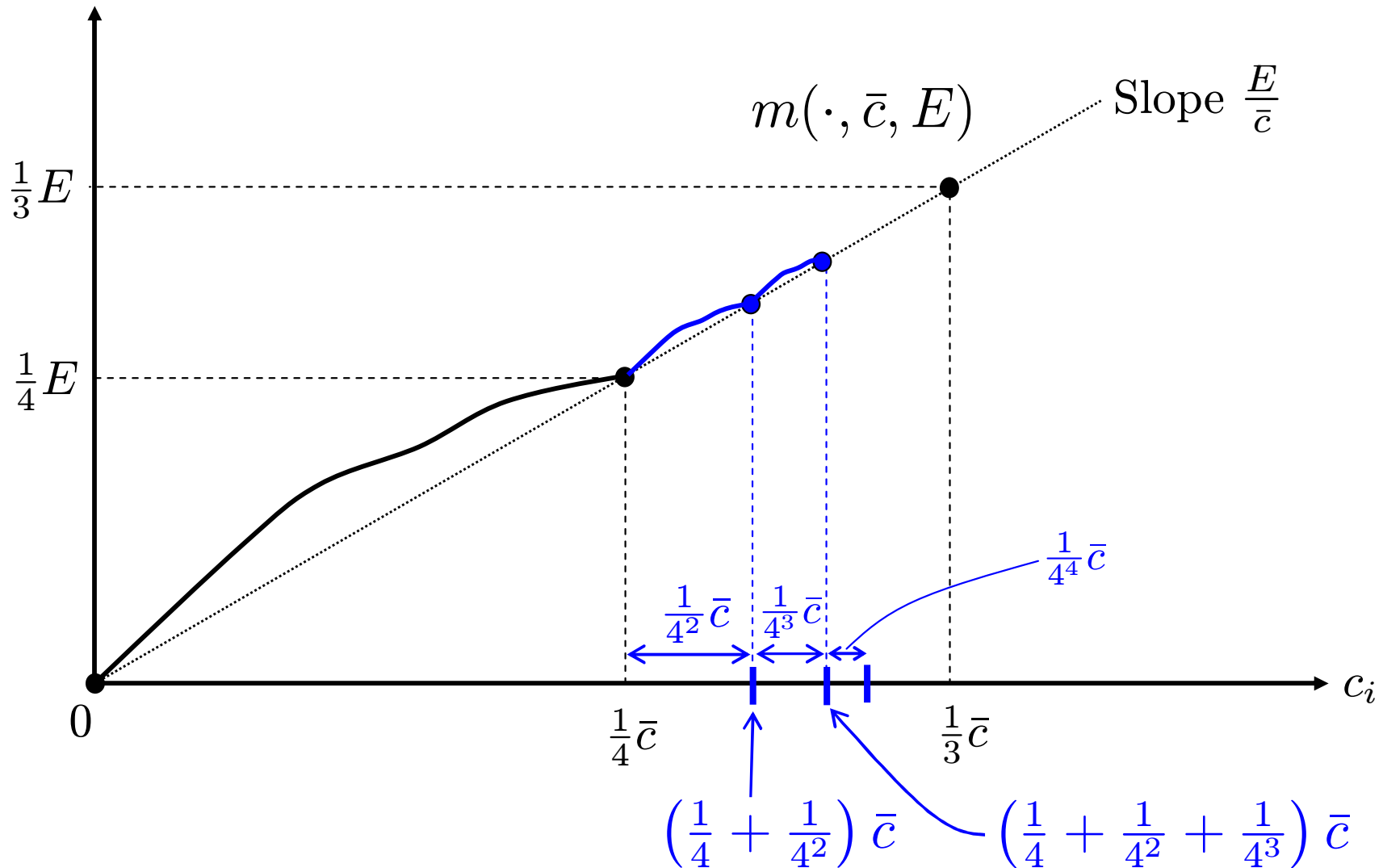
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- Given  $\bar{c}$  and  $E$ ,



# Results (n=3)

$$m(c_i, \bar{c}, E)$$

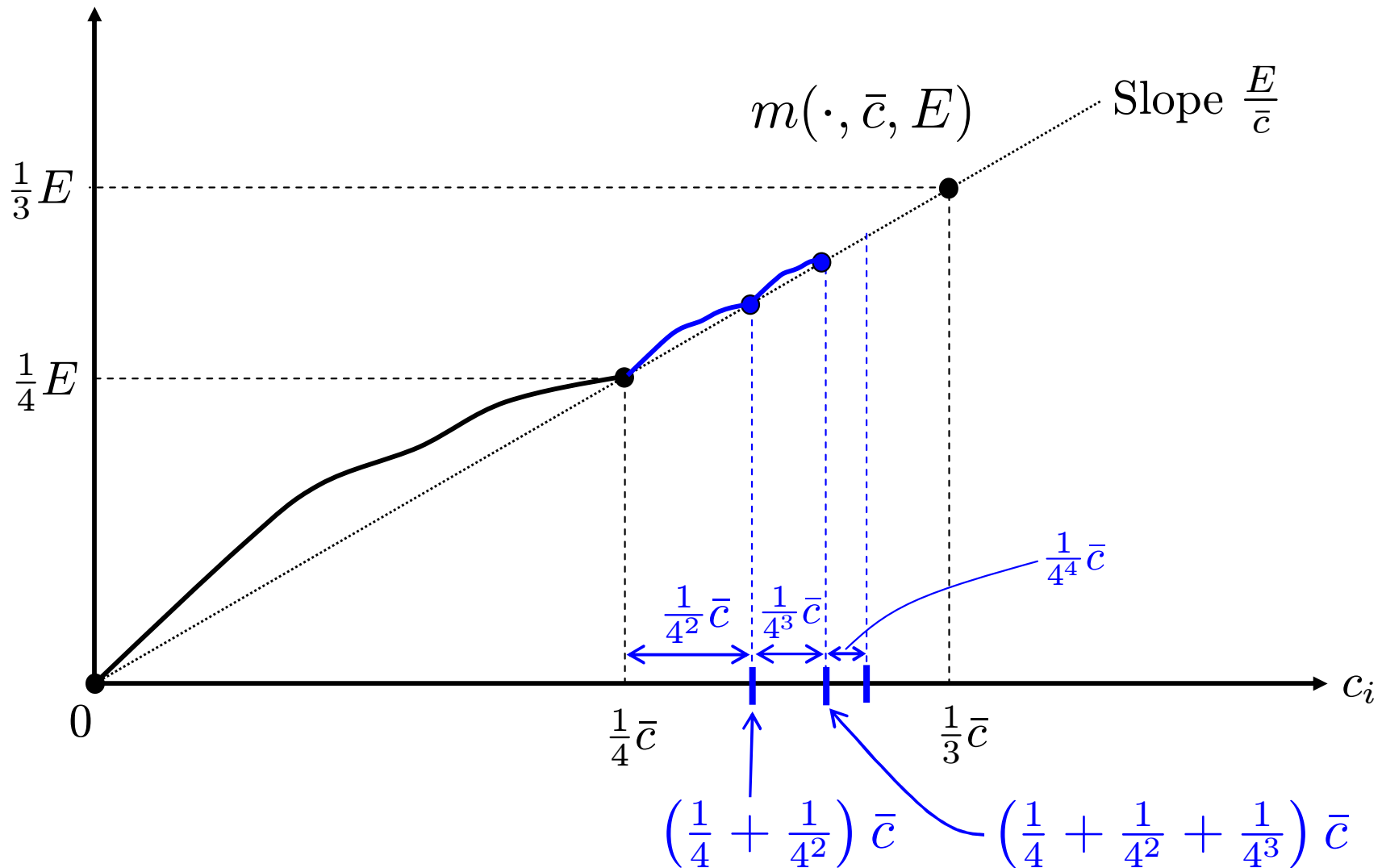
- Continuous
- Given  $\bar{c}$  and  $E$ ,



# Results (n=3)

$$m(c_i, \bar{c}, E)$$

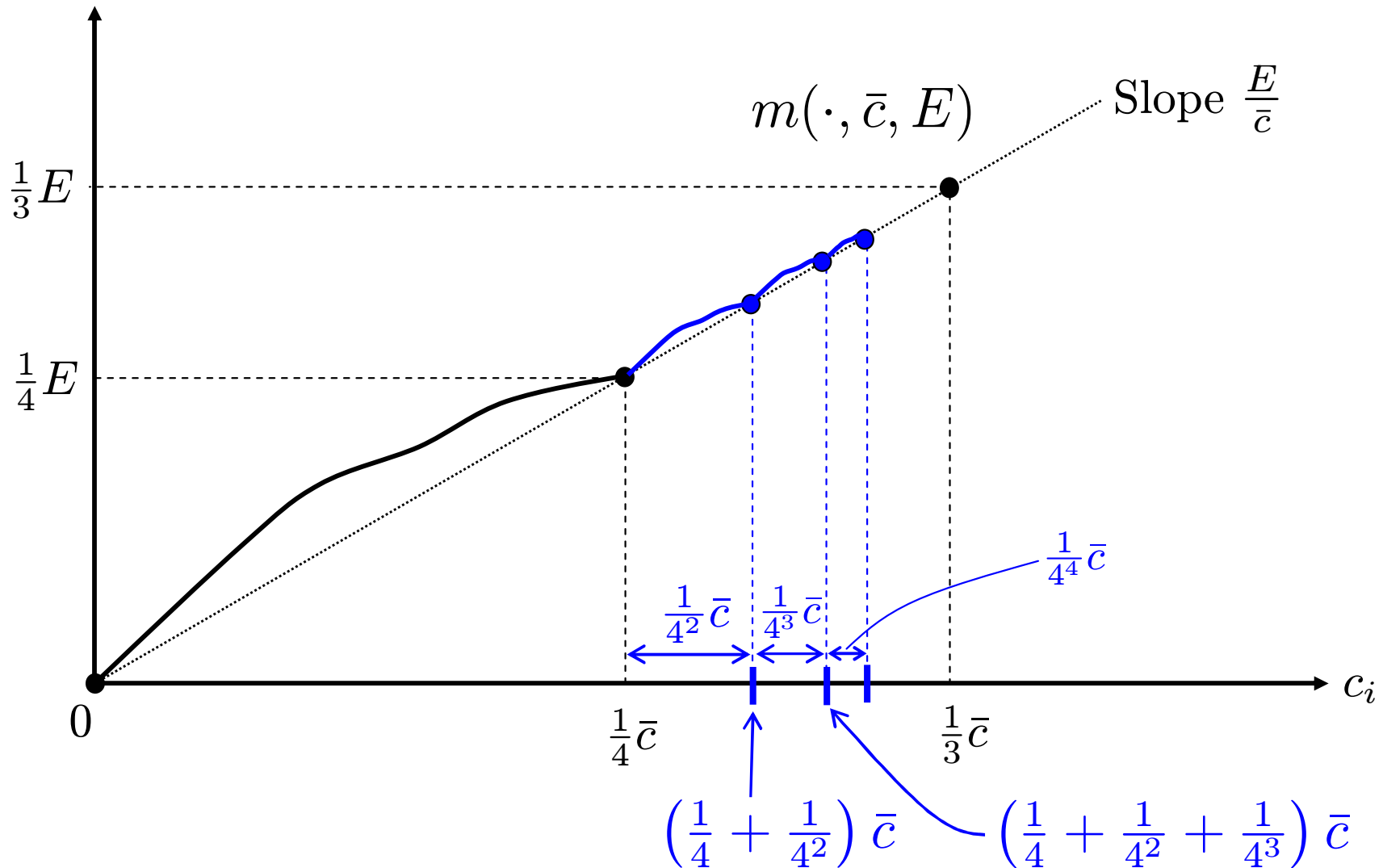
- Continuous
- Given  $\bar{c}$  and  $E$ ,



# Results (n=3)

$$m(c_i, \bar{c}, E)$$

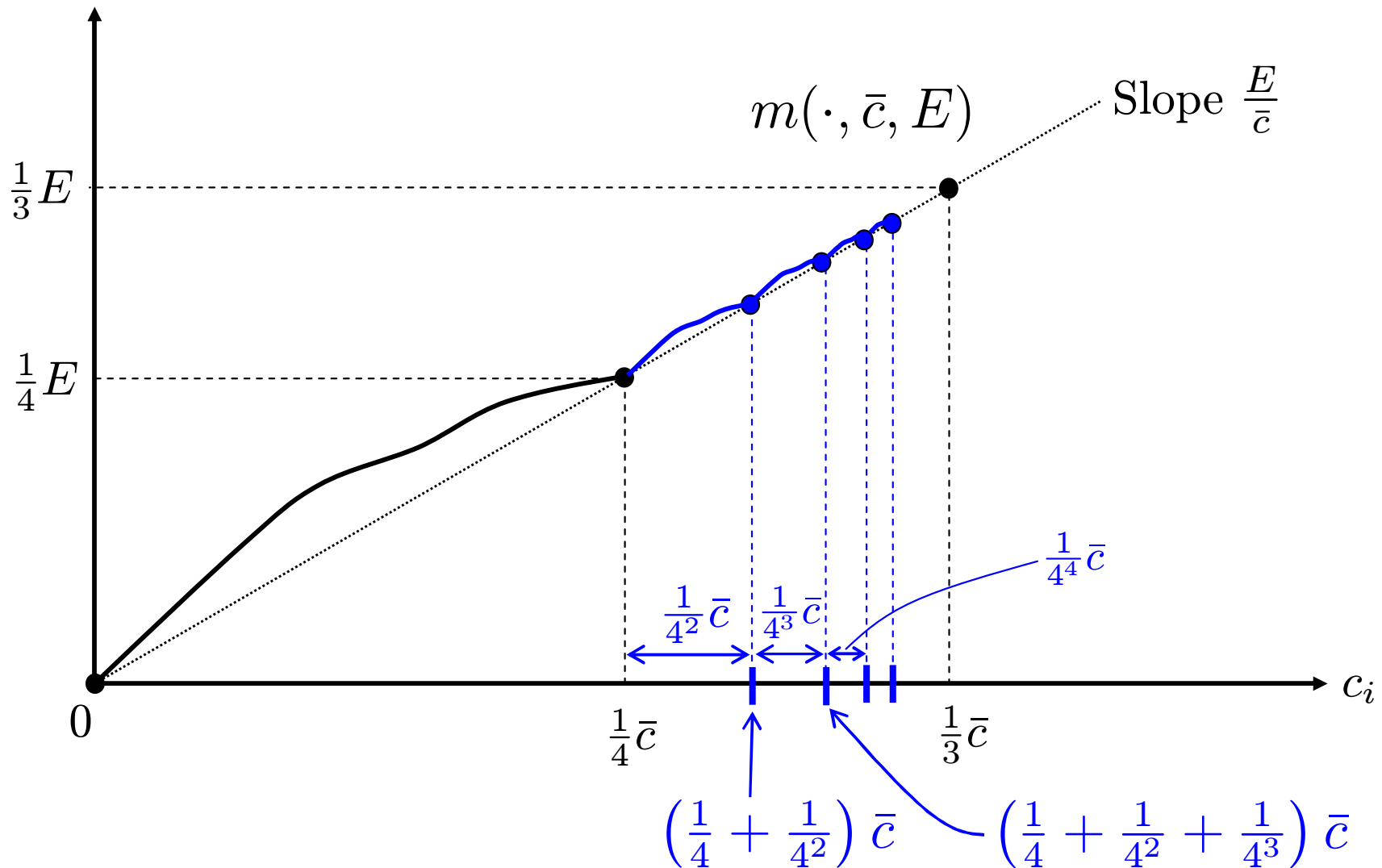
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# Results (n=3)

$$m(c_i, \bar{c}, E)$$

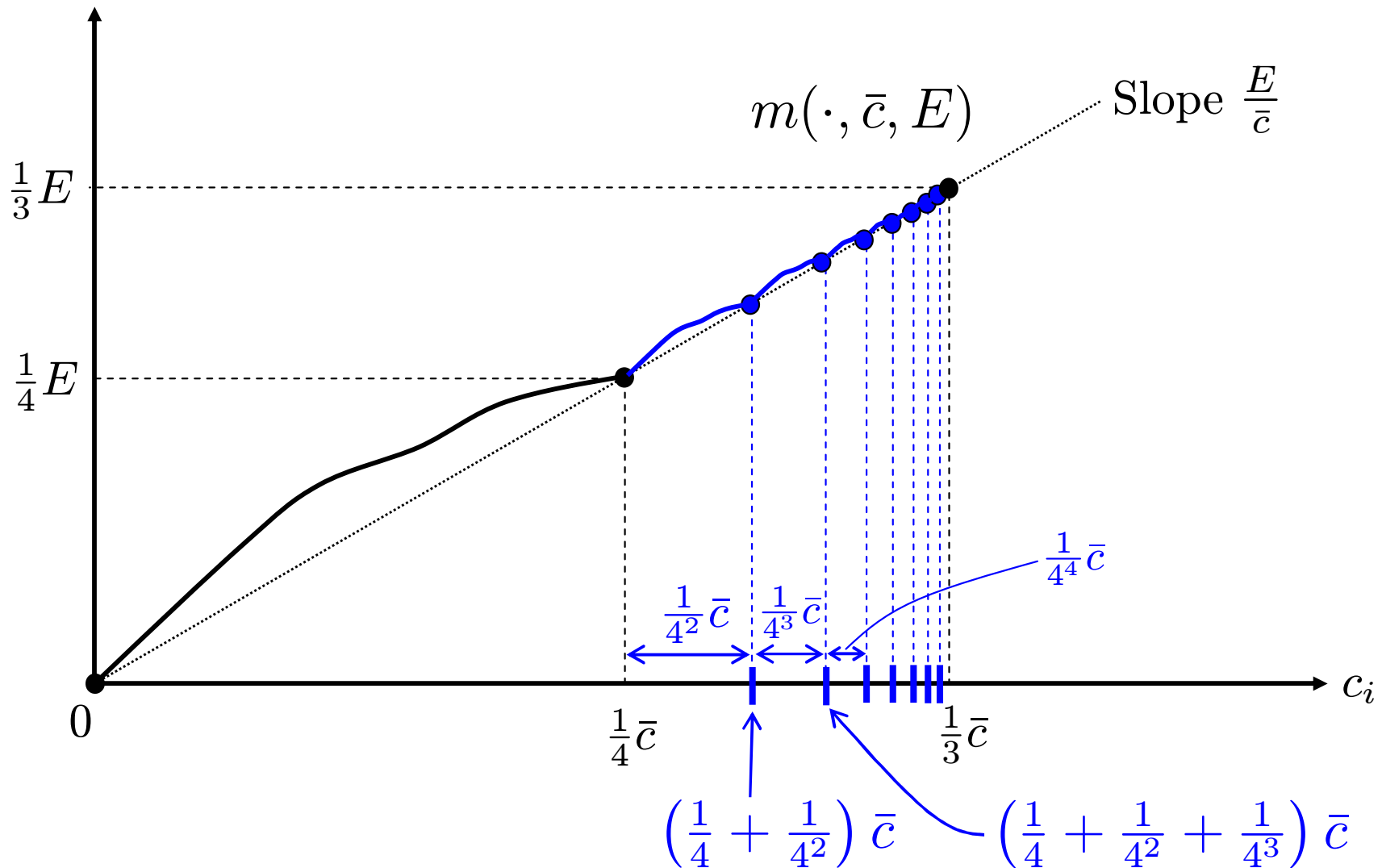
- Continuous
- Given  $\bar{c}$  and  $E$ ,



# Results (n=3)

$$m(c_i, \bar{c}, E)$$

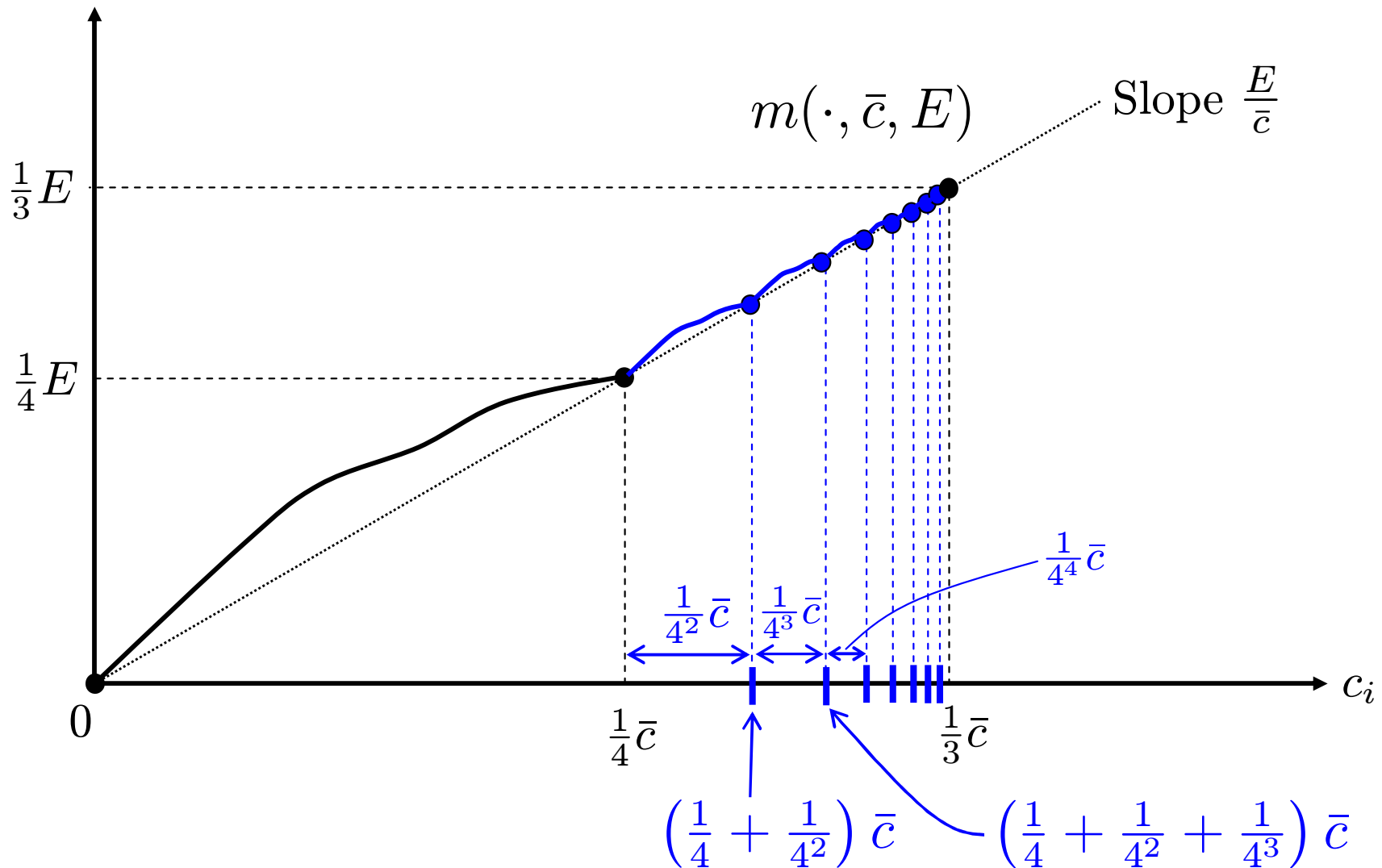
- Continuous
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# Results (n=3)

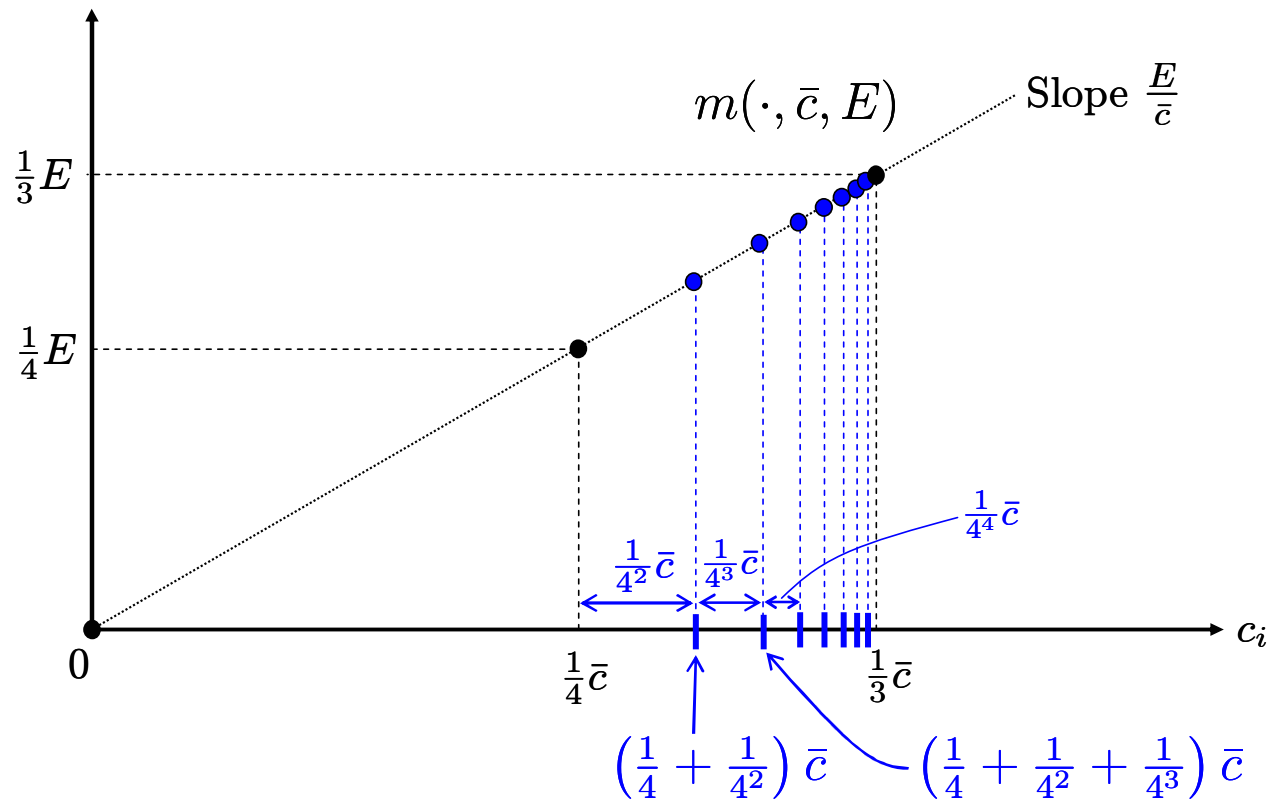
- Minimal award function,  $m$ :

- Continuous
- Given  $\bar{c}$  and  $E$ ,



# Results (n=3)

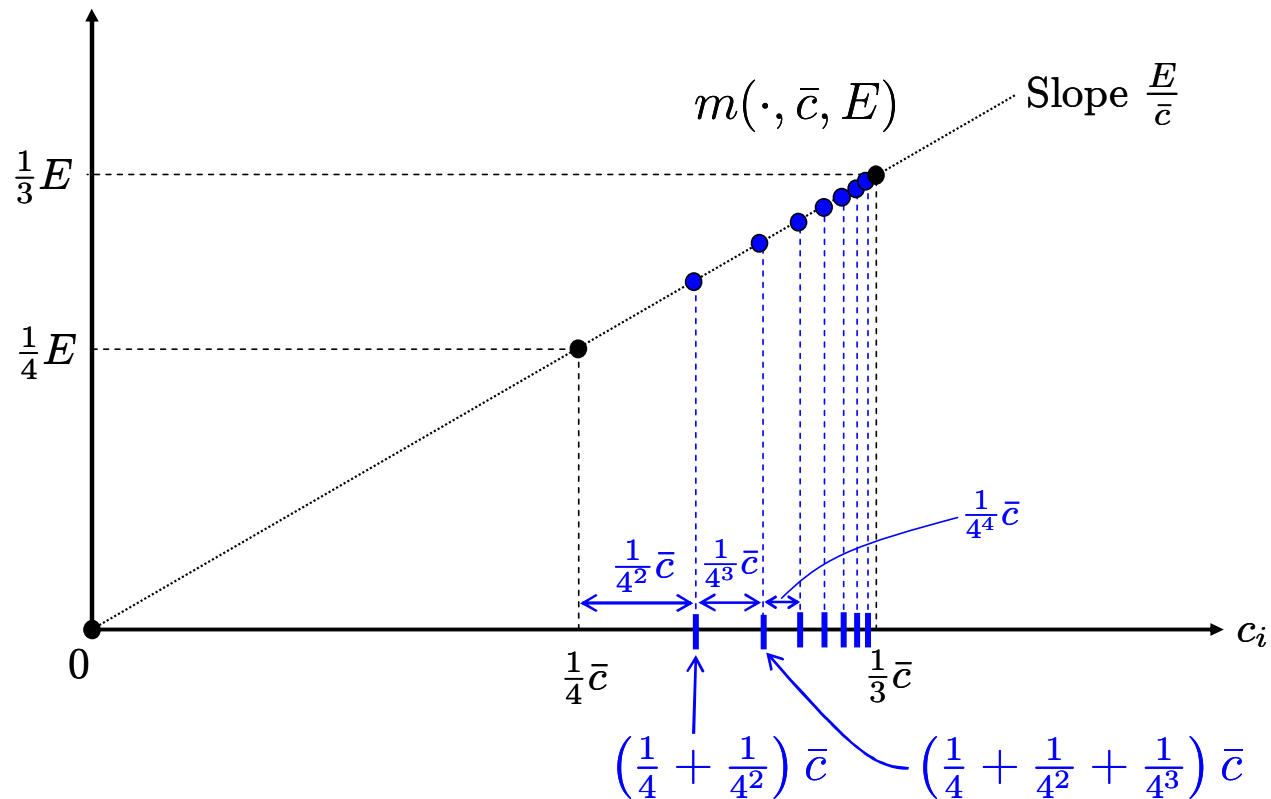
- Given  $\bar{c}$  and  $E$ ,



# Results (n=3)

$ord-pres-g \implies$  “equal treatment of equals”

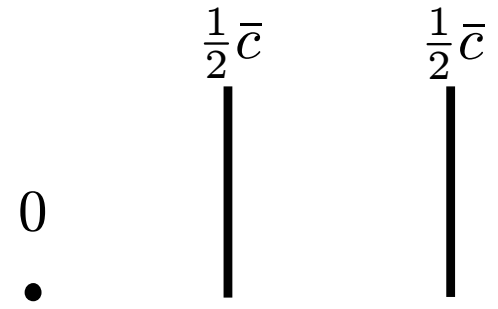
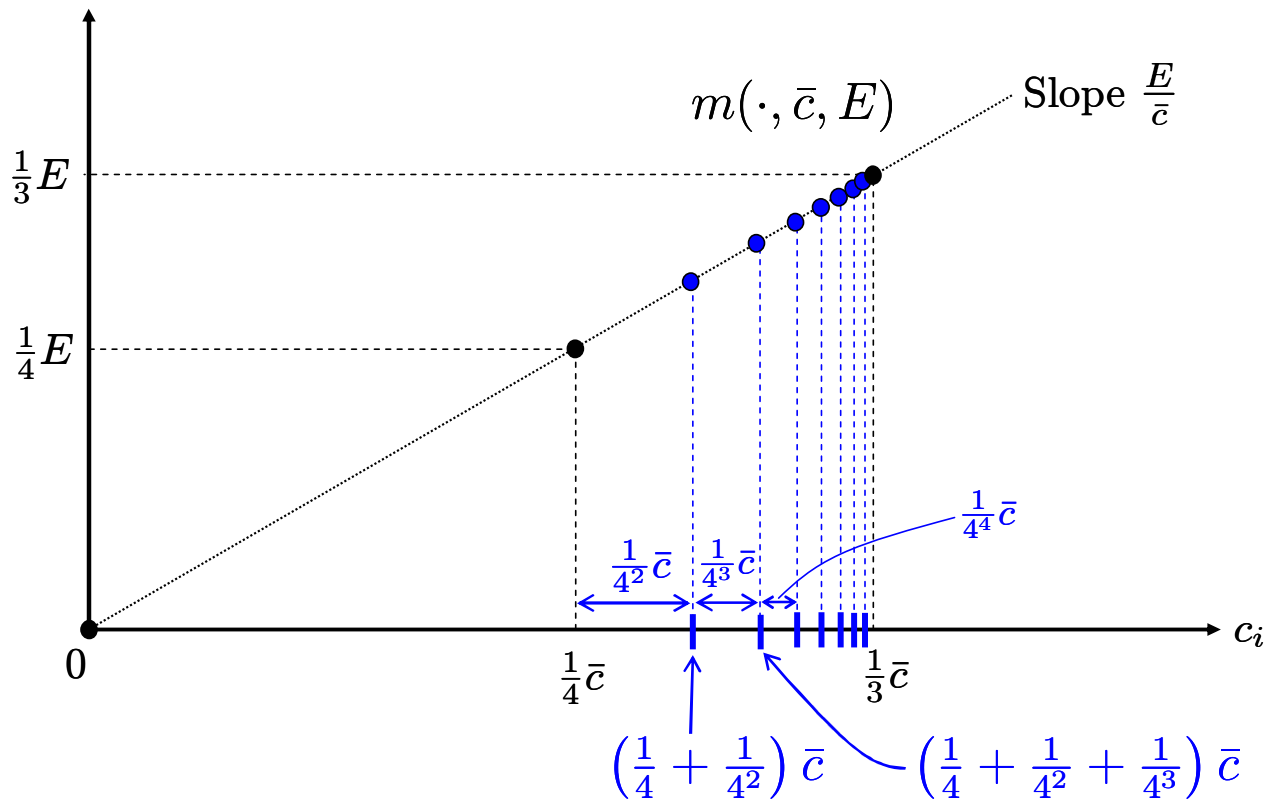
- Given  $\bar{c}$  and  $E$ ,



# Results (n=3)

*ord-pres-g*  $\implies$  “equal treatment of equals”

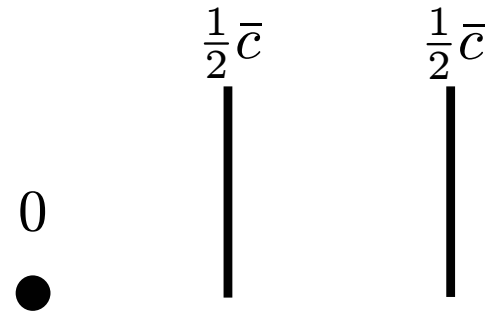
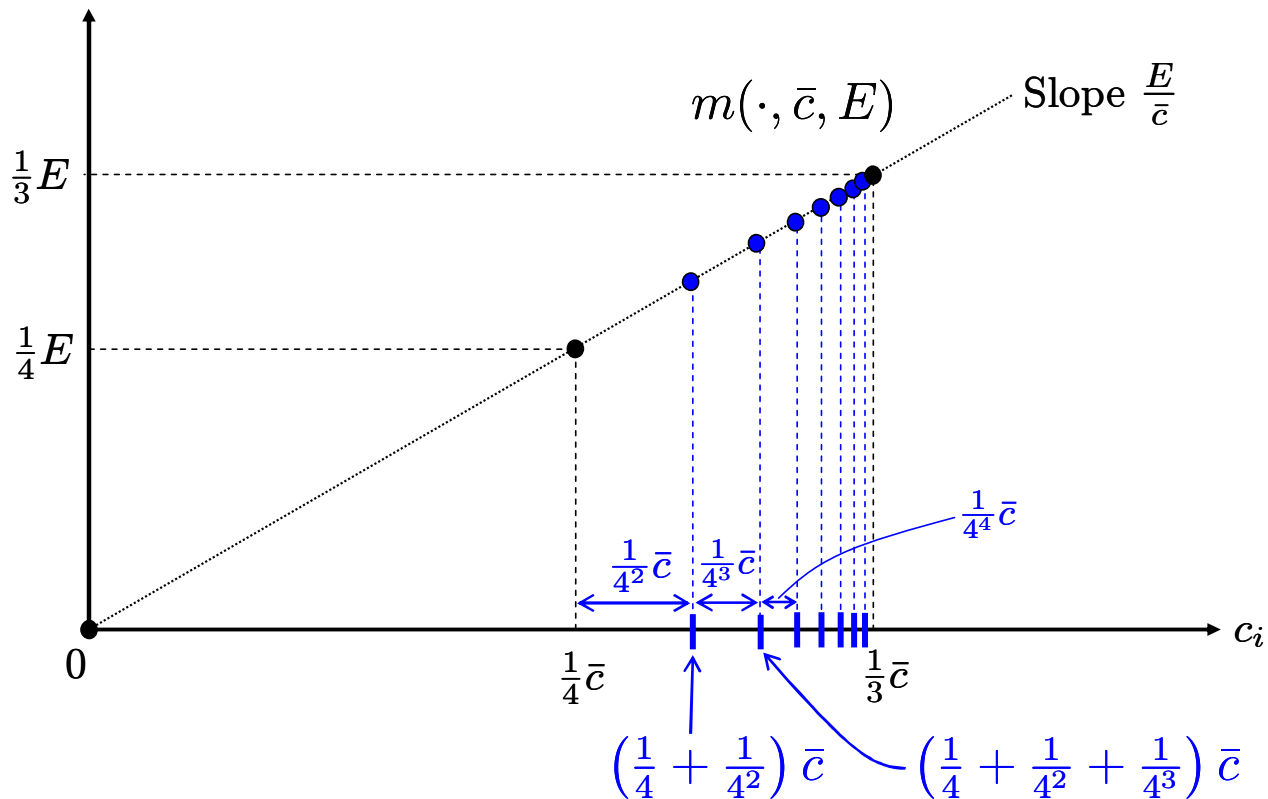
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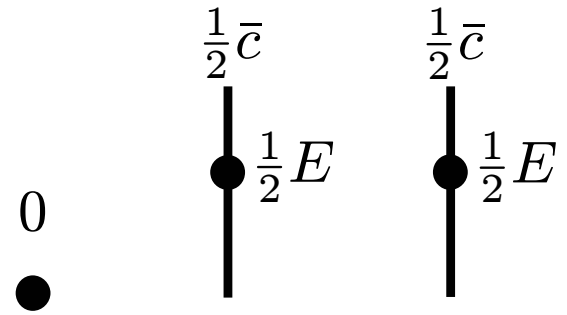
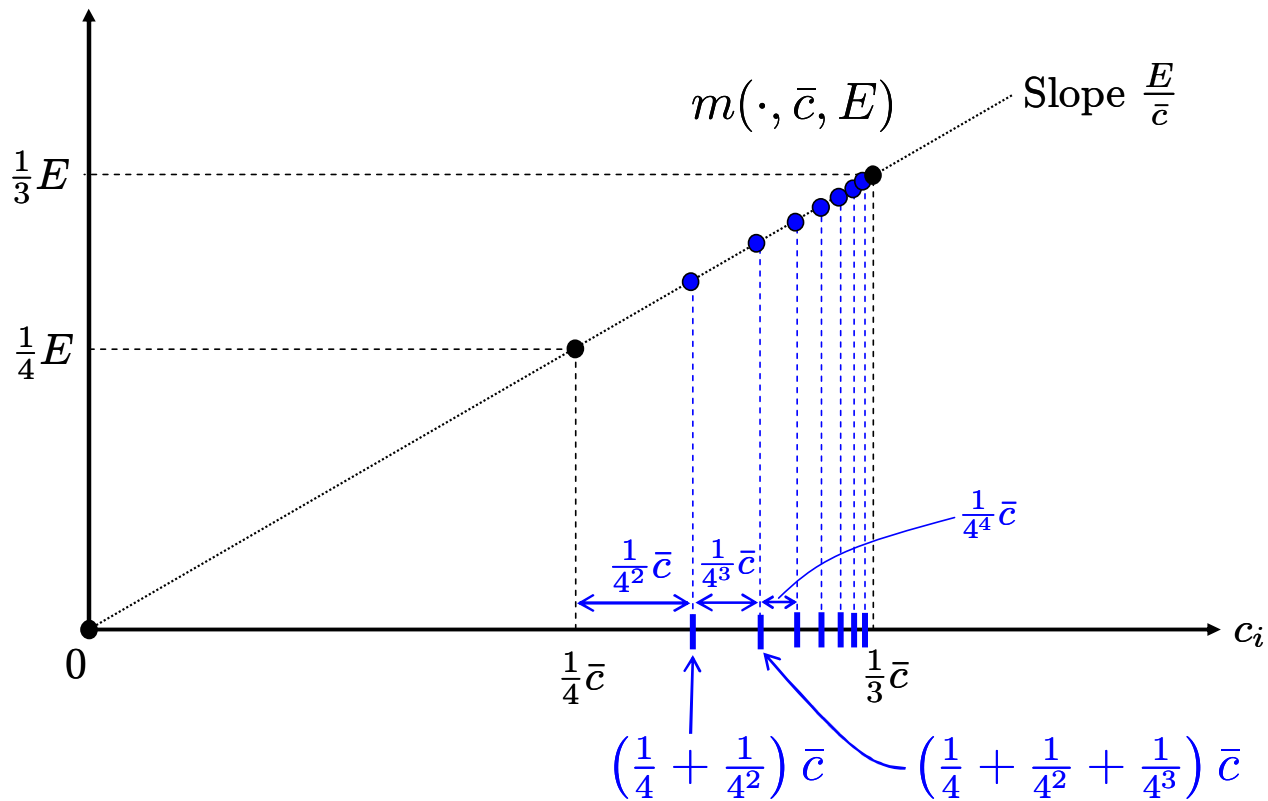
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$ord-pres-g \implies$  “equal treatment of equals”

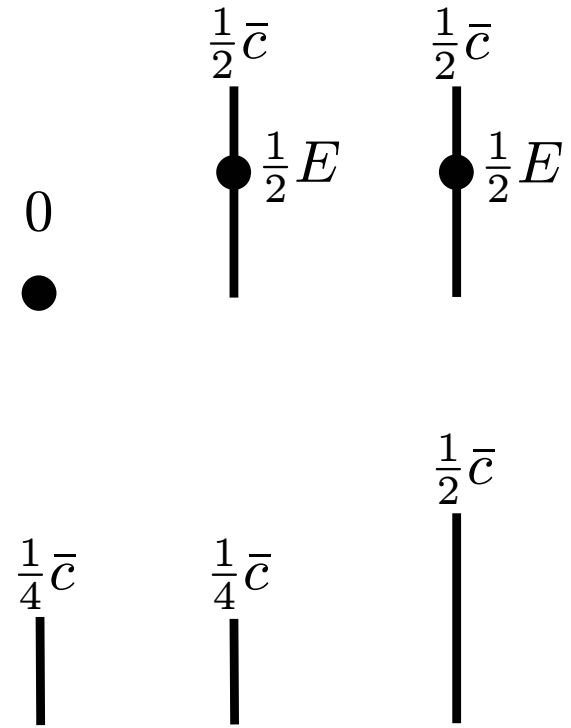
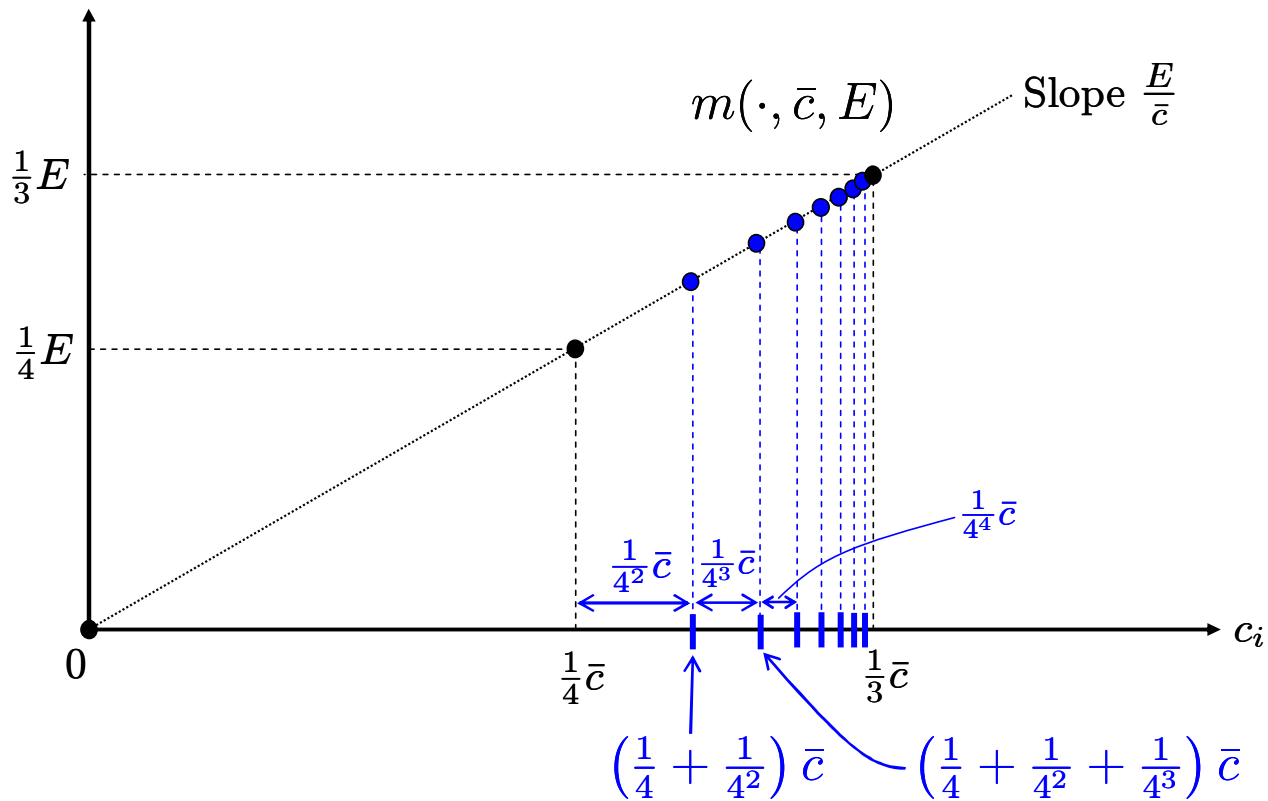
- Given  $\bar{c}$  and  $E$ ,



# Results (n=3)

*ord-pres-g*  $\implies$  “equal treatment of equals”

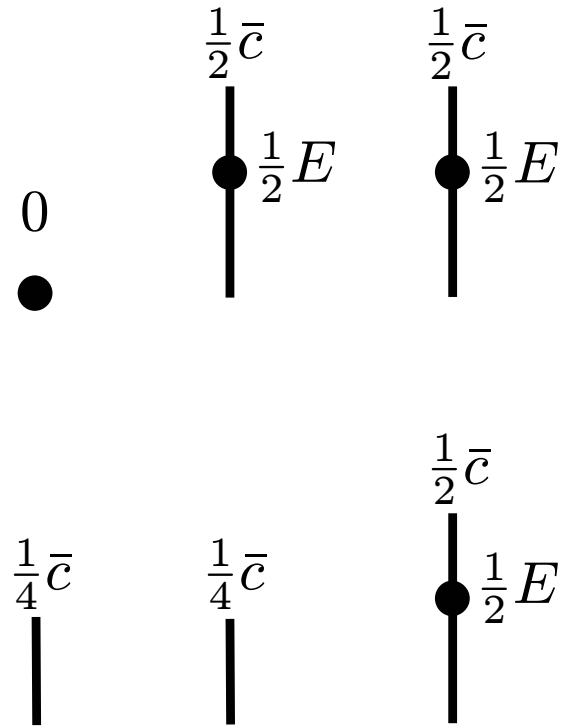
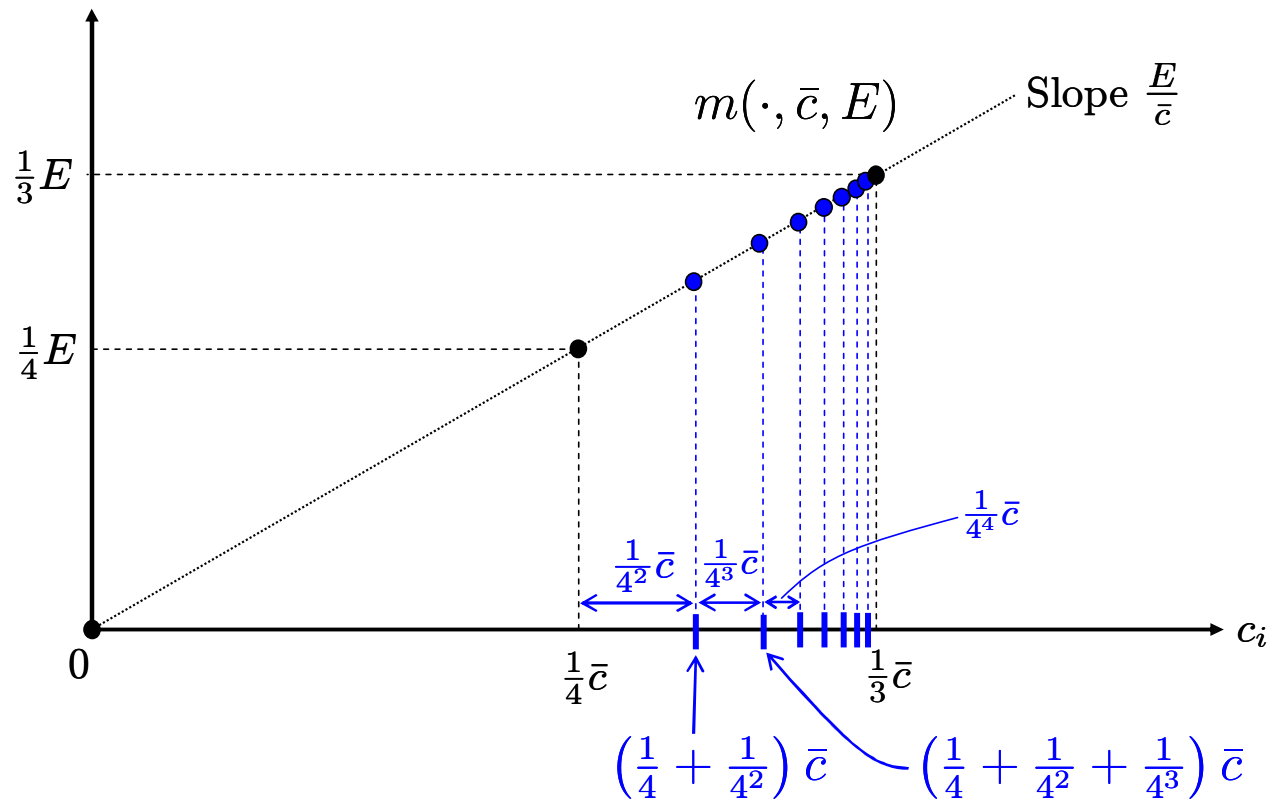
- Given  $\bar{c}$  and  $E$ ,



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$ord-pres-g \implies$  “equal treatment of equals”

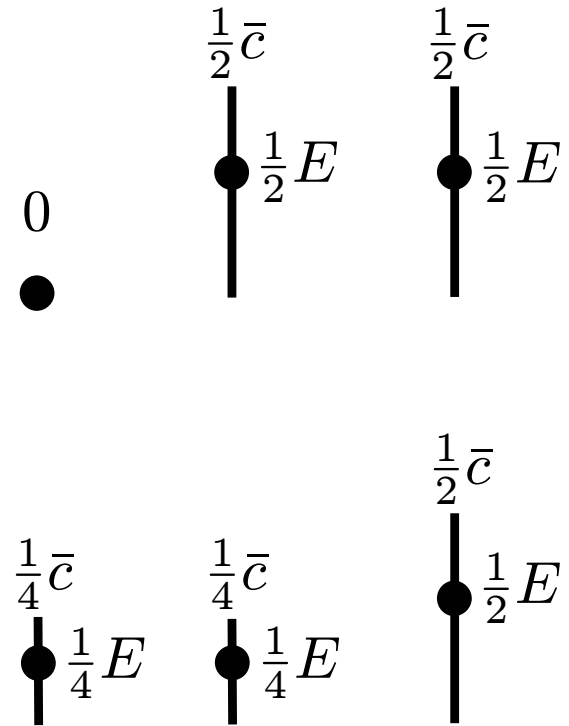
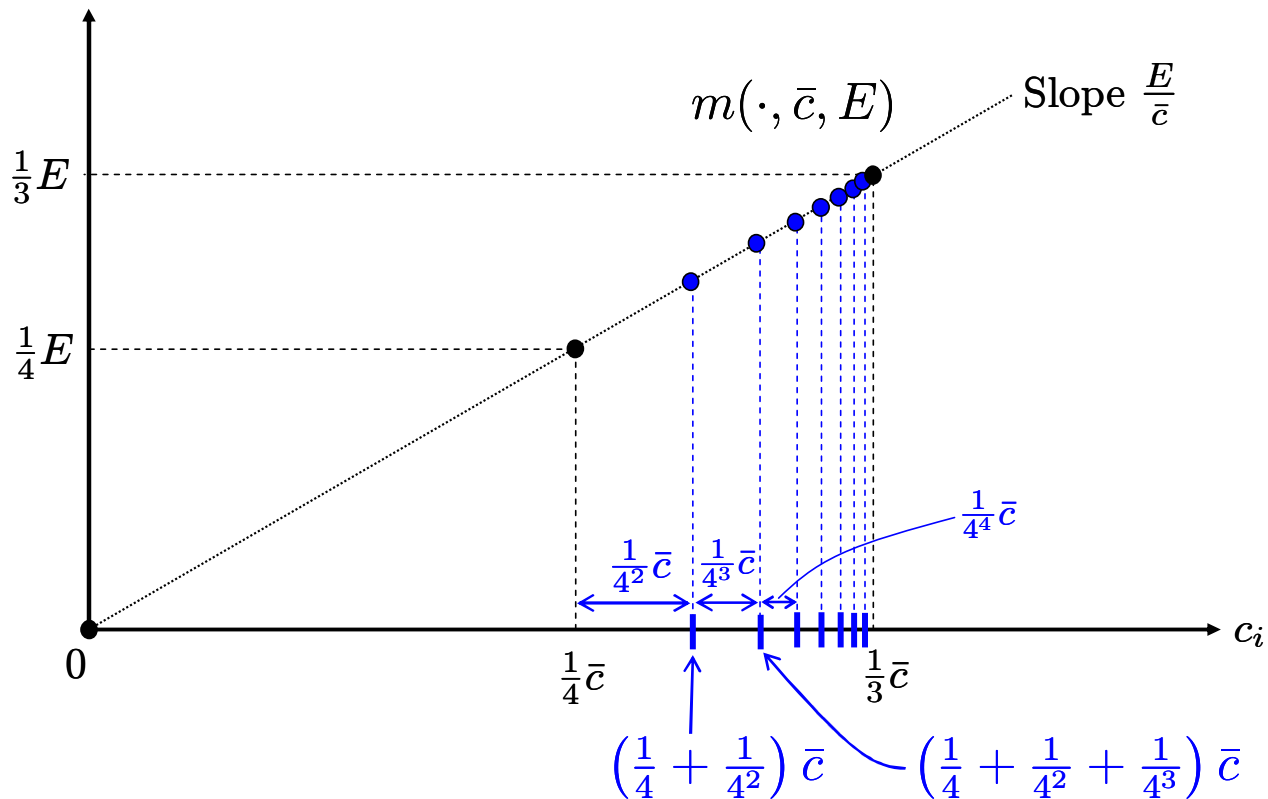
- Given  $\bar{c}$  and  $E$ ,



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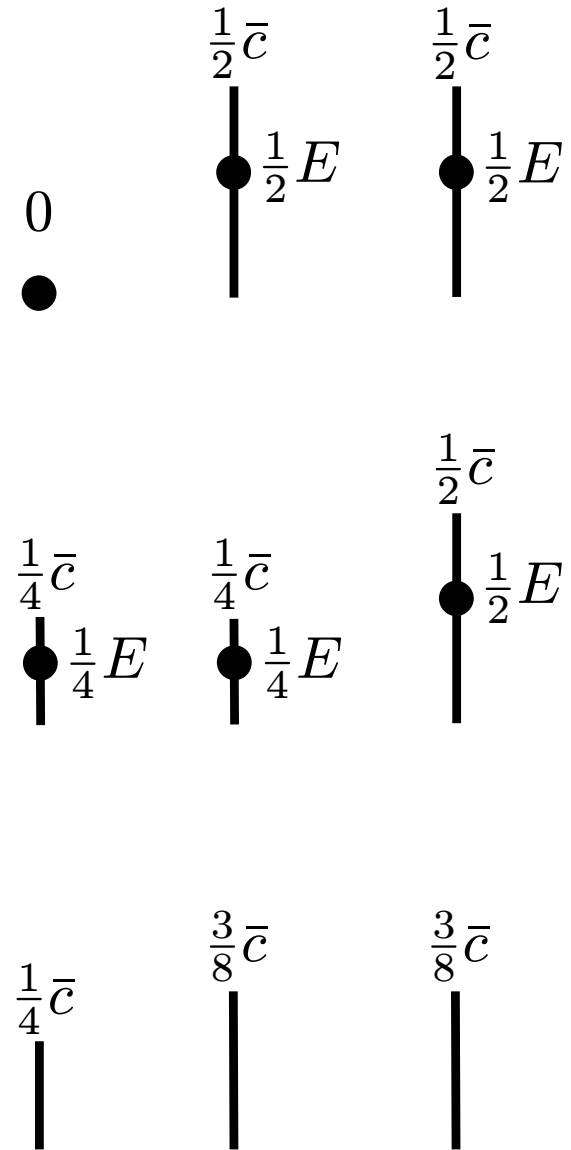
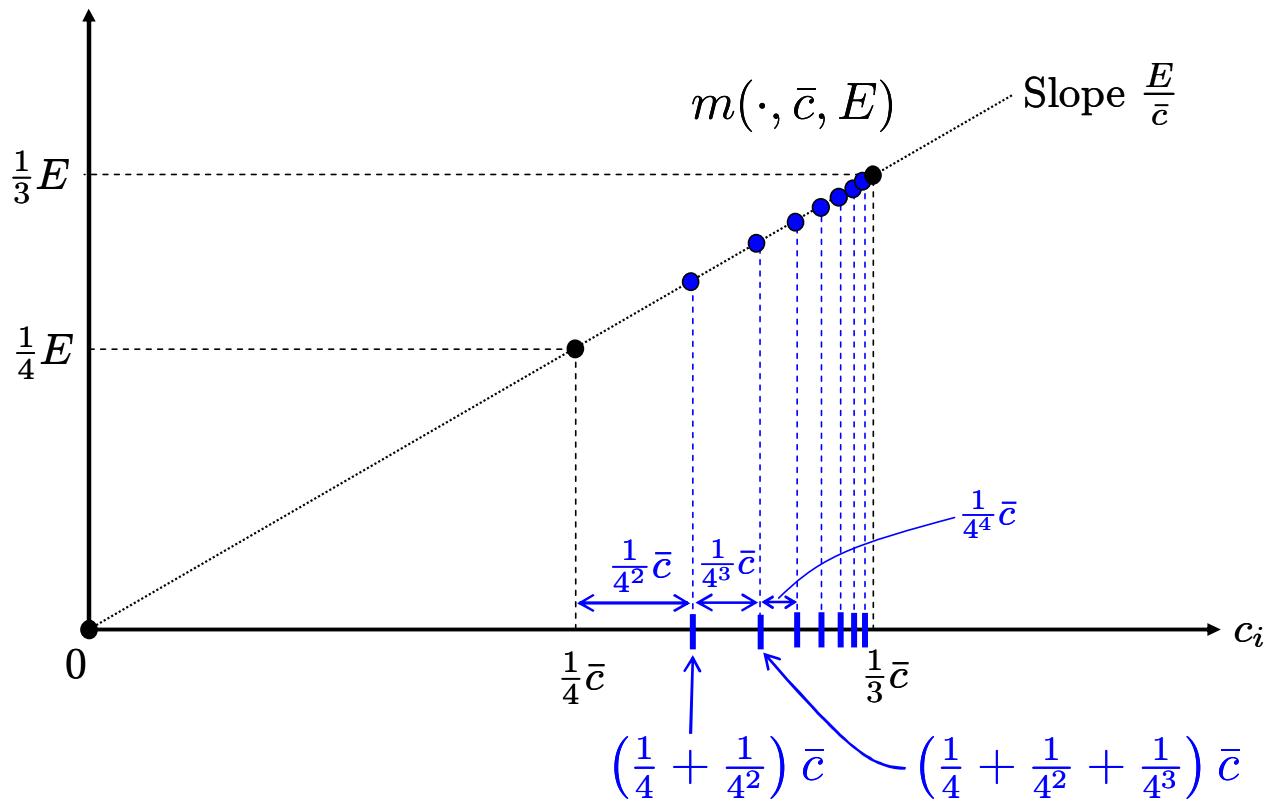
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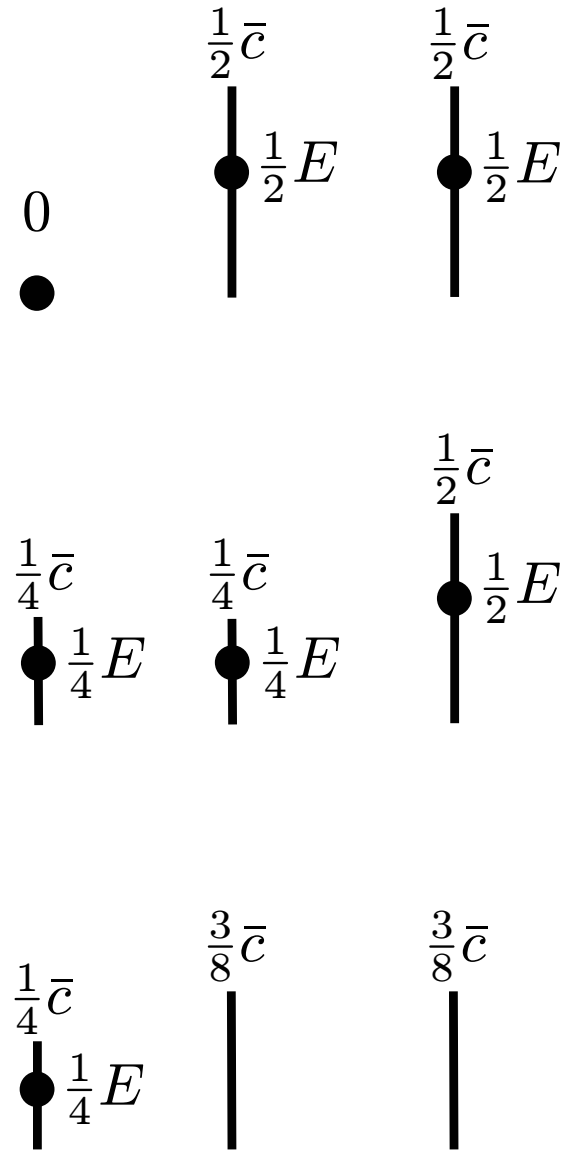
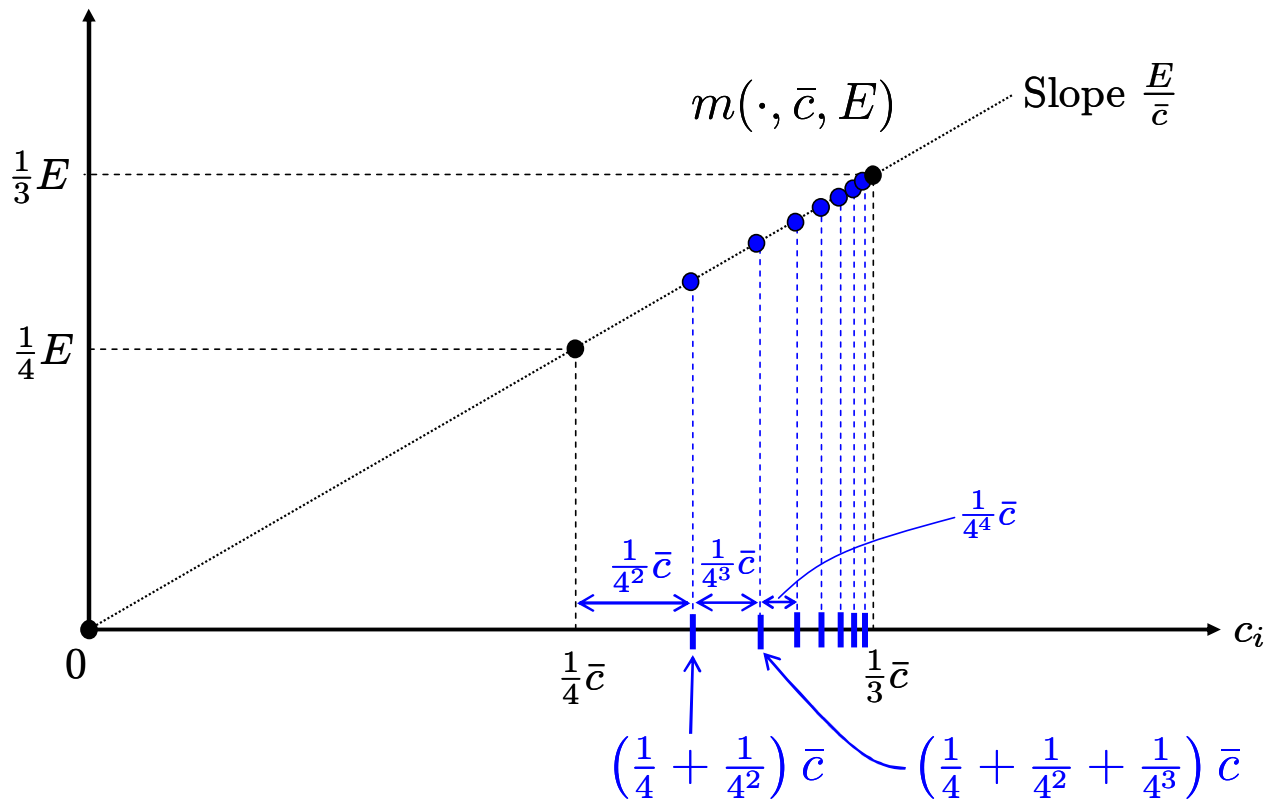
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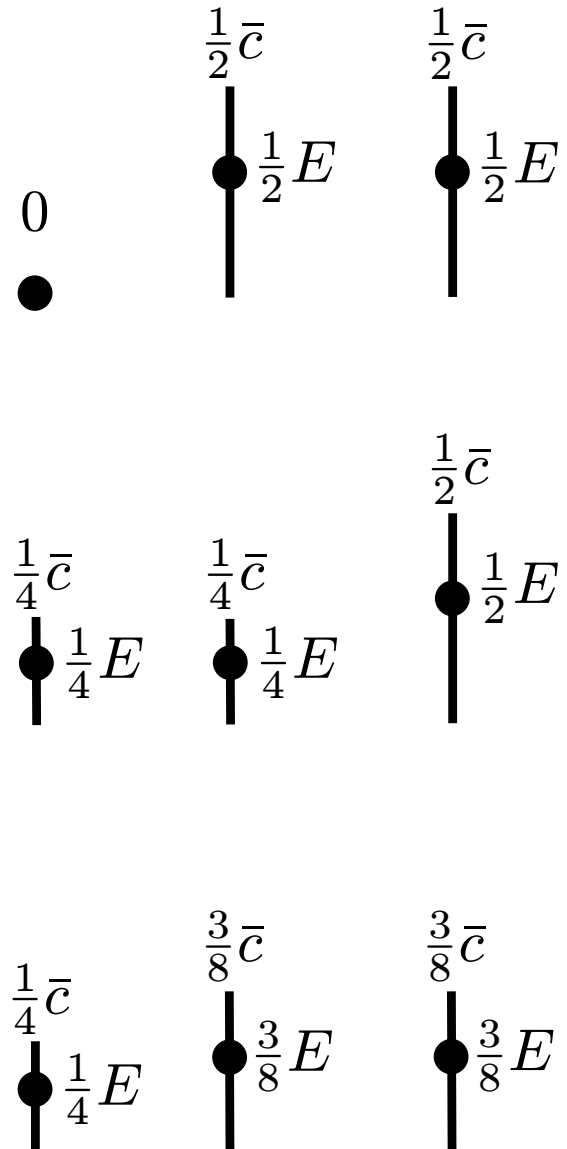
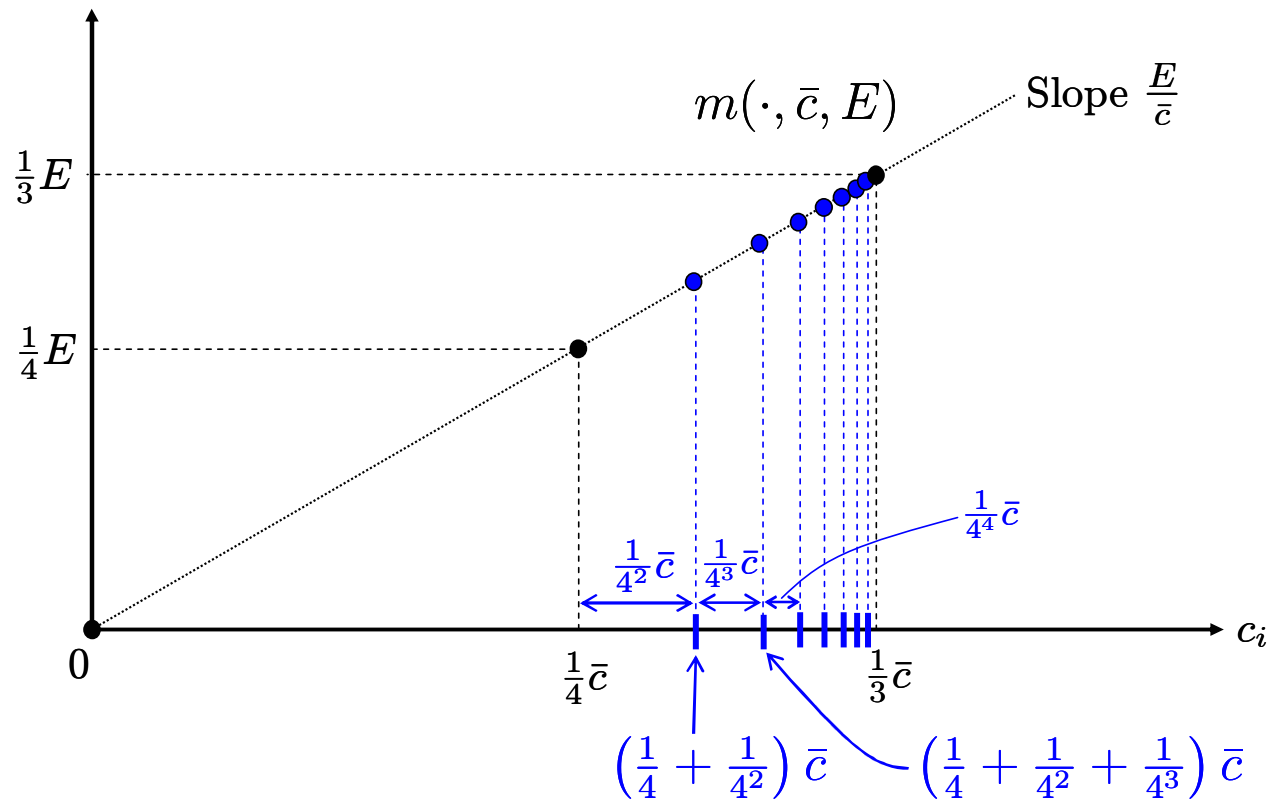
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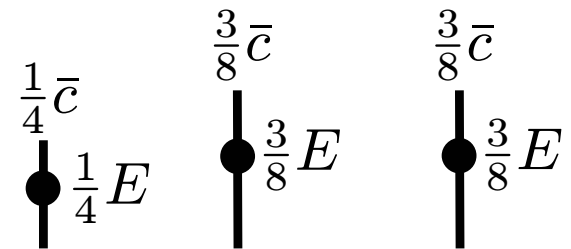
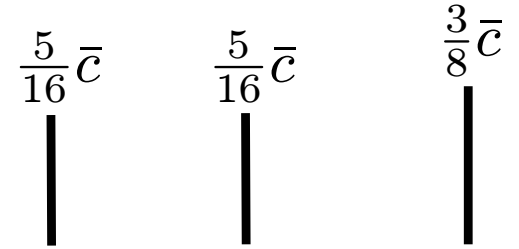
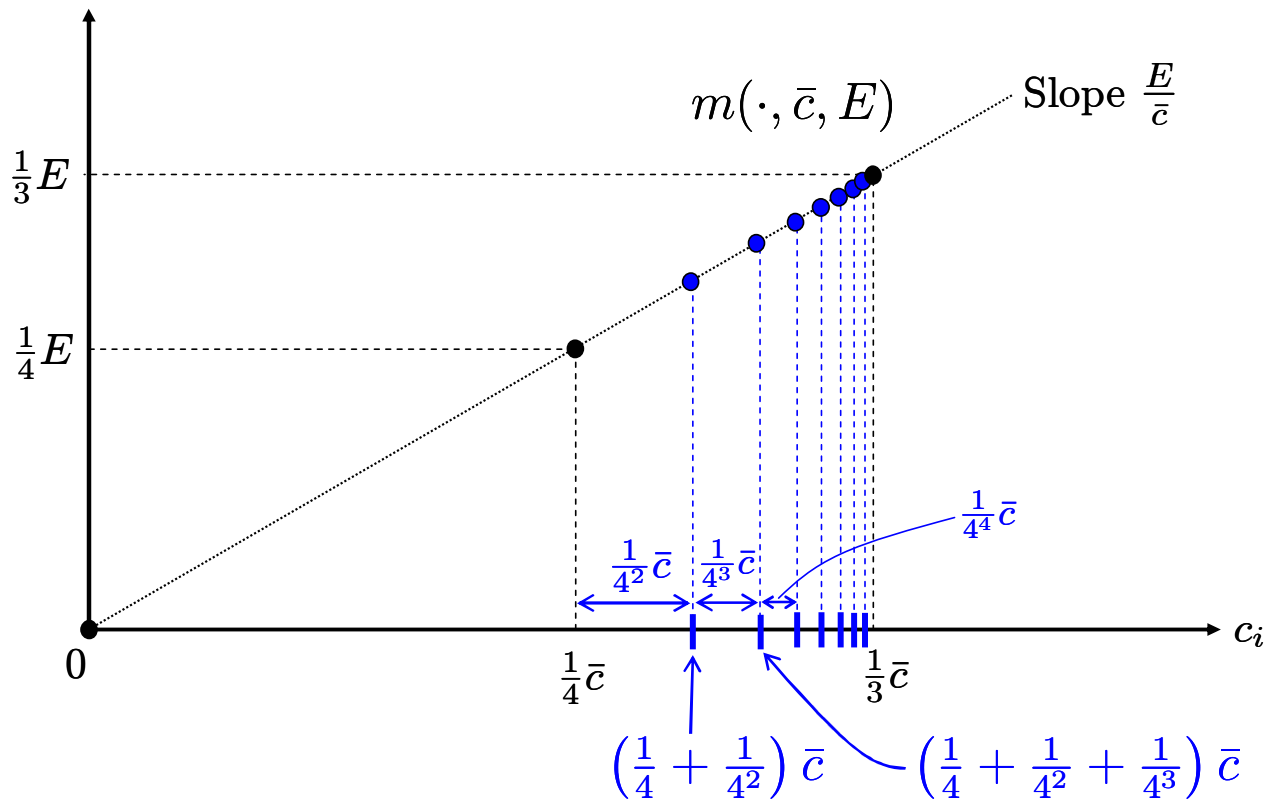
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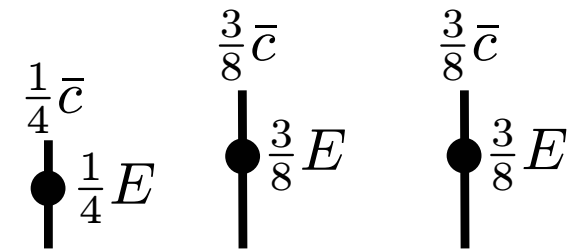
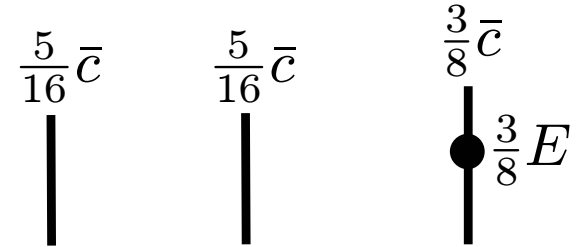
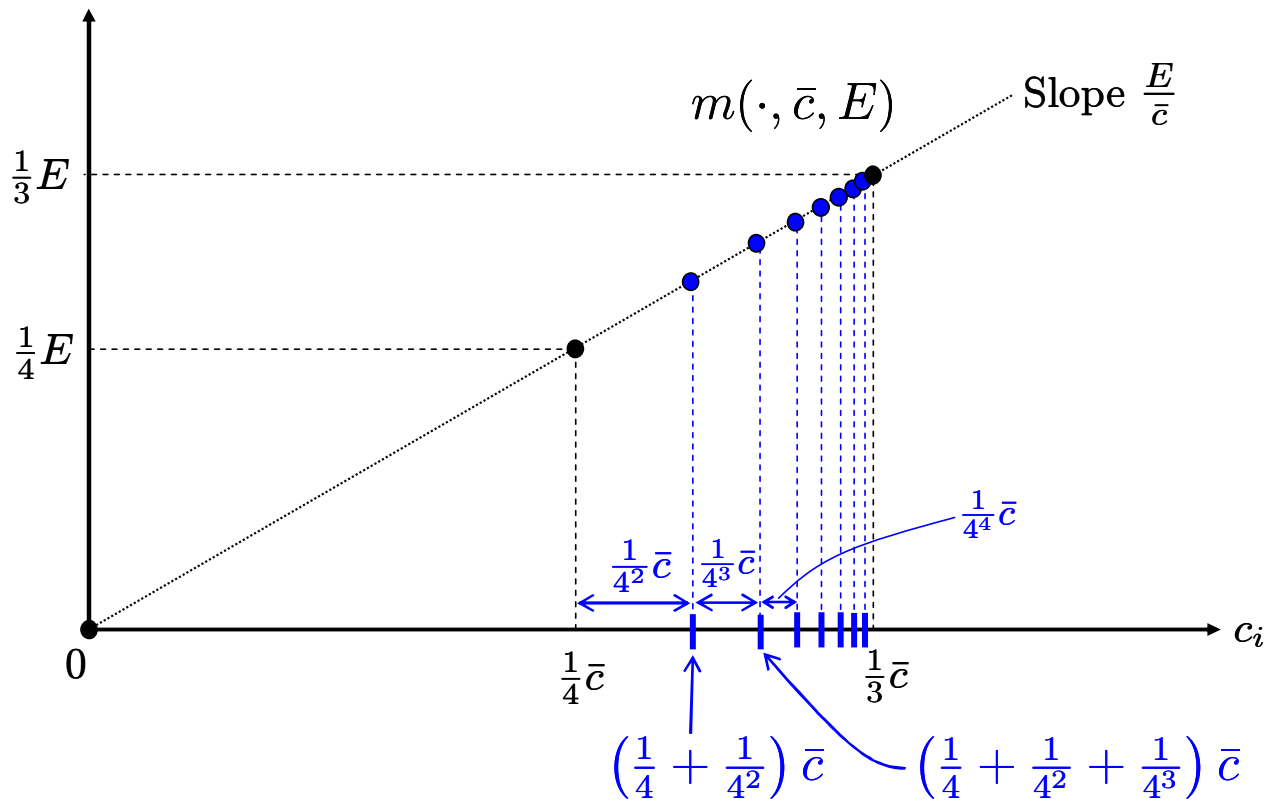
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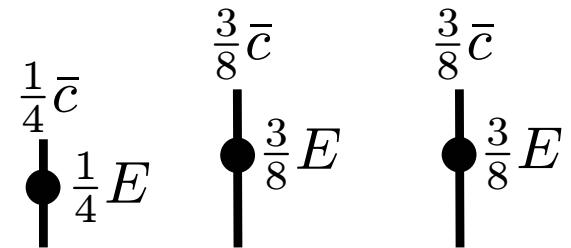
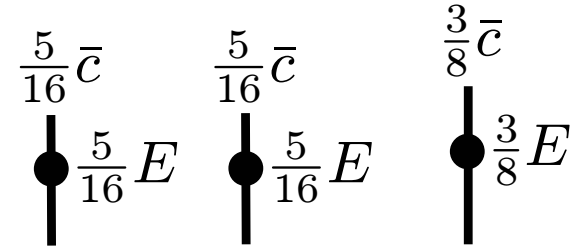
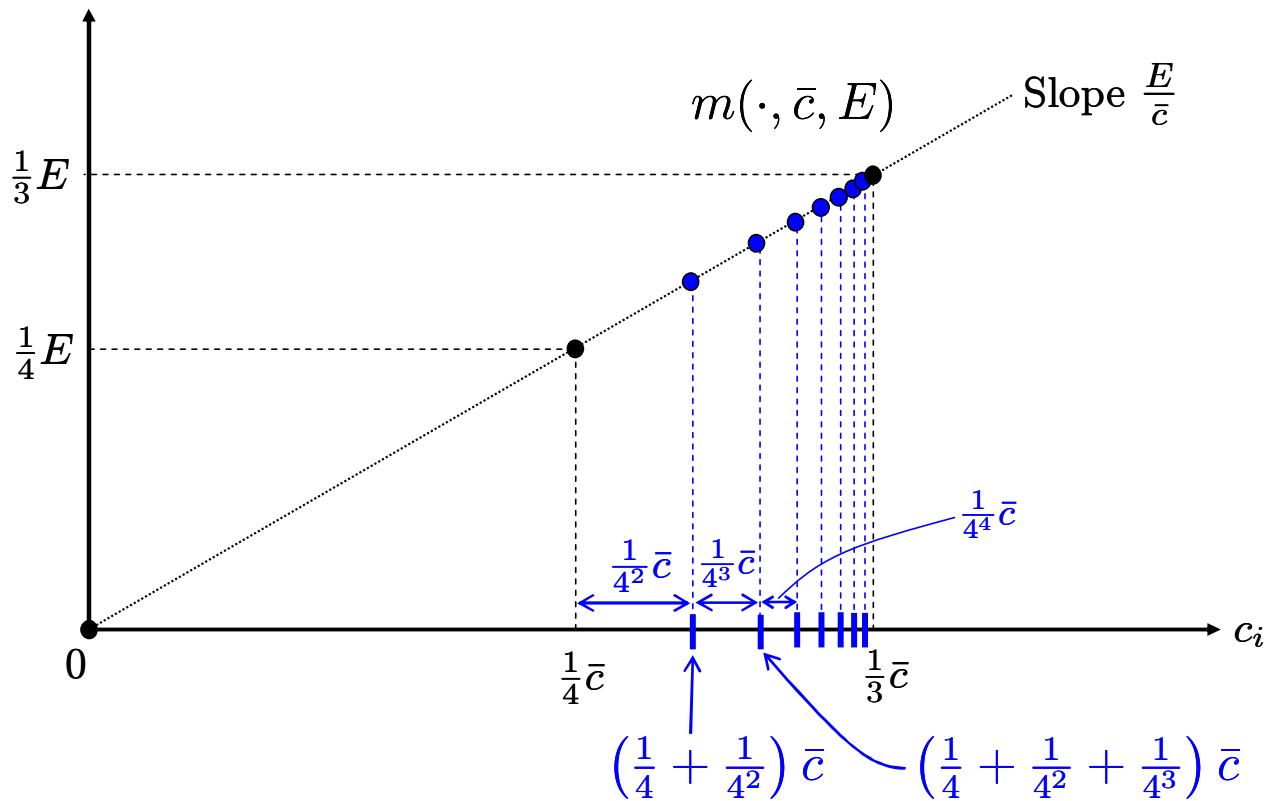
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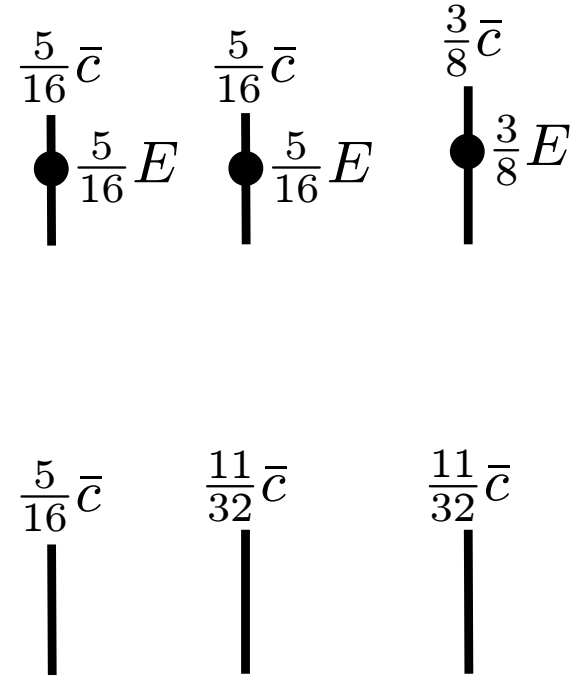
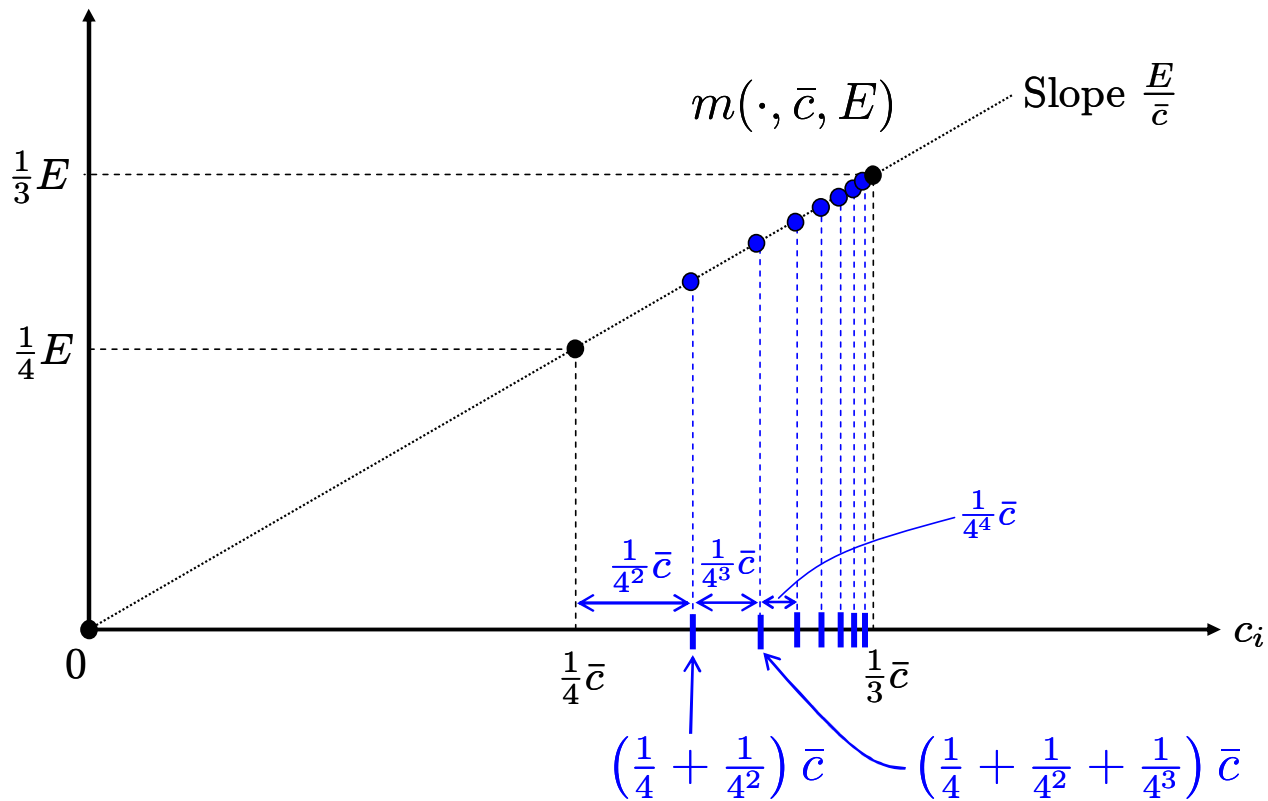
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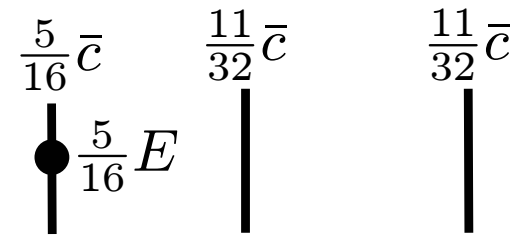
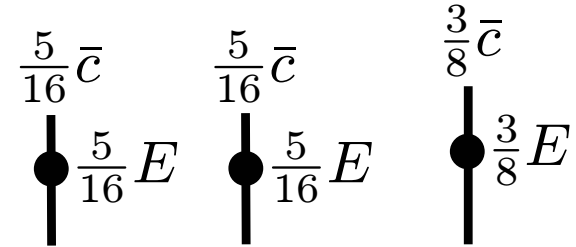
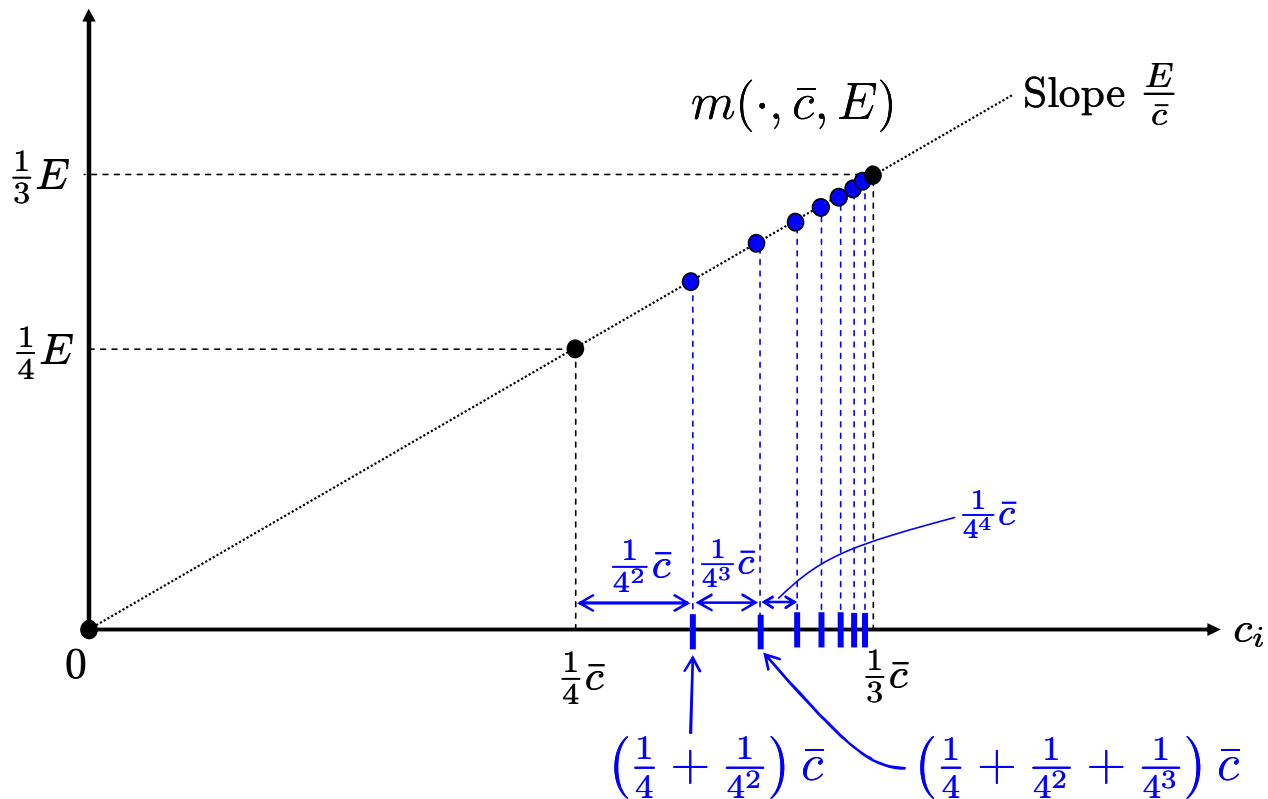
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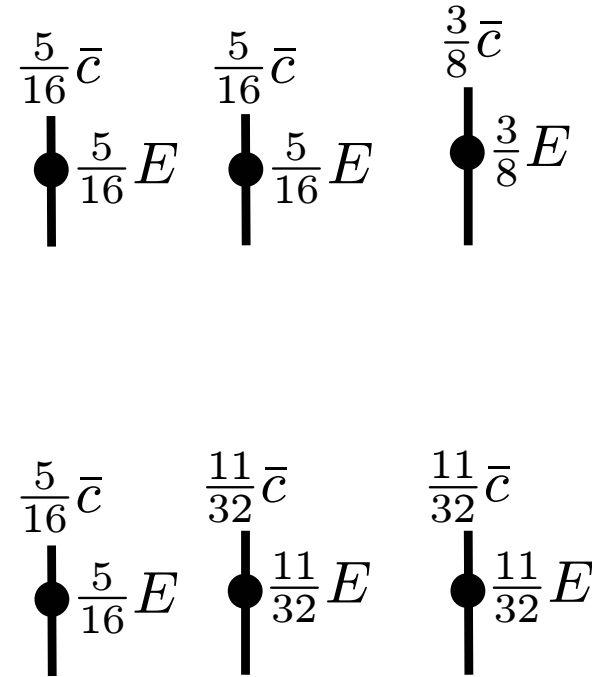
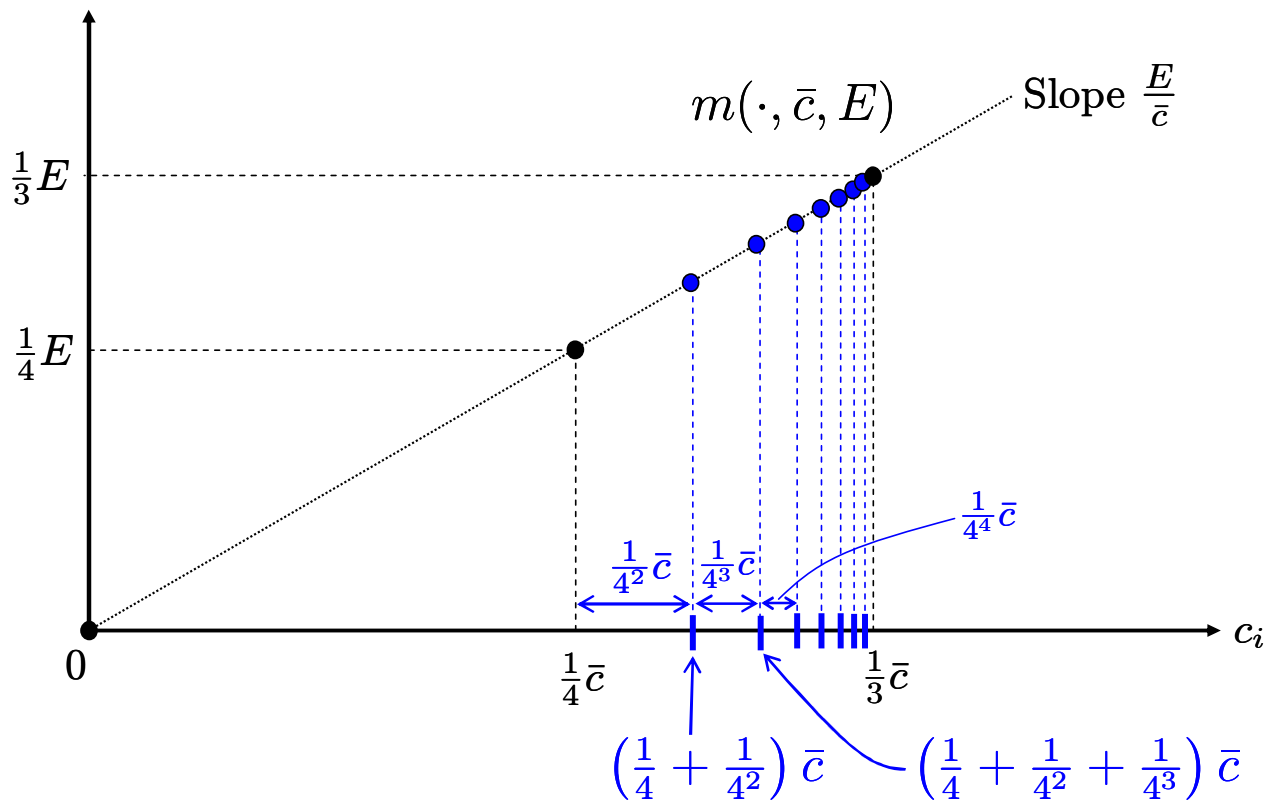
- Given  $\bar{c}$  and  $E$ ,



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*ord-pres-g*  $\implies$  “equal treatment of equals”

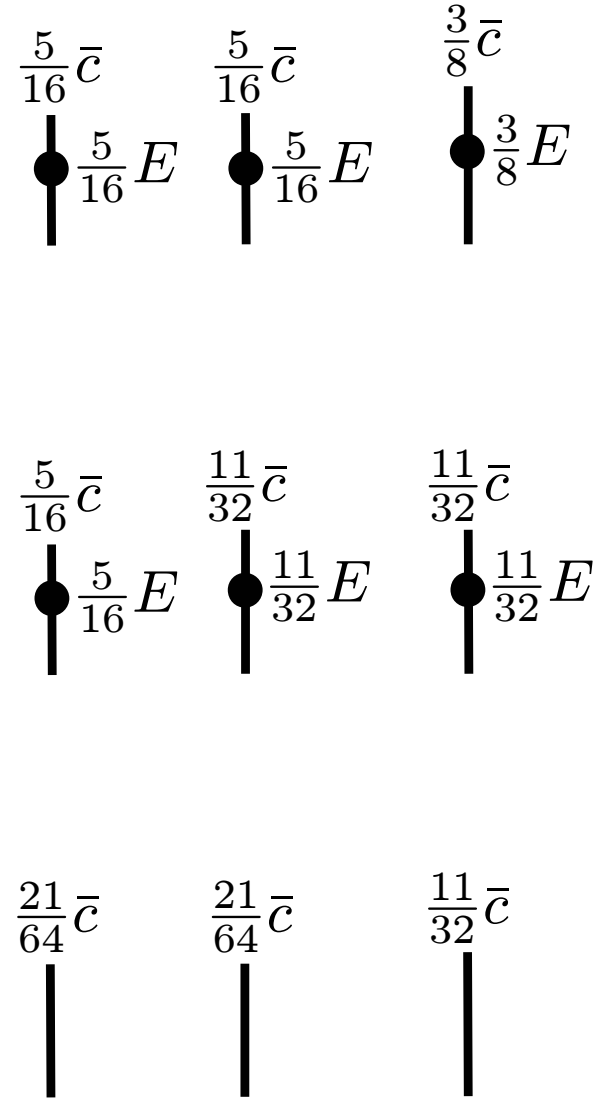
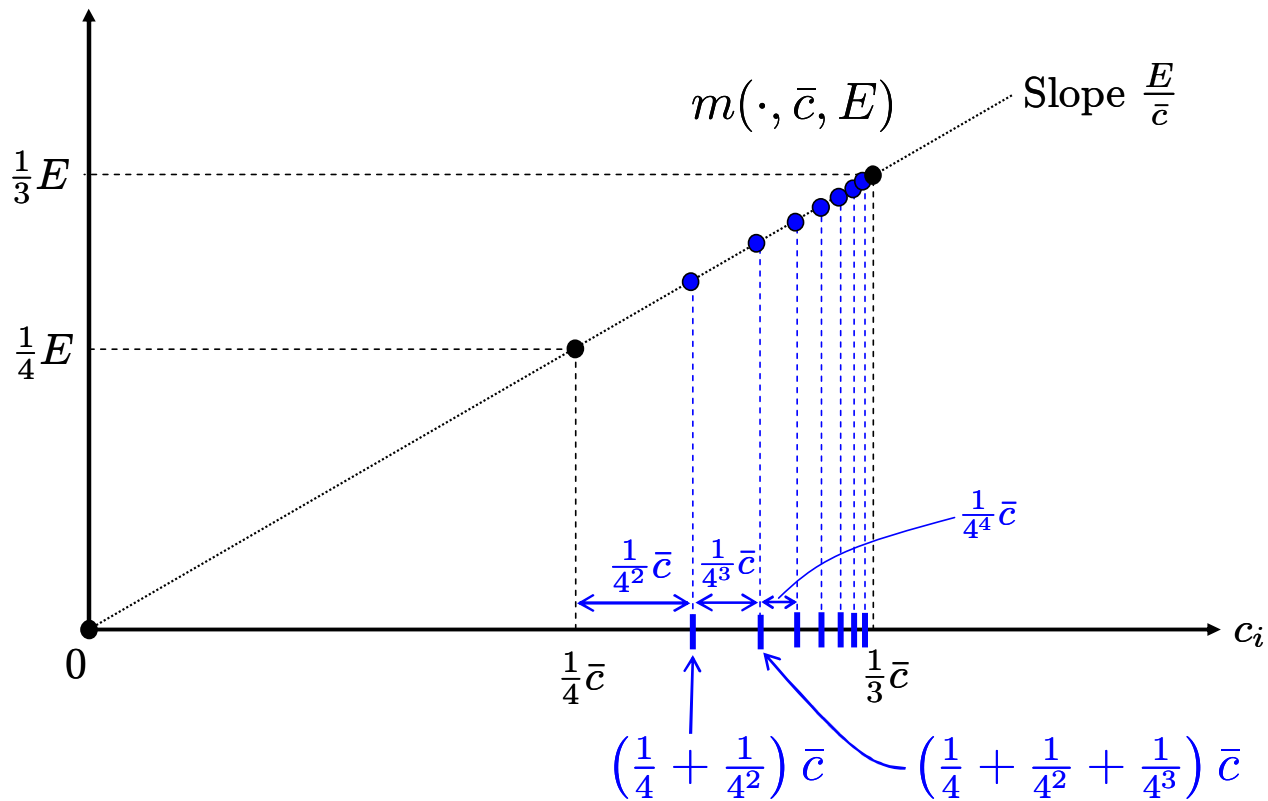
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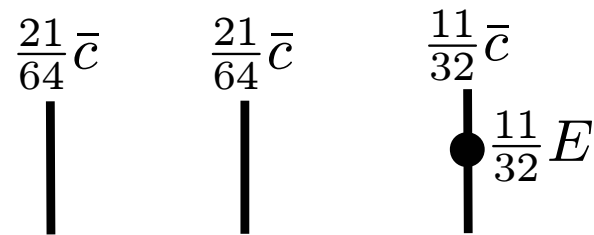
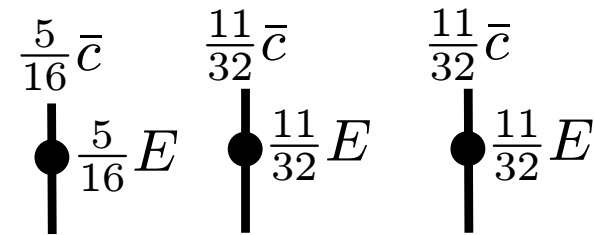
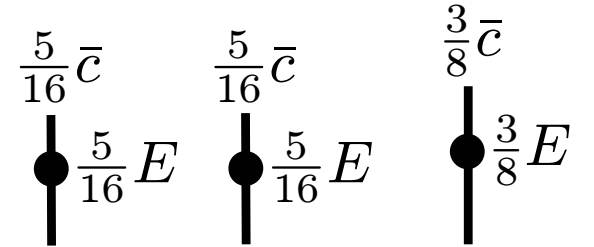
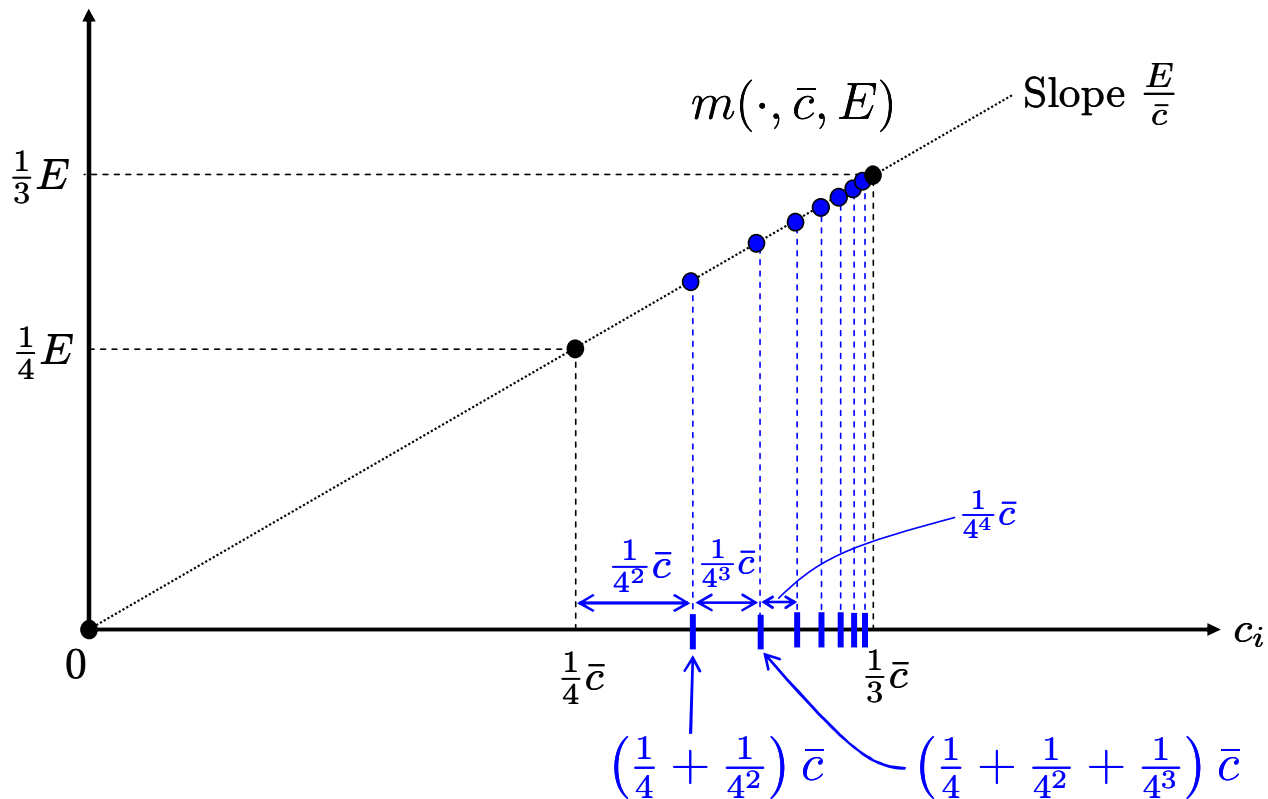
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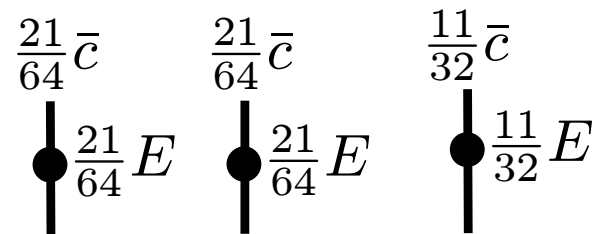
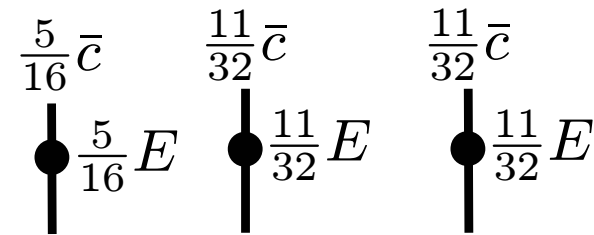
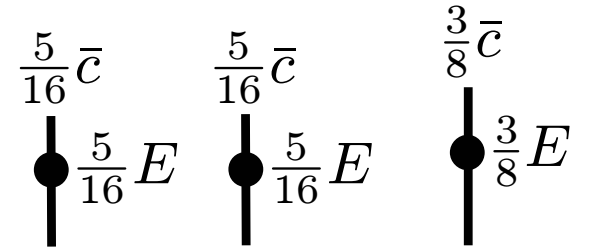
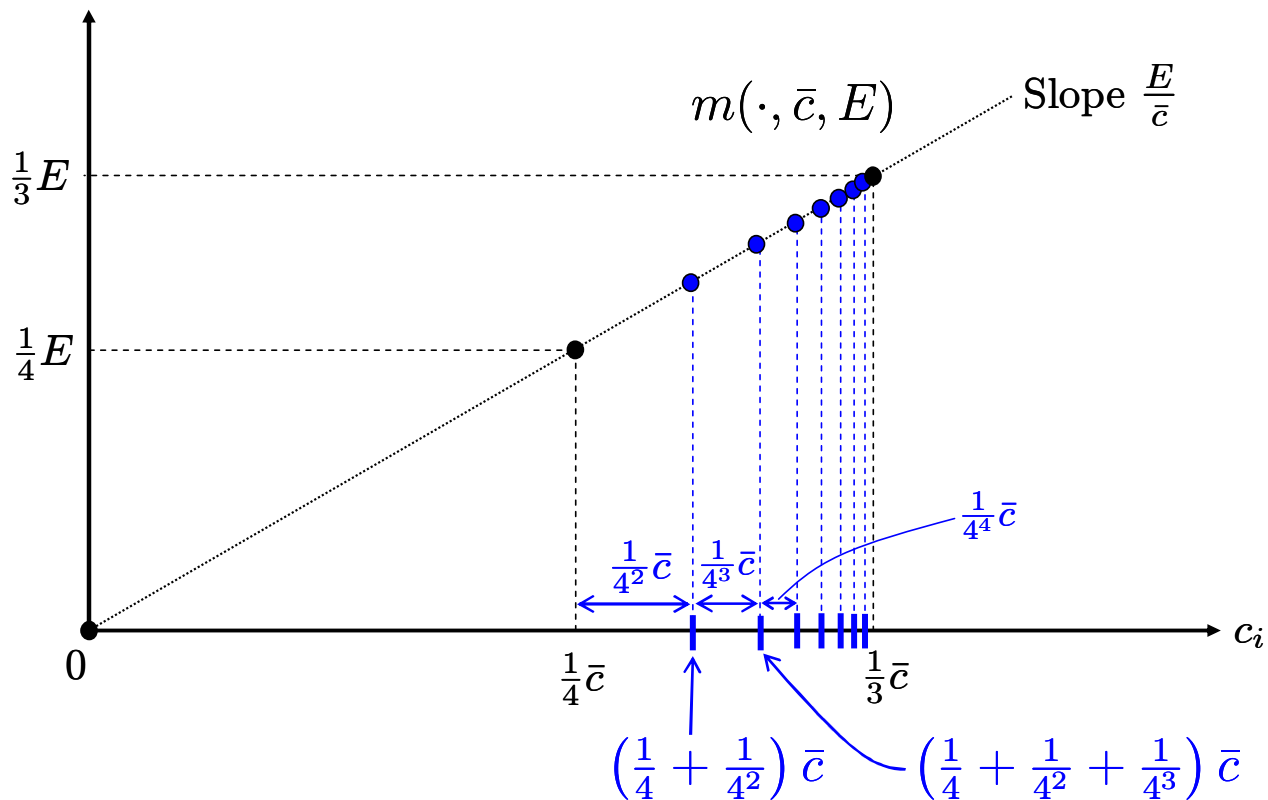
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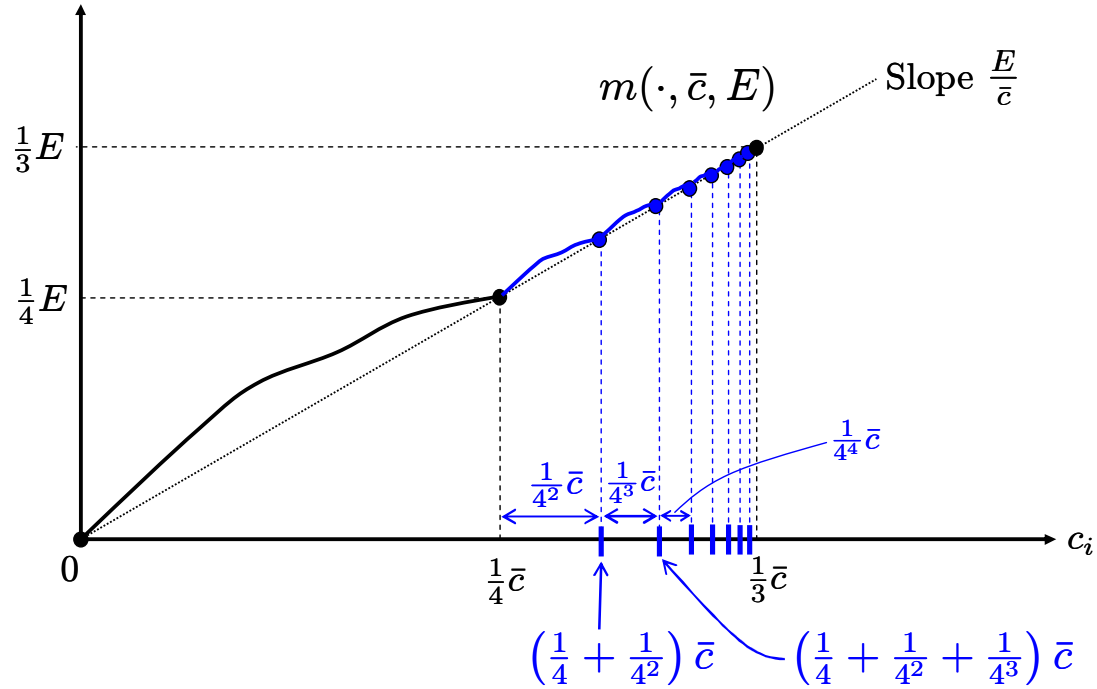
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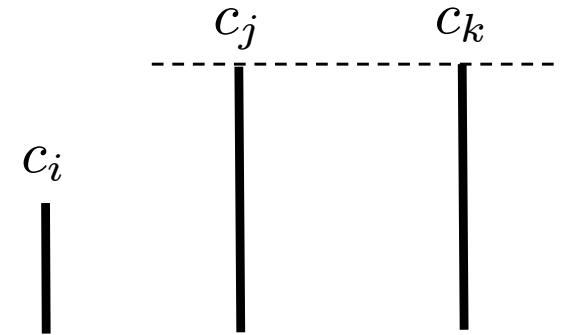
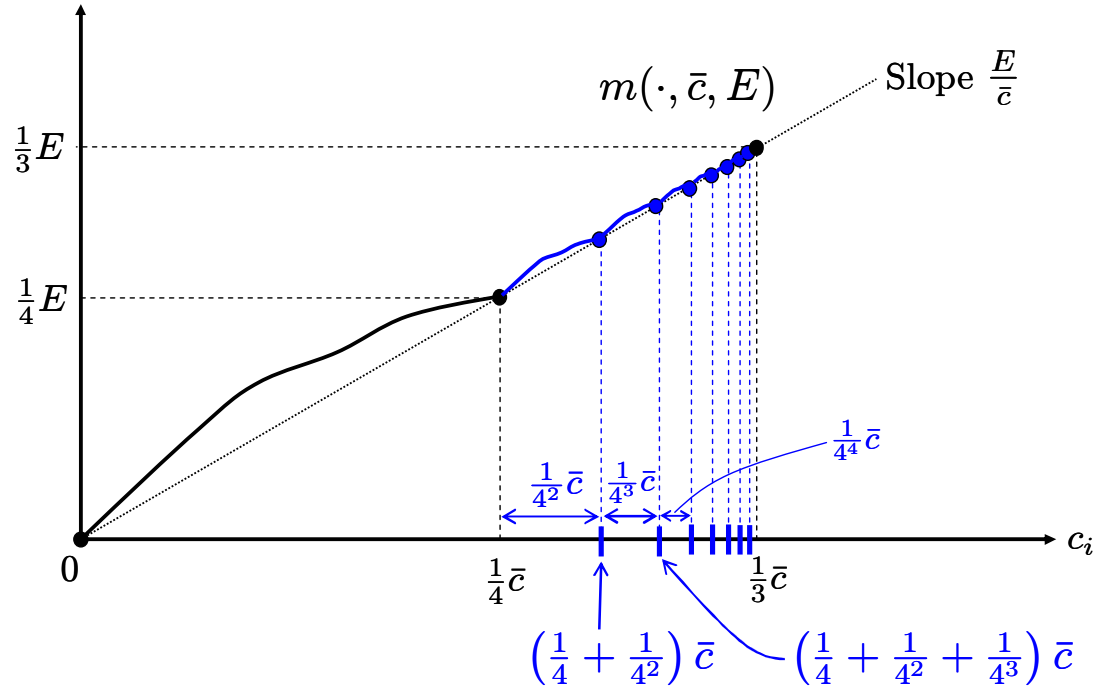
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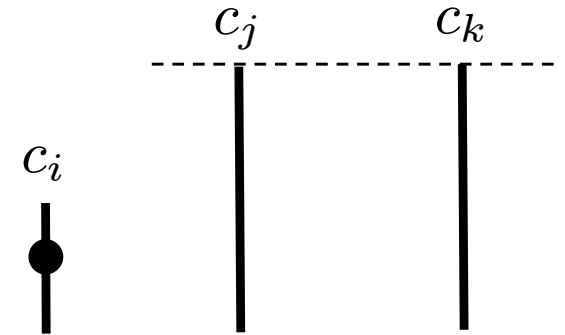
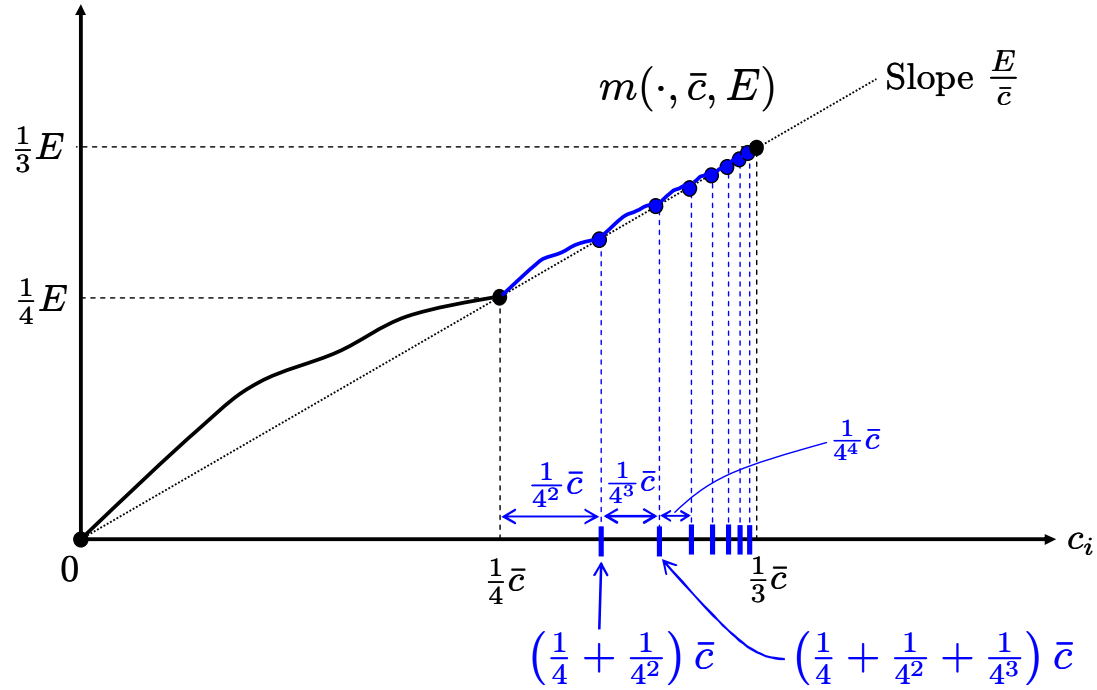
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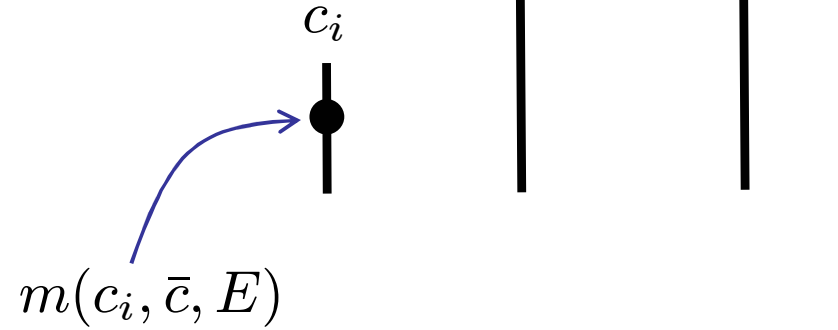
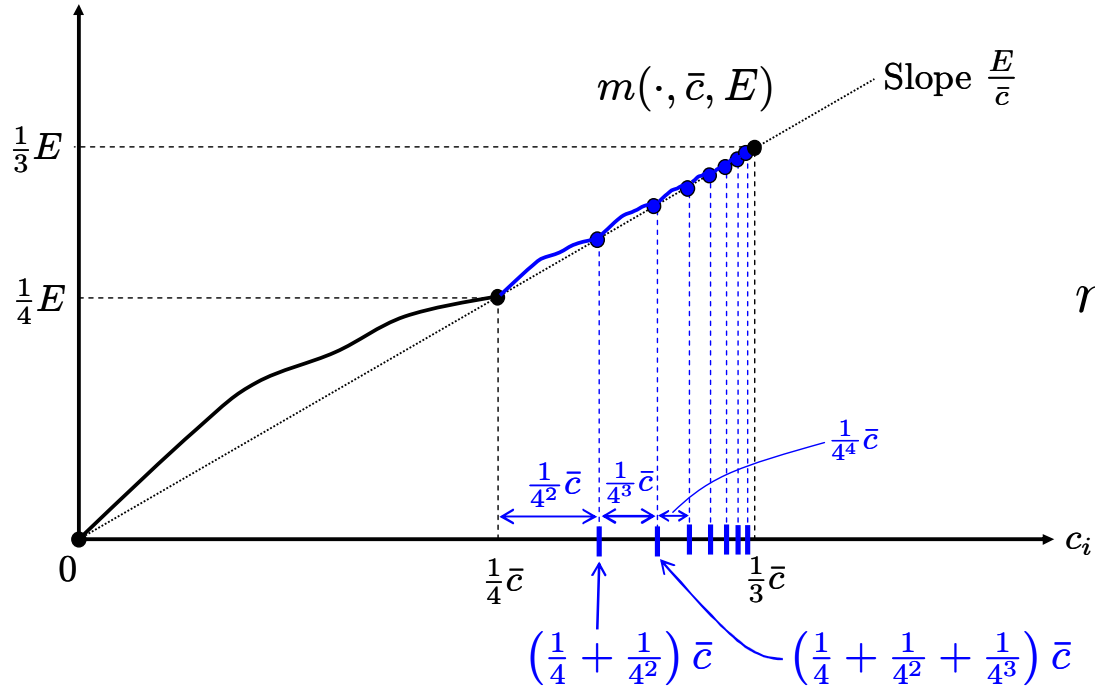
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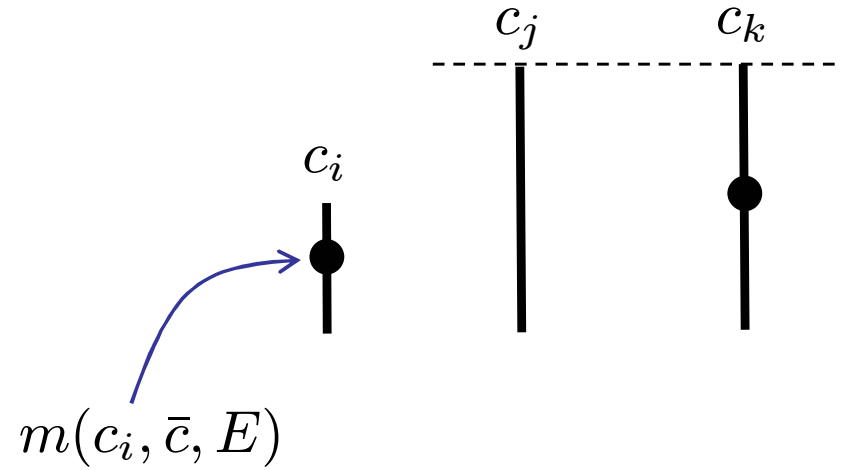
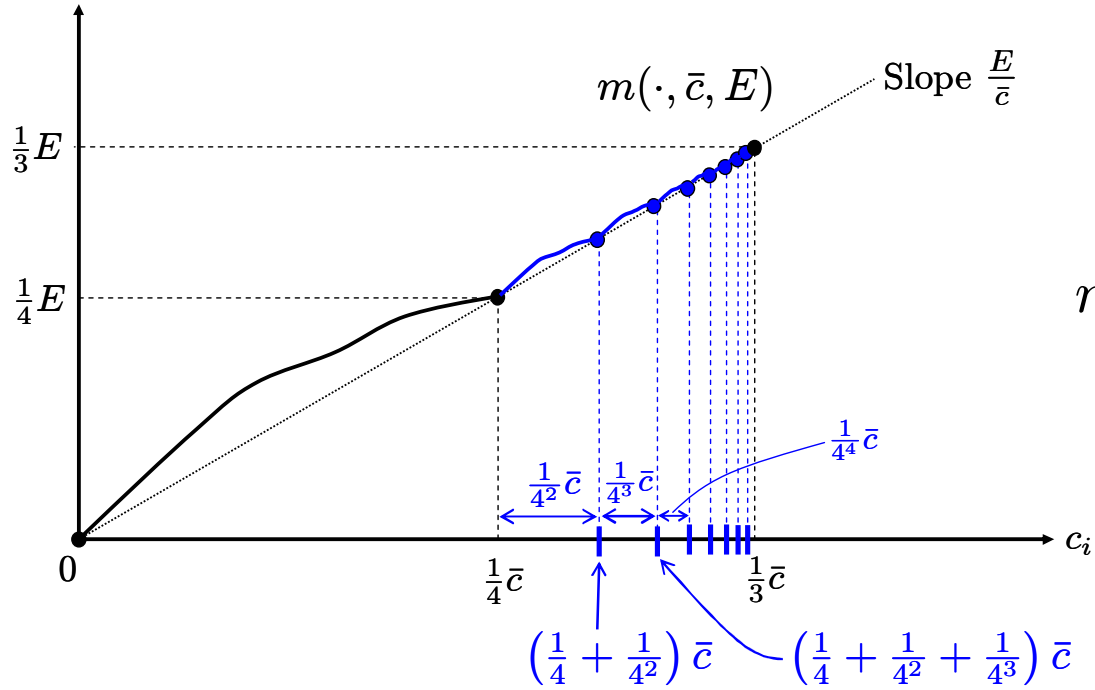
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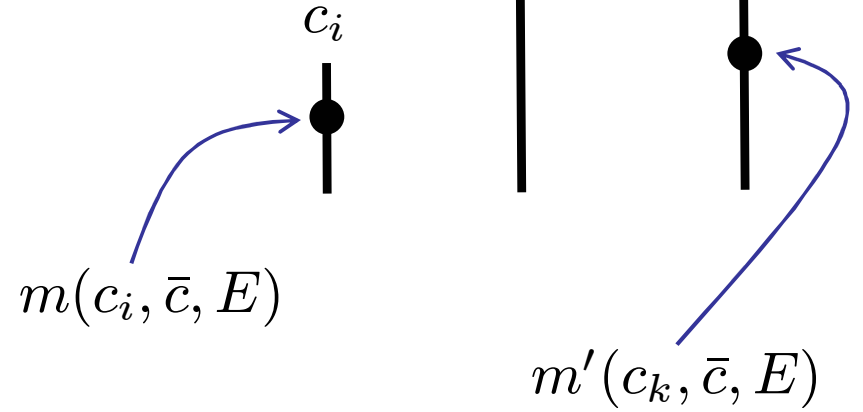
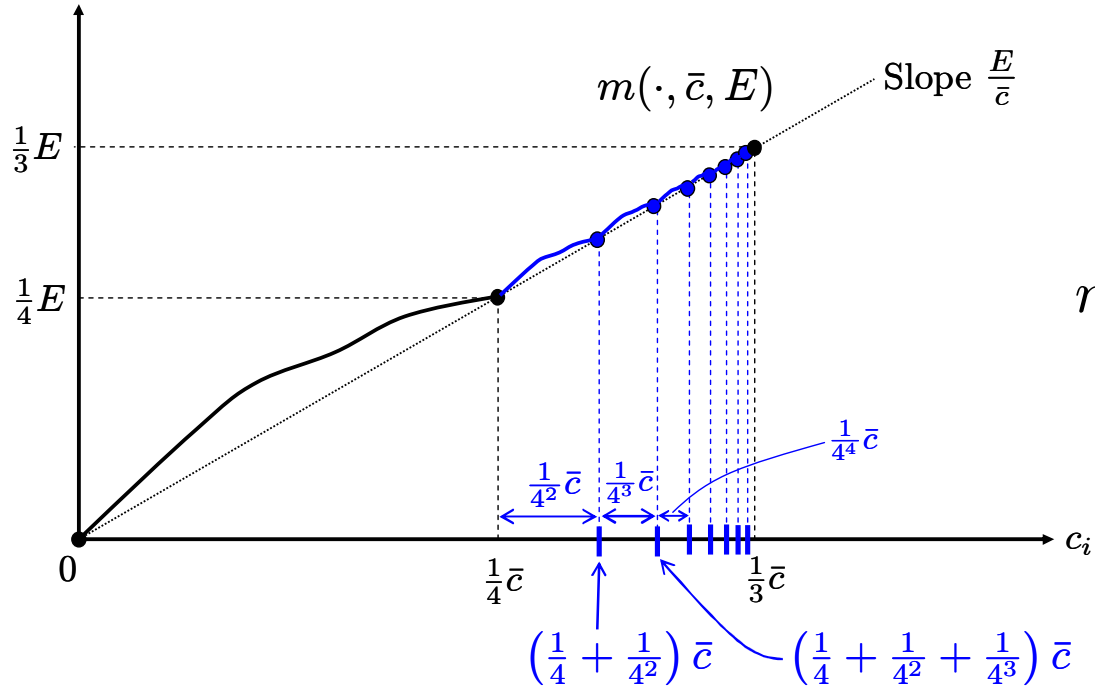
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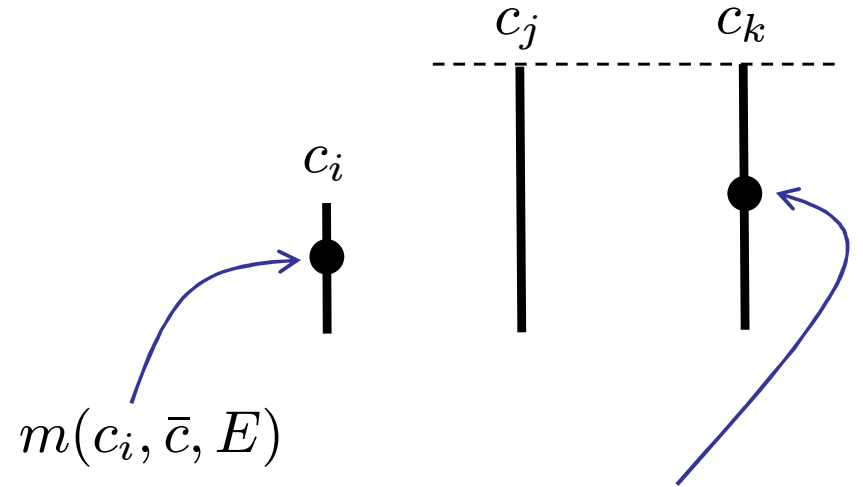
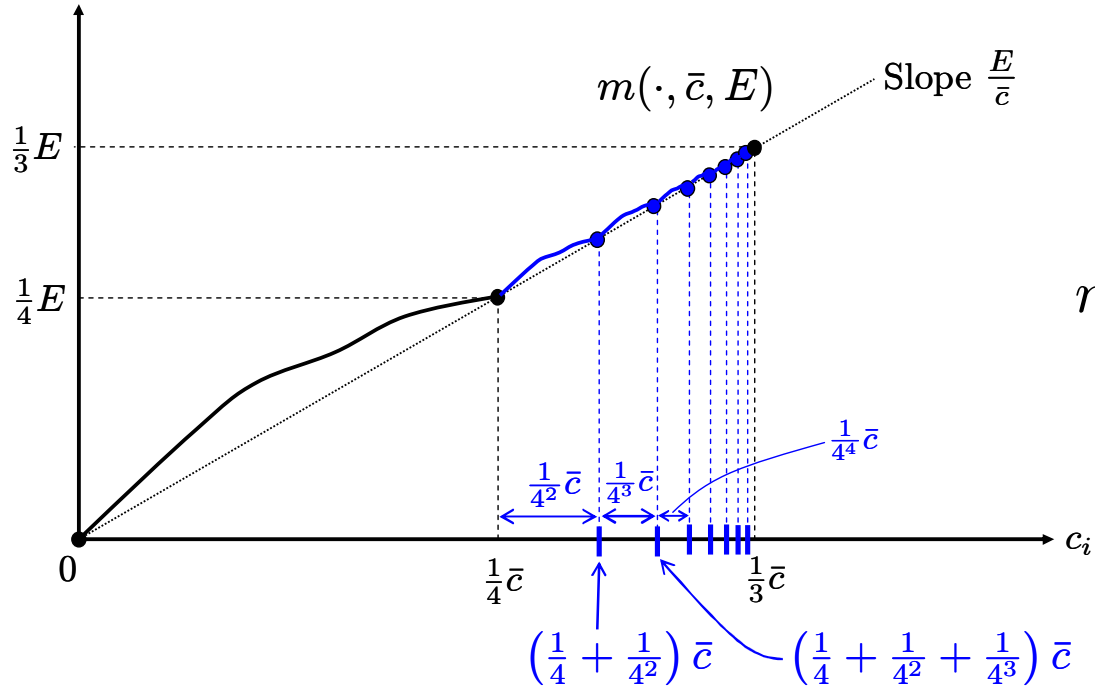
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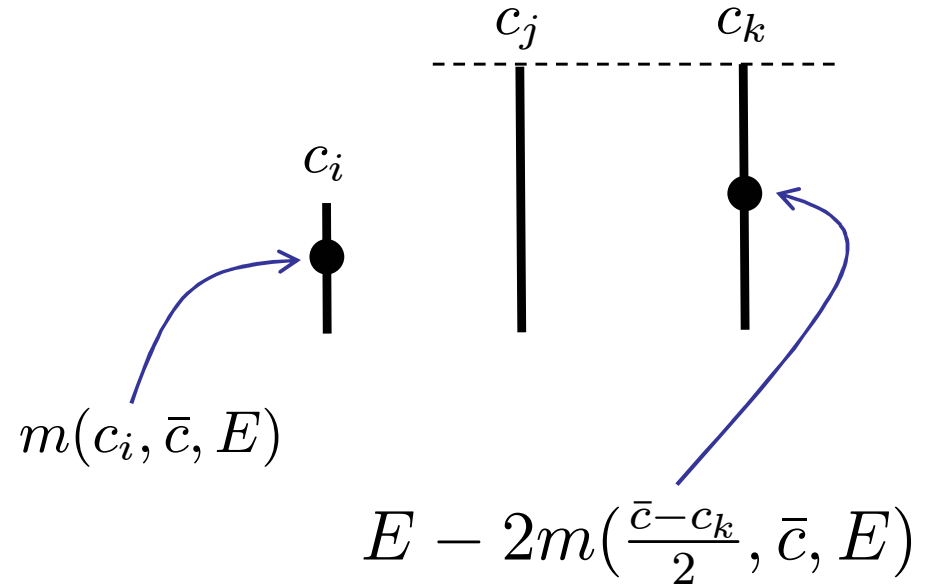
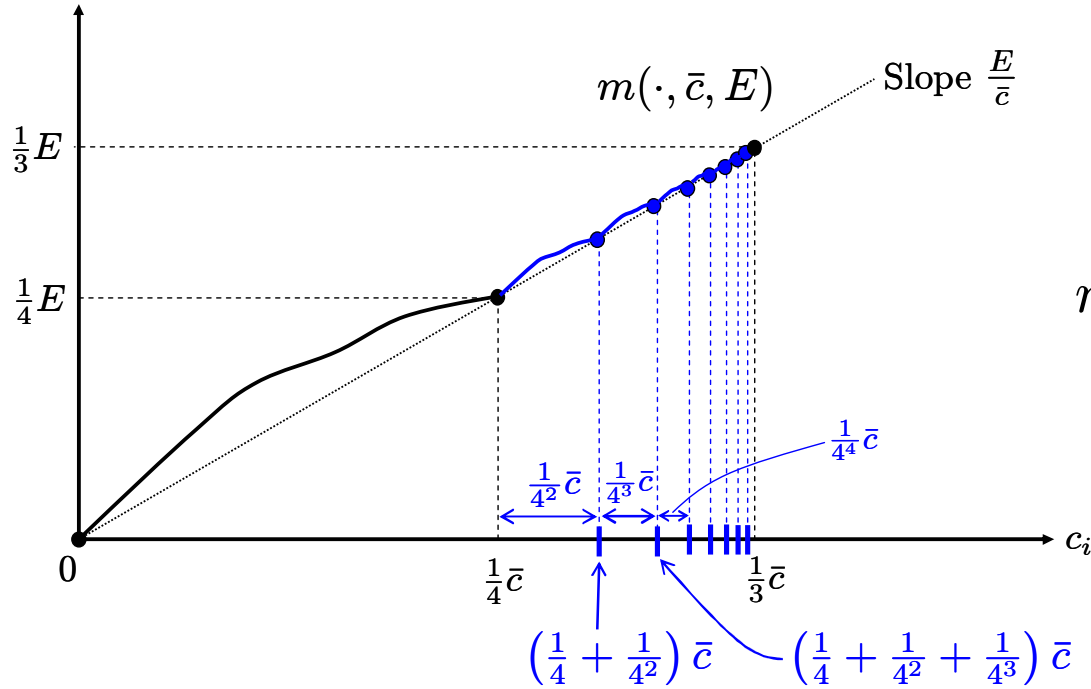
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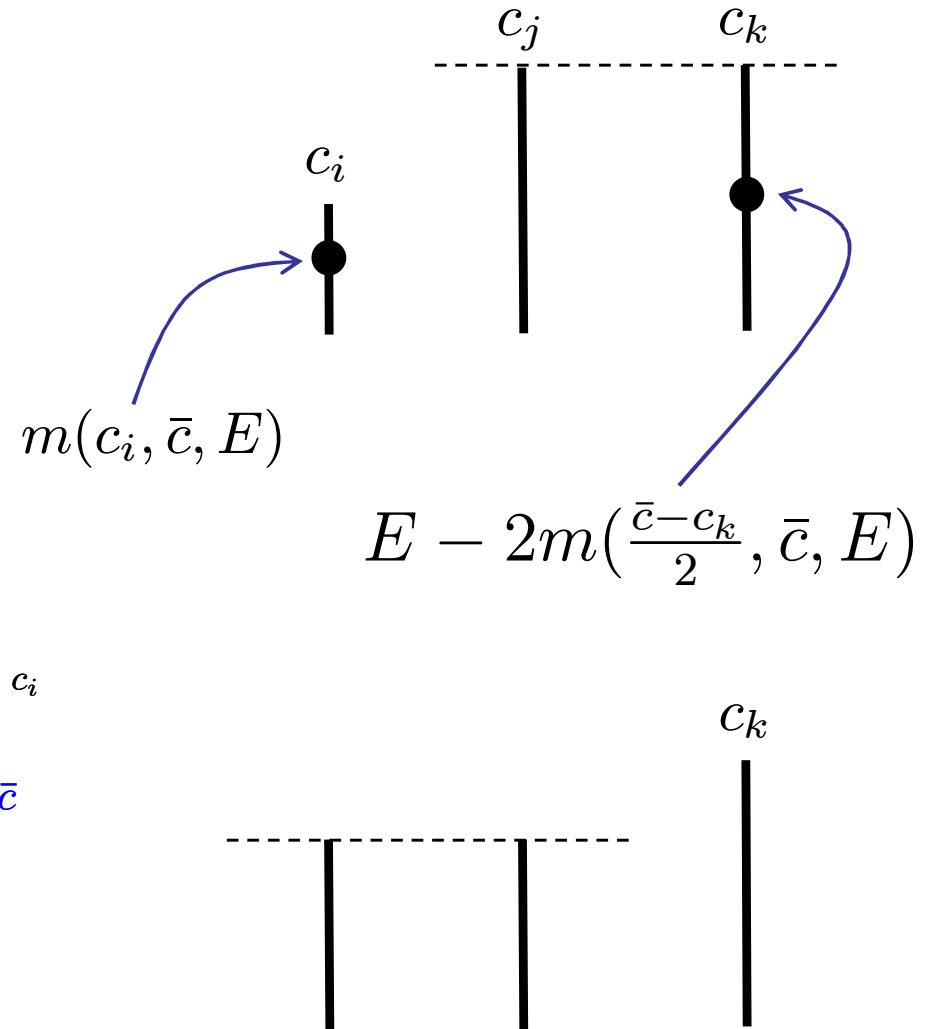
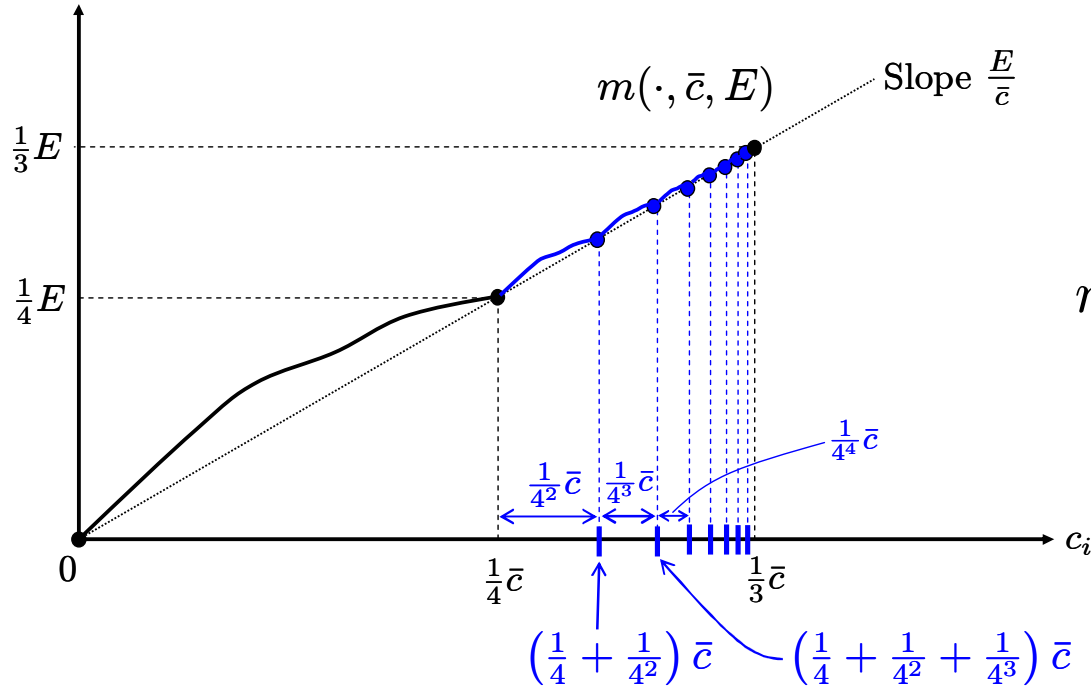
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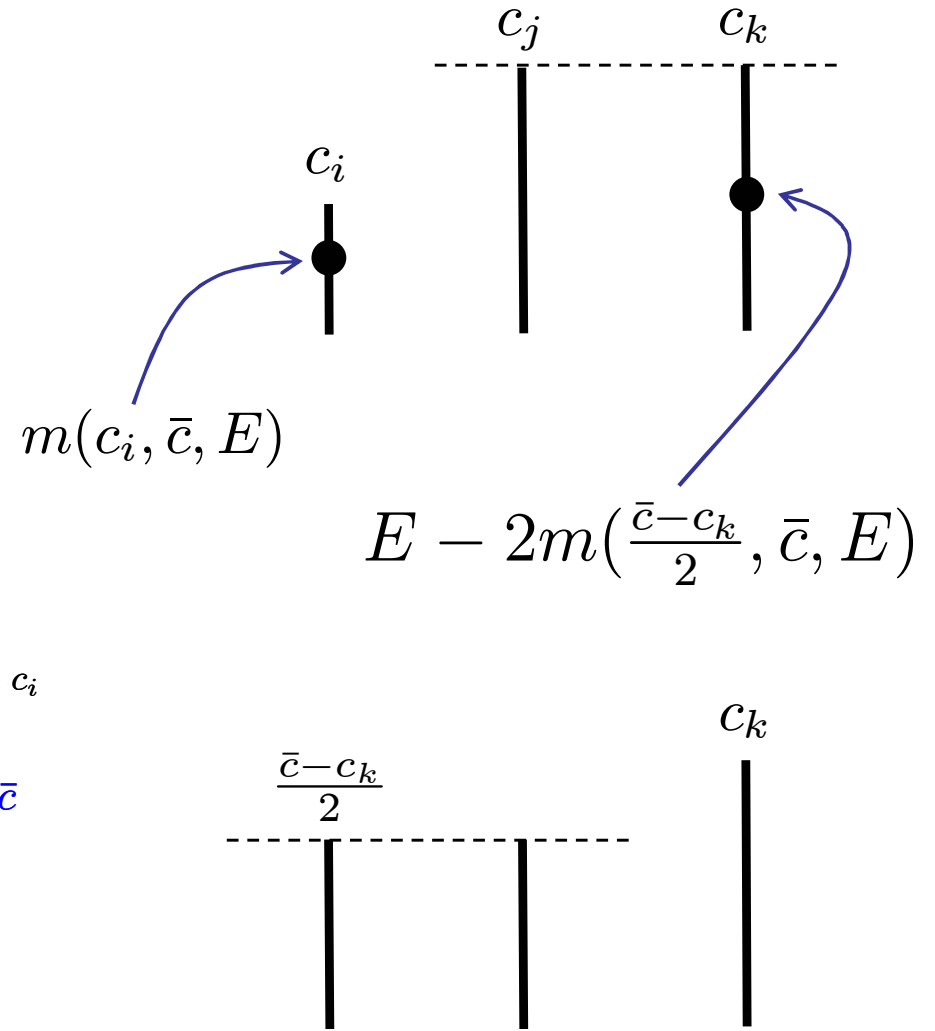
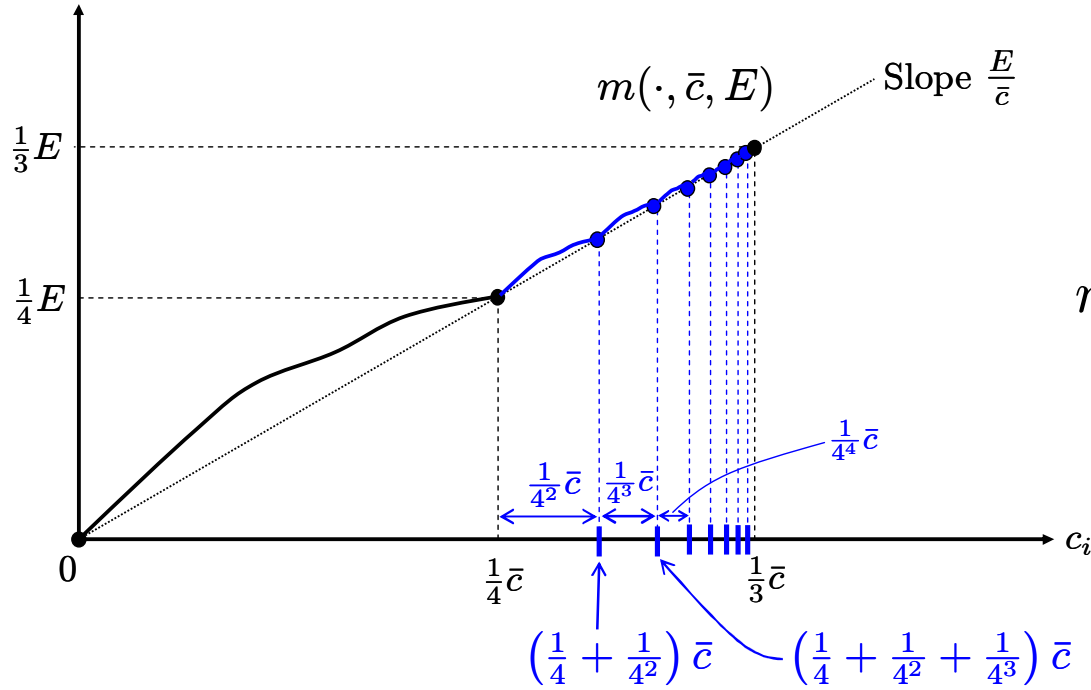
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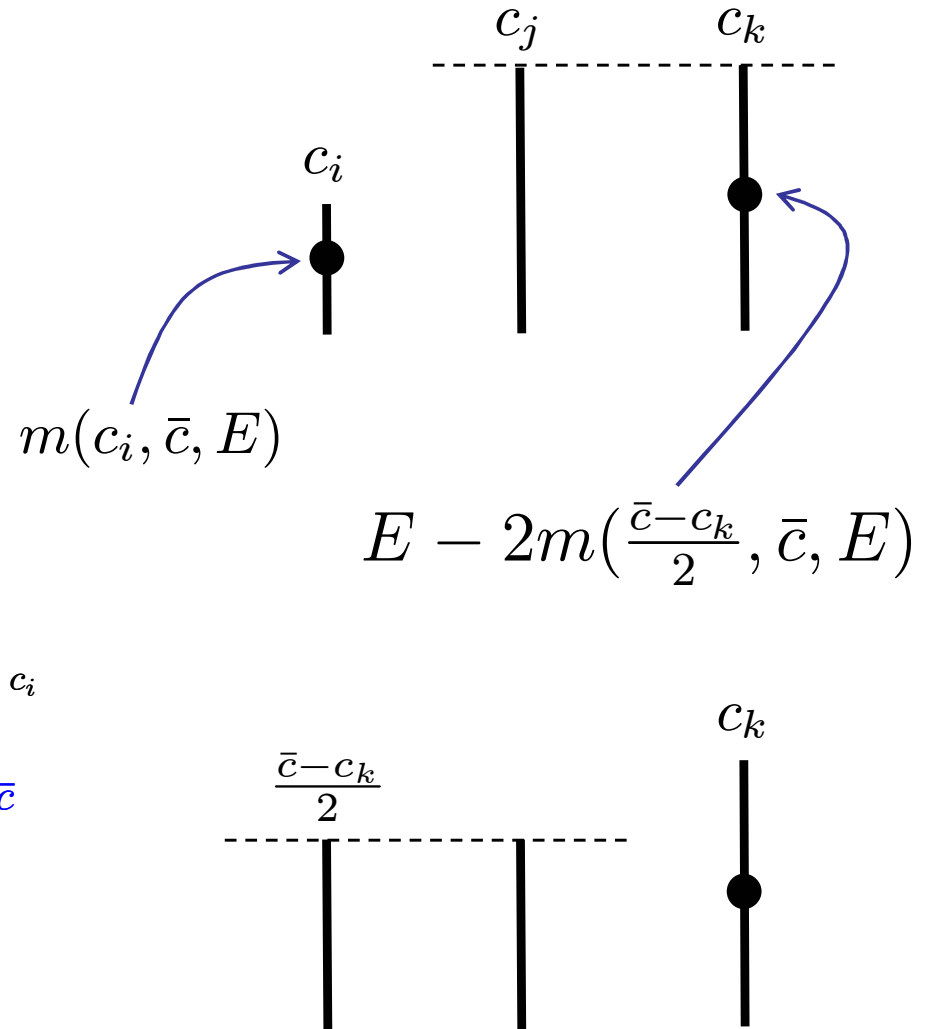
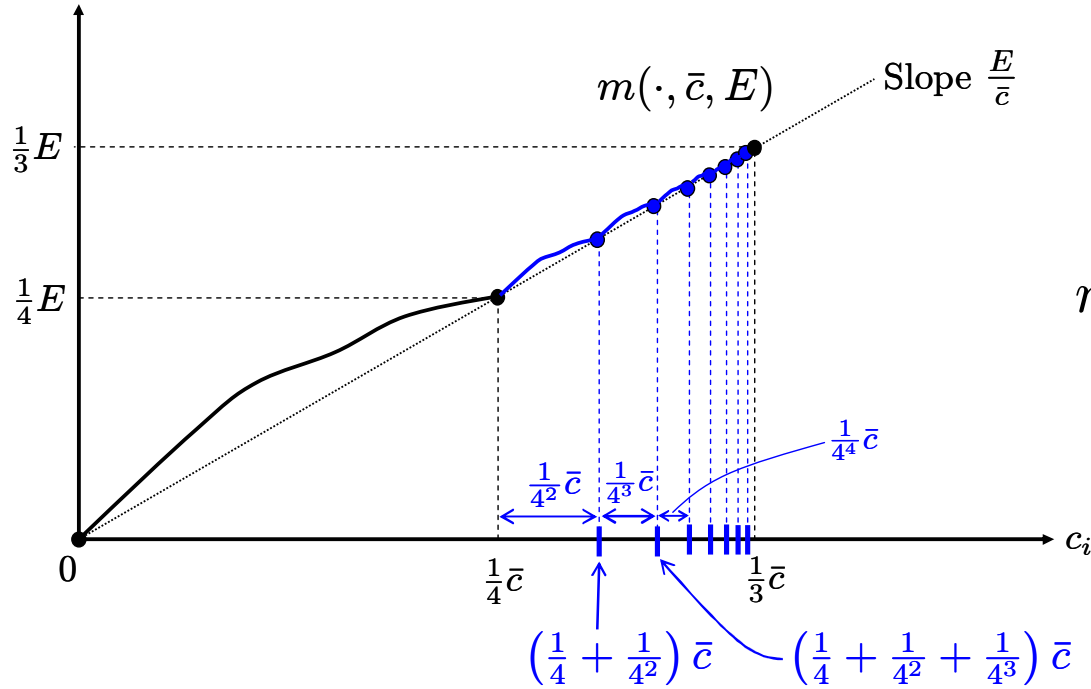
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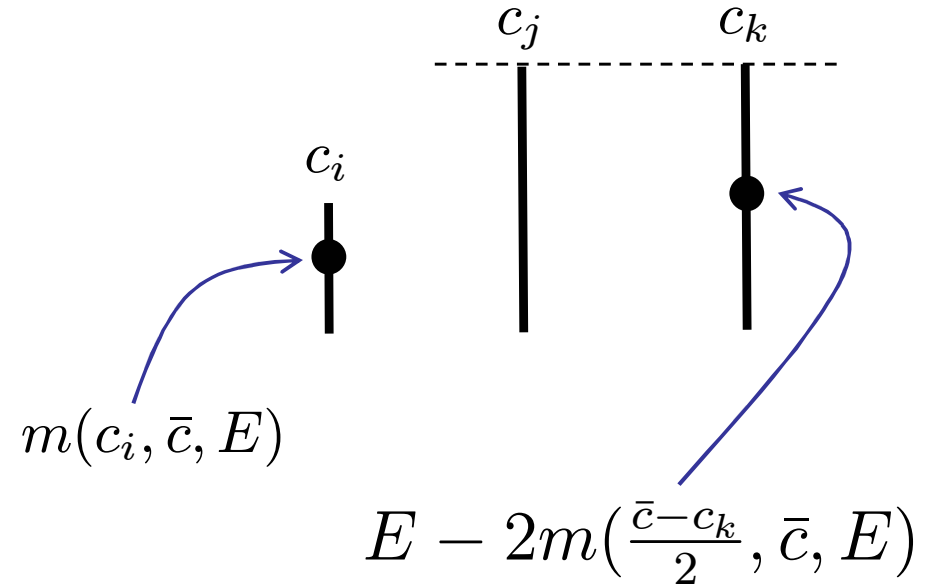
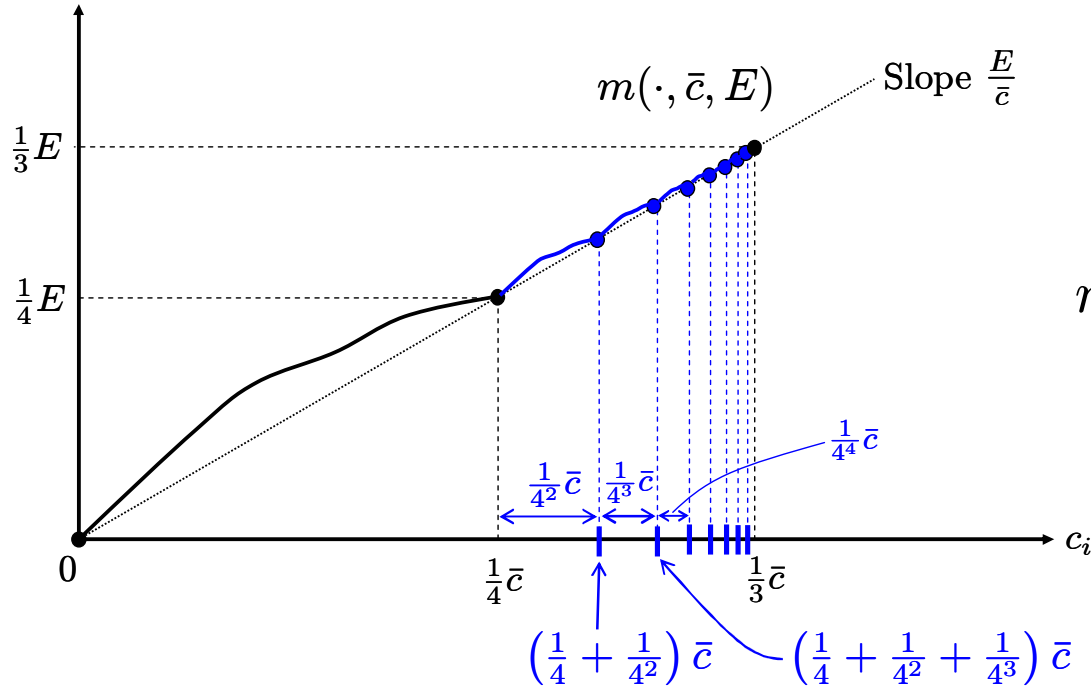
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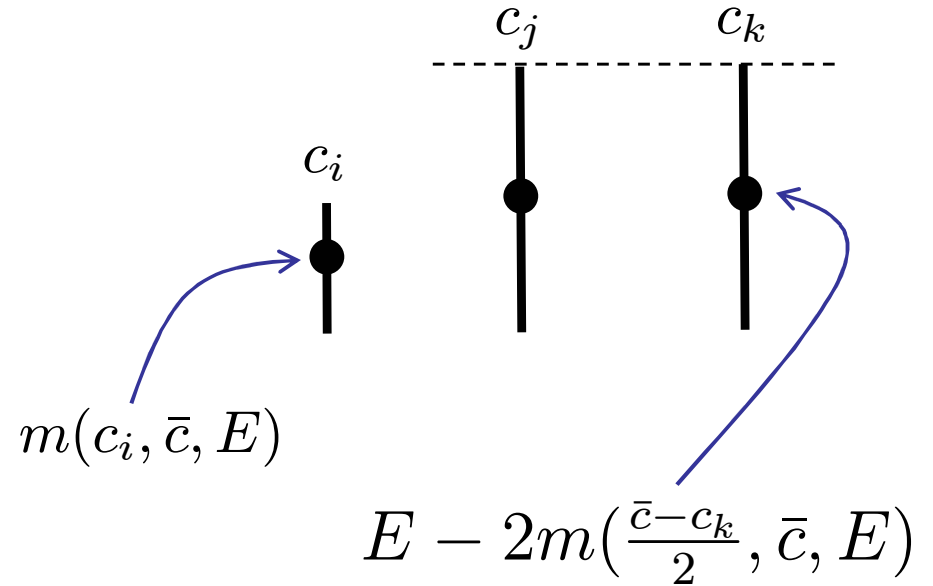
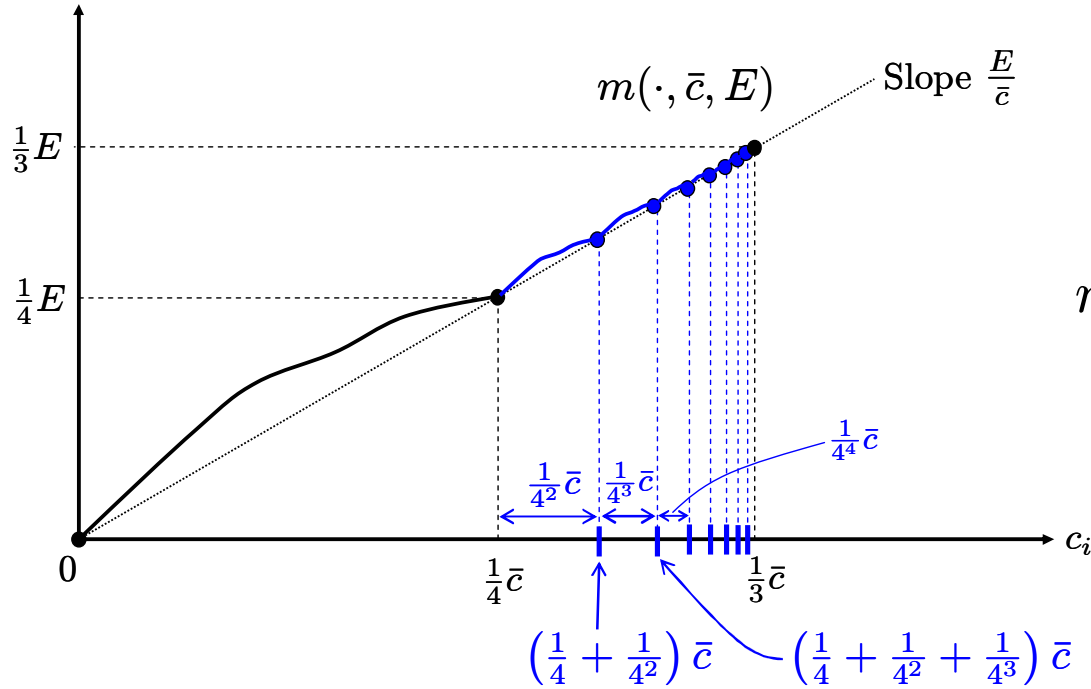
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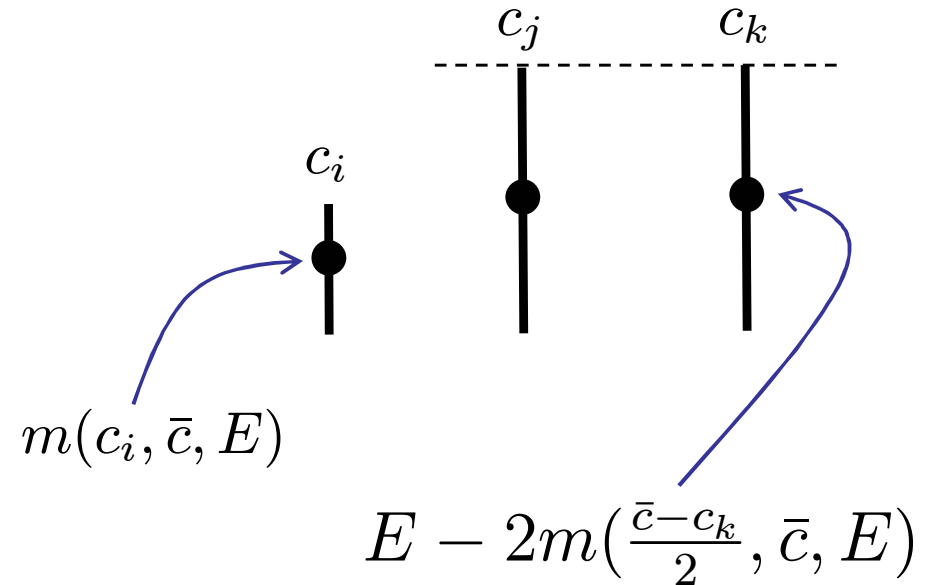
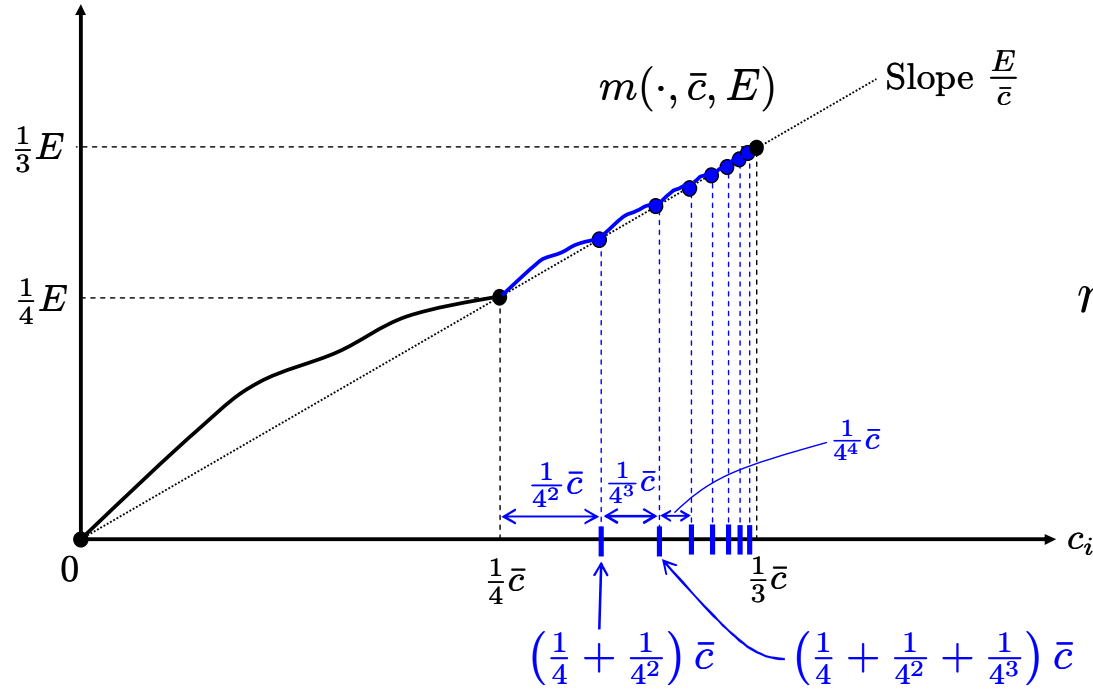
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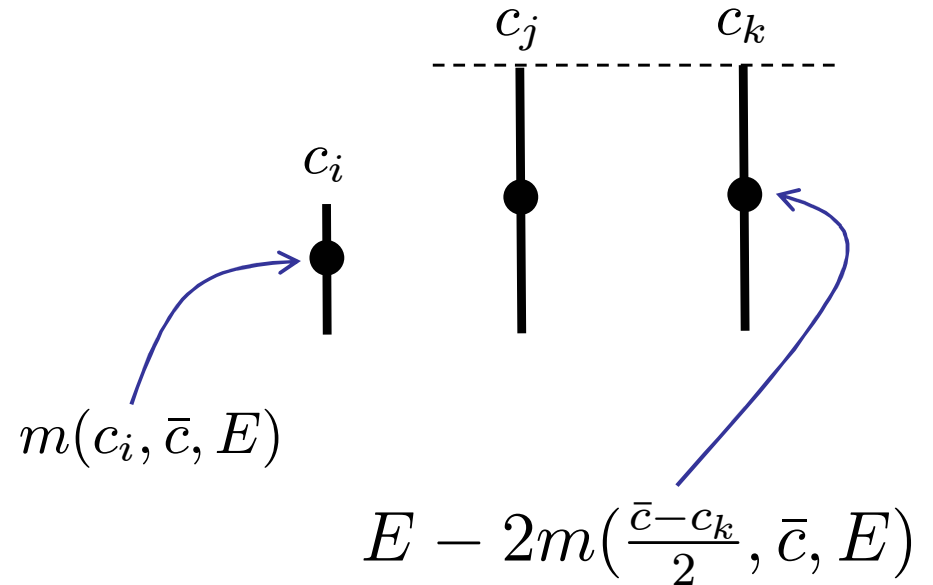
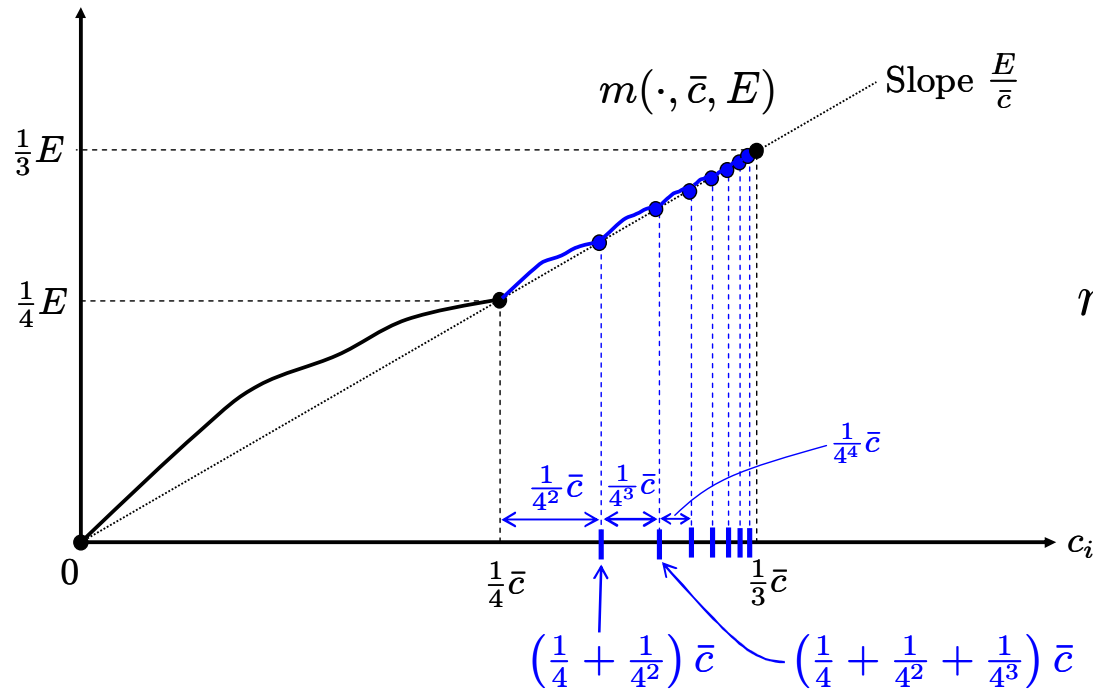
- Given  $\bar{c}$  and  $E$ ,



$$m(c_i, \bar{c}, E) = E - \left[ E - 2m\left(\frac{\bar{c} - c_j}{2}, \bar{c}, E\right) \right] - \left[ E - 2m\left(\frac{\bar{c} - c_k}{2}, \bar{c}, E\right) \right]$$

# Results (n=3)

- Given  $\bar{c}$  and  $E$ ,

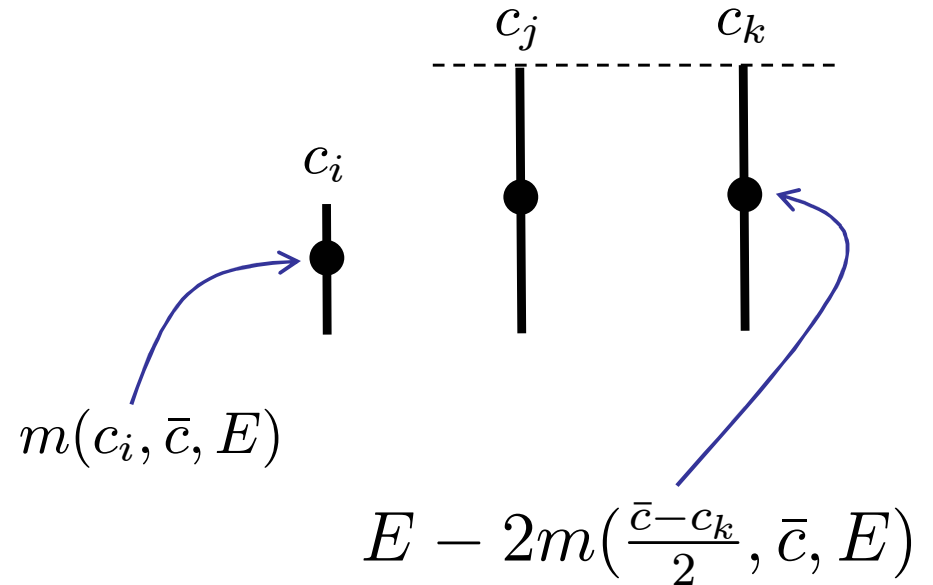
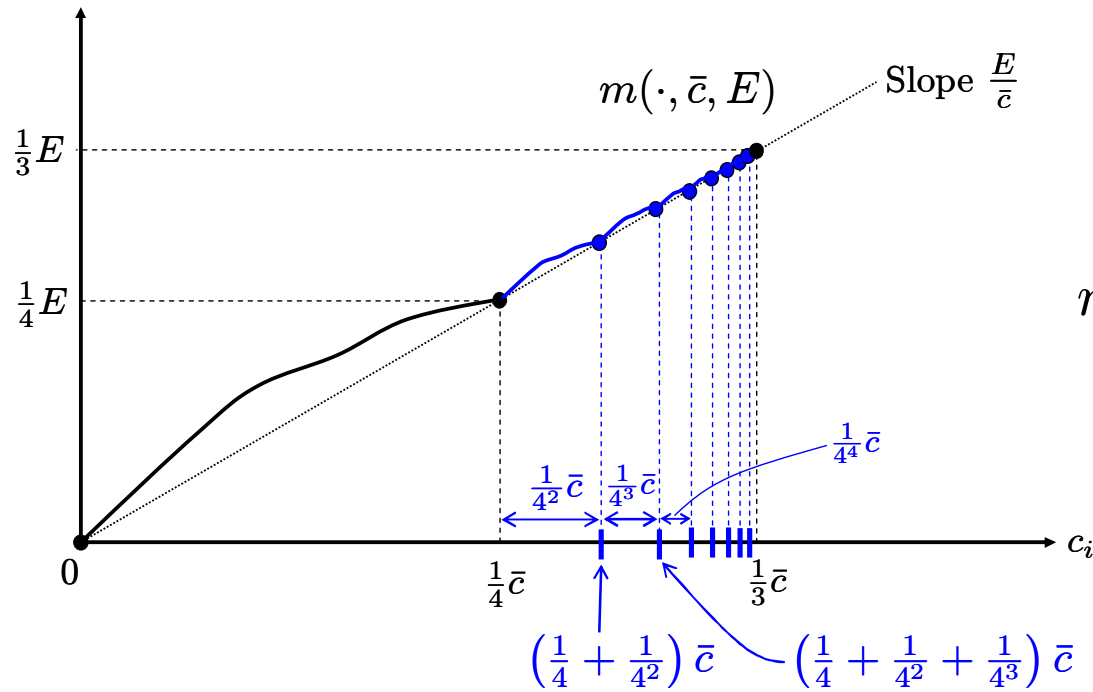


$$m(c_i, \bar{c}, E) = E - [E - 2m(\frac{\bar{c}-c_j}{2}, \bar{c}, E)] - [E - 2m(\frac{\bar{c}-c_k}{2}, \bar{c}, E)]$$

$$m(\frac{\bar{c}+c_i}{4}, \bar{c}, E) = \frac{1}{4}E + \frac{1}{4}m(c_i, \bar{c}, E)$$

# Results (n=3)

- Given  $\bar{c}$  and  $E$ ,



$$E - 2m\left(\frac{\bar{c} - c_k}{2}, \bar{c}, E\right)$$

$$m(c_i, \bar{c}, E) = E - \left[ E - 2m\left(\frac{\bar{c} - c_j}{2}, \bar{c}, E\right) \right] - \left[ E - 2m\left(\frac{\bar{c} - c_k}{2}, \bar{c}, E\right) \right]$$

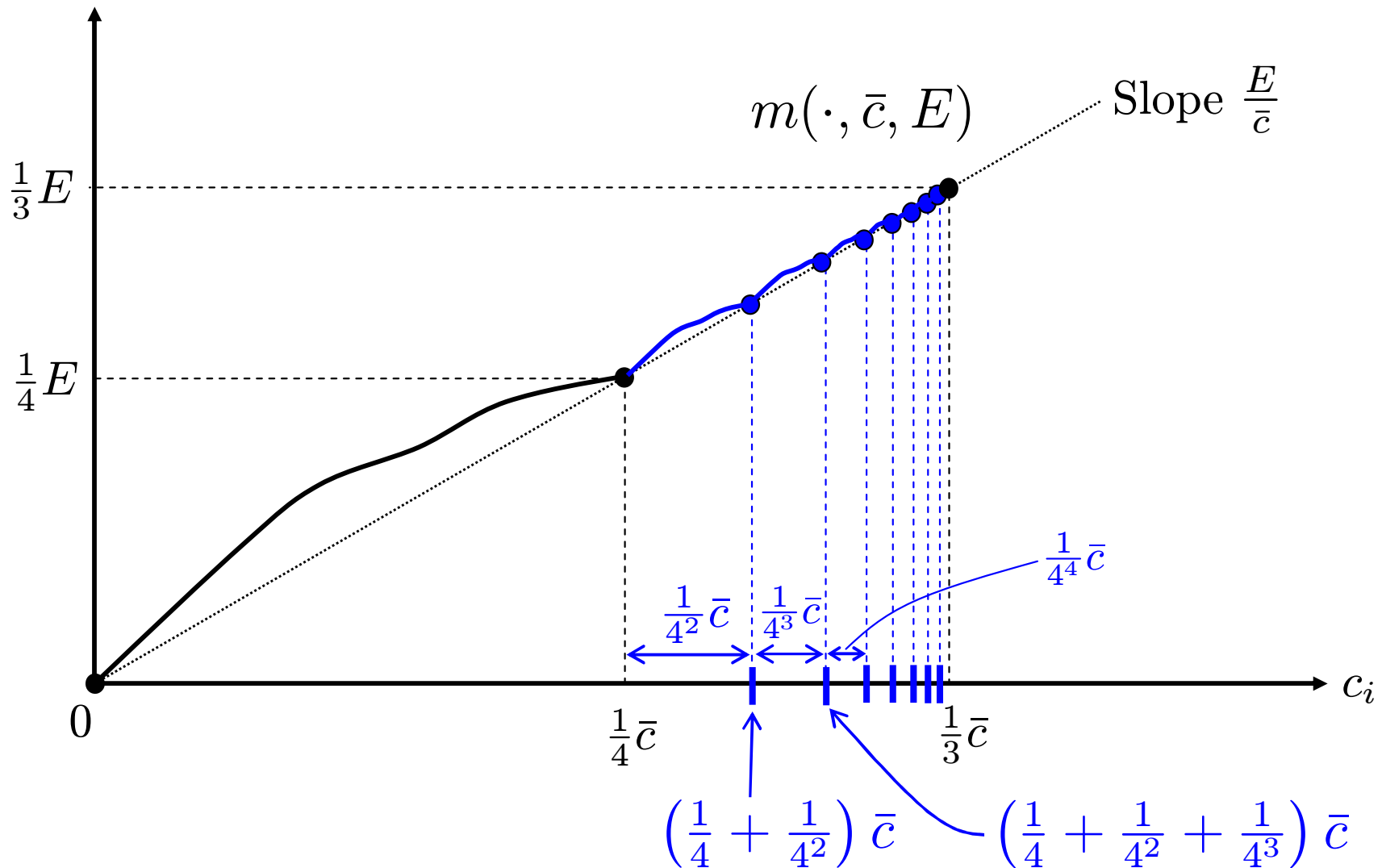
$$m\left(\frac{\bar{c} + c_i}{4}, \bar{c}, E\right) = \frac{1}{4}E + \frac{1}{4}m(c_i, \bar{c}, E)$$

$$\frac{\bar{c} - c_i}{2}$$

# Results (n=3)

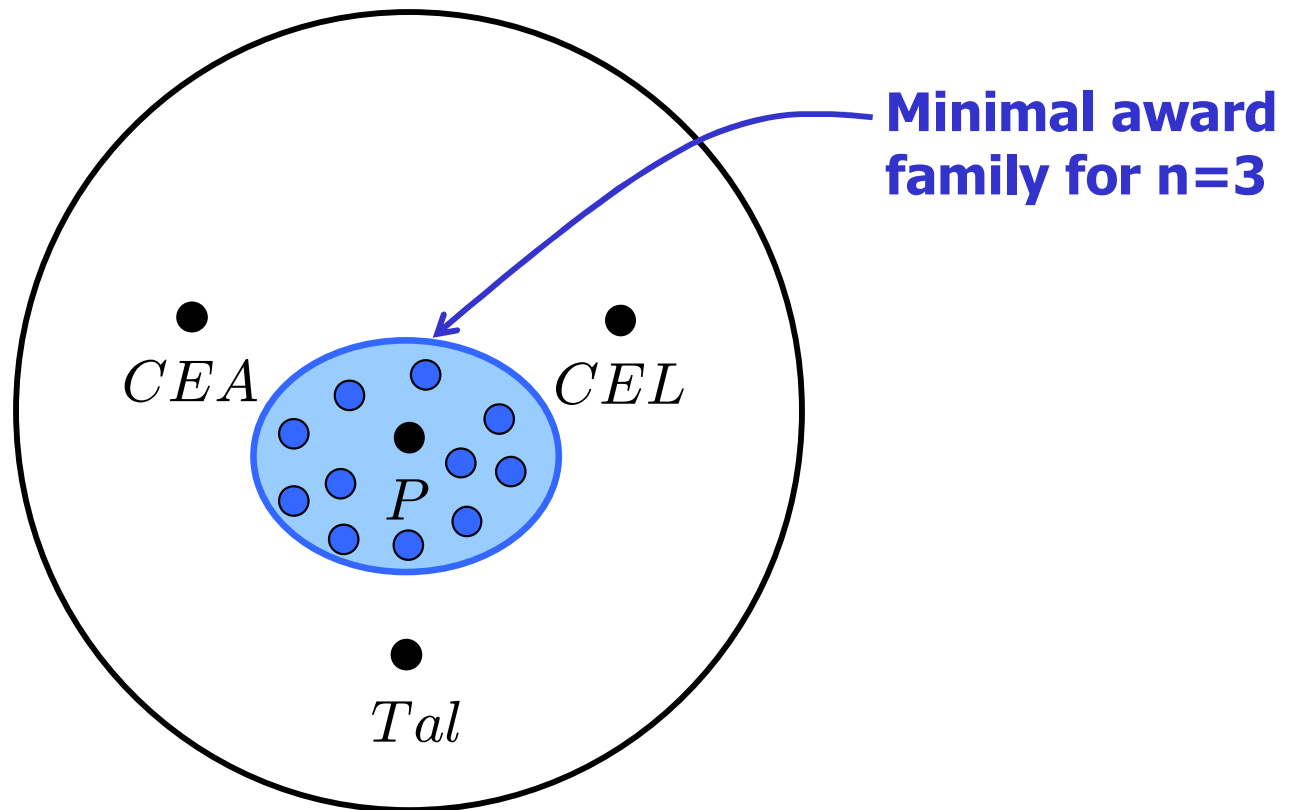
- Minimal award function,  $m$ :

- Continuous
- Given  $\bar{c}$  and  $E$ ,



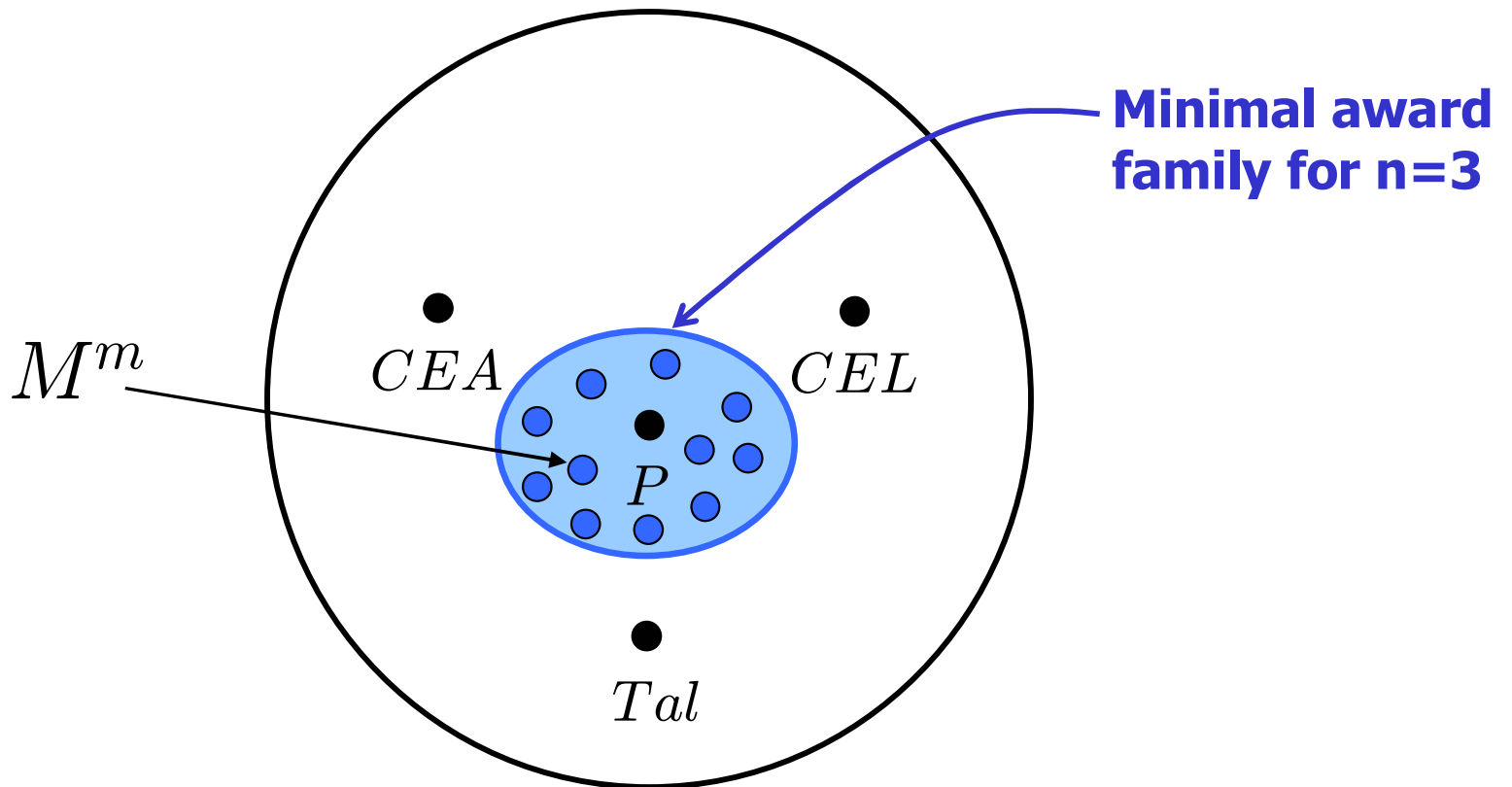
# Results (n=3)

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# Results (n=3)

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# Results ( $n=3$ )

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- Minimal award rule associated to  $m$ ,  $M^m$ :

# Results ( $n=3$ )

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- Minimal award rule associated to  $m$ ,  $M^m$ :

For each  $(c, E) \in \mathcal{C}$ , and each  $i \in N$ ,



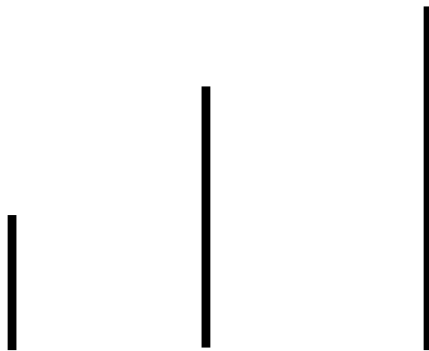
# Results ( $n=3$ )

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- Minimal award rule associated to  $m$ ,  $M^m$ :

For each  $(c, E) \in \mathcal{C}$ , and each  $i \in N$ ,

$$M_i^m(c, E) \equiv \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right.$$





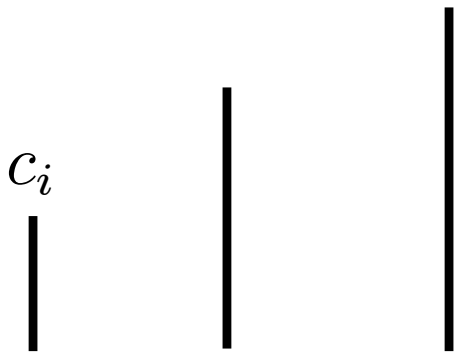
# Results ( $n=3$ )

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- Minimal award rule associated to  $m$ ,  $M^m$ :

For each  $(c, E) \in \mathcal{C}$ , and each  $i \in N$ ,

$$M_i^m(c, E) \equiv \begin{cases} & \text{if } c_i = \min c < \max c \end{cases}$$



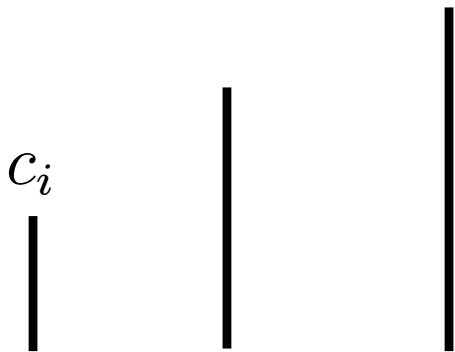
# Results ( $n=3$ )

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For each  $(c, E) \in \mathcal{C}$ , and each  $i \in N$ ,

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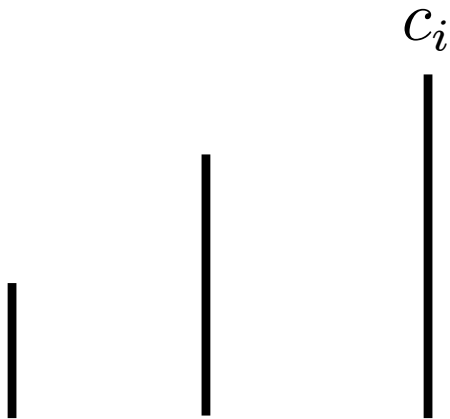
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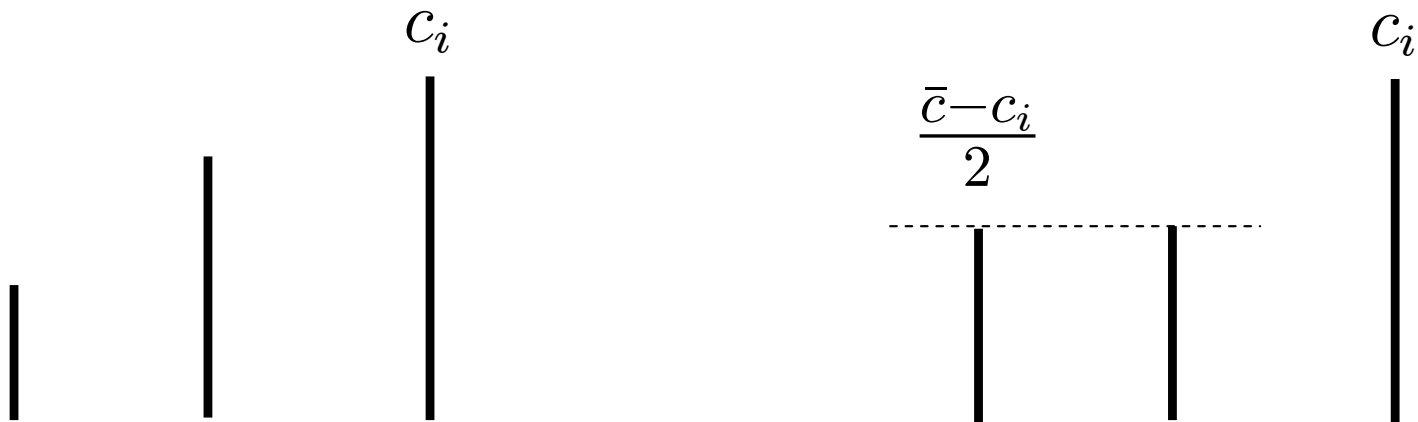
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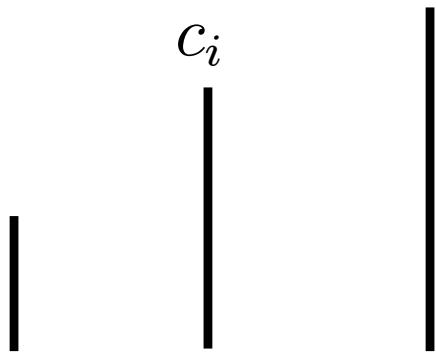
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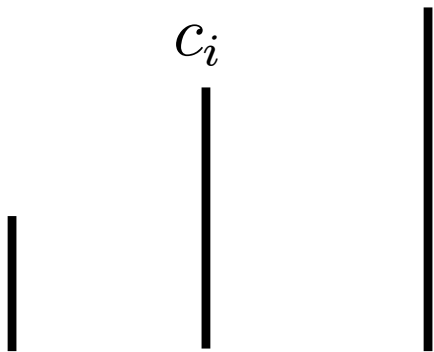
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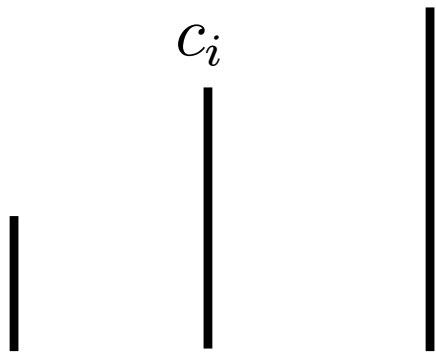
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# Results ( $n=3$ )

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**Theorem 2:** Assume  $n = 3$ .

$S: \text{cont}, \begin{matrix} \text{ord-pres-g} & \text{ineq-pres-g} \\ \text{ord-pres-l} & \text{ineq-pres-l} \end{matrix} \iff$

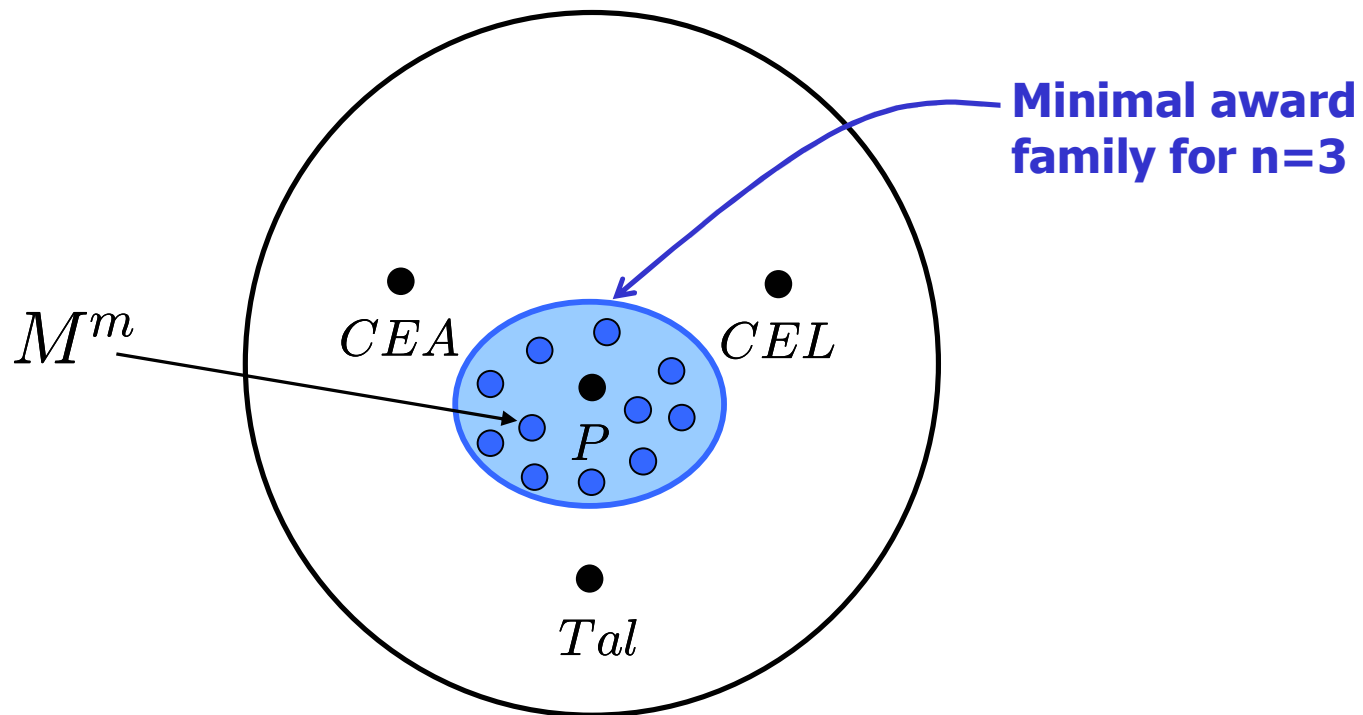
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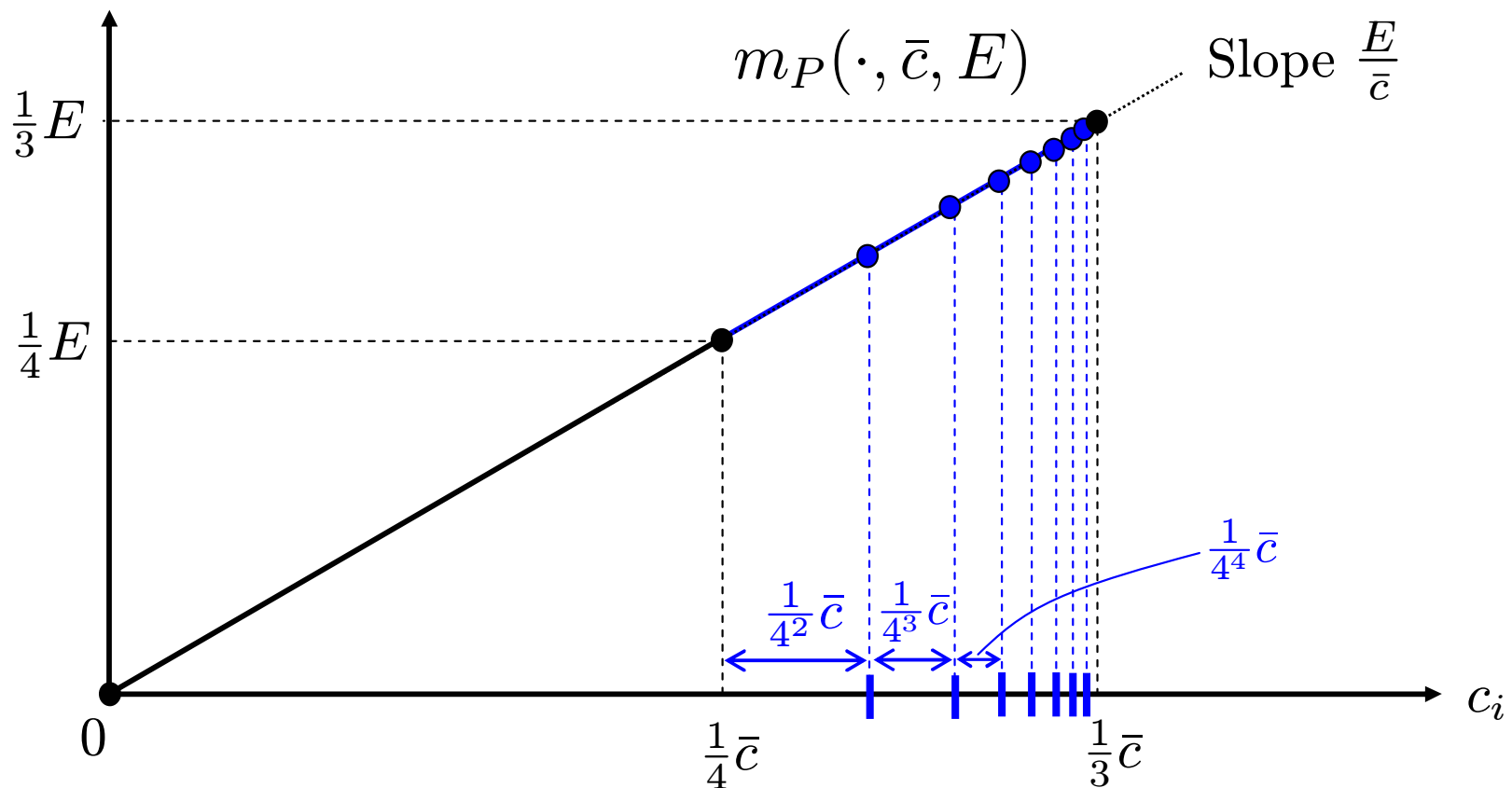


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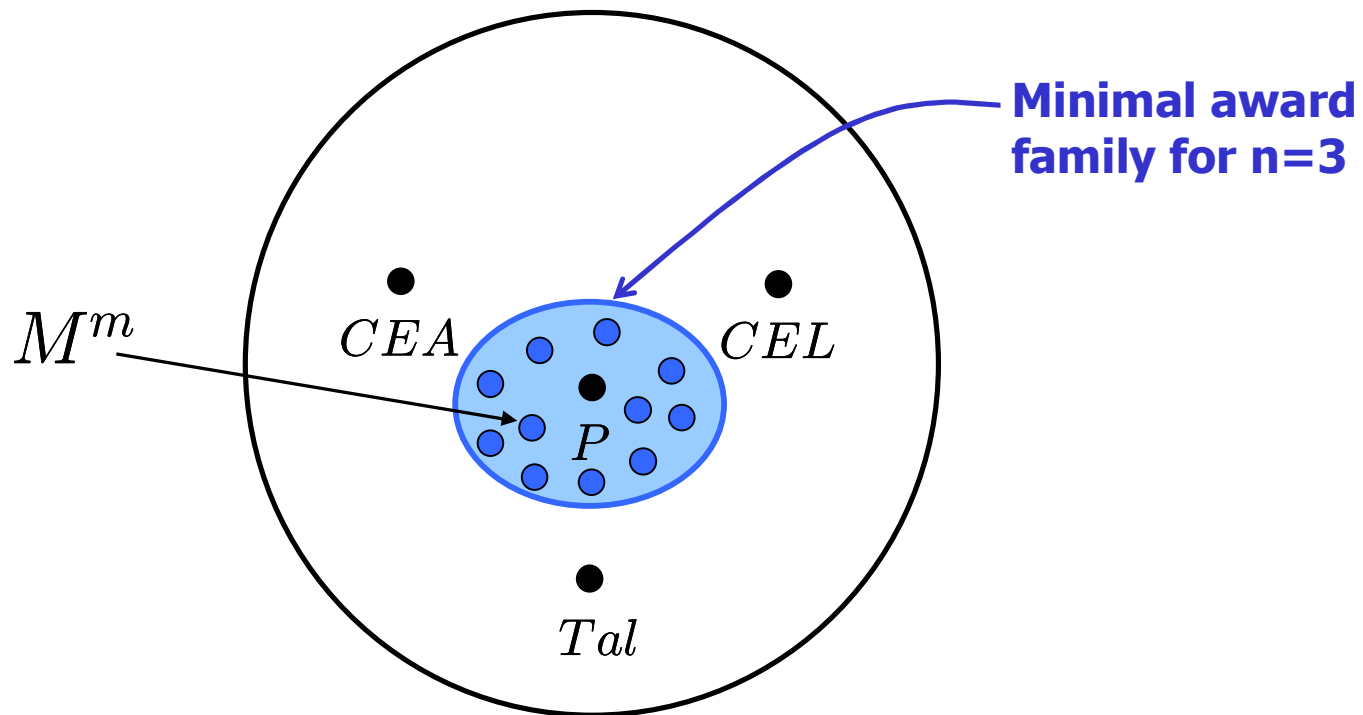


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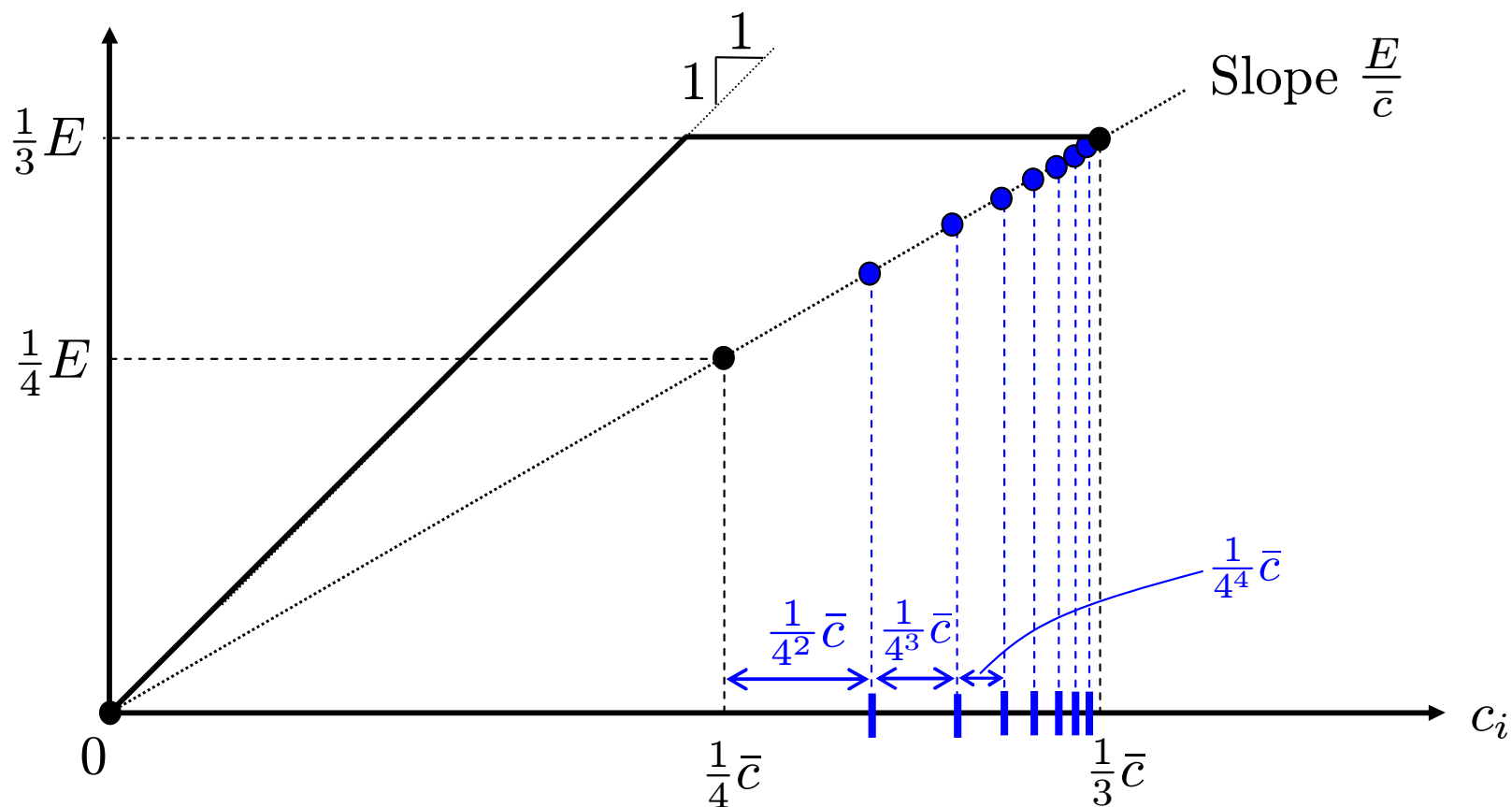


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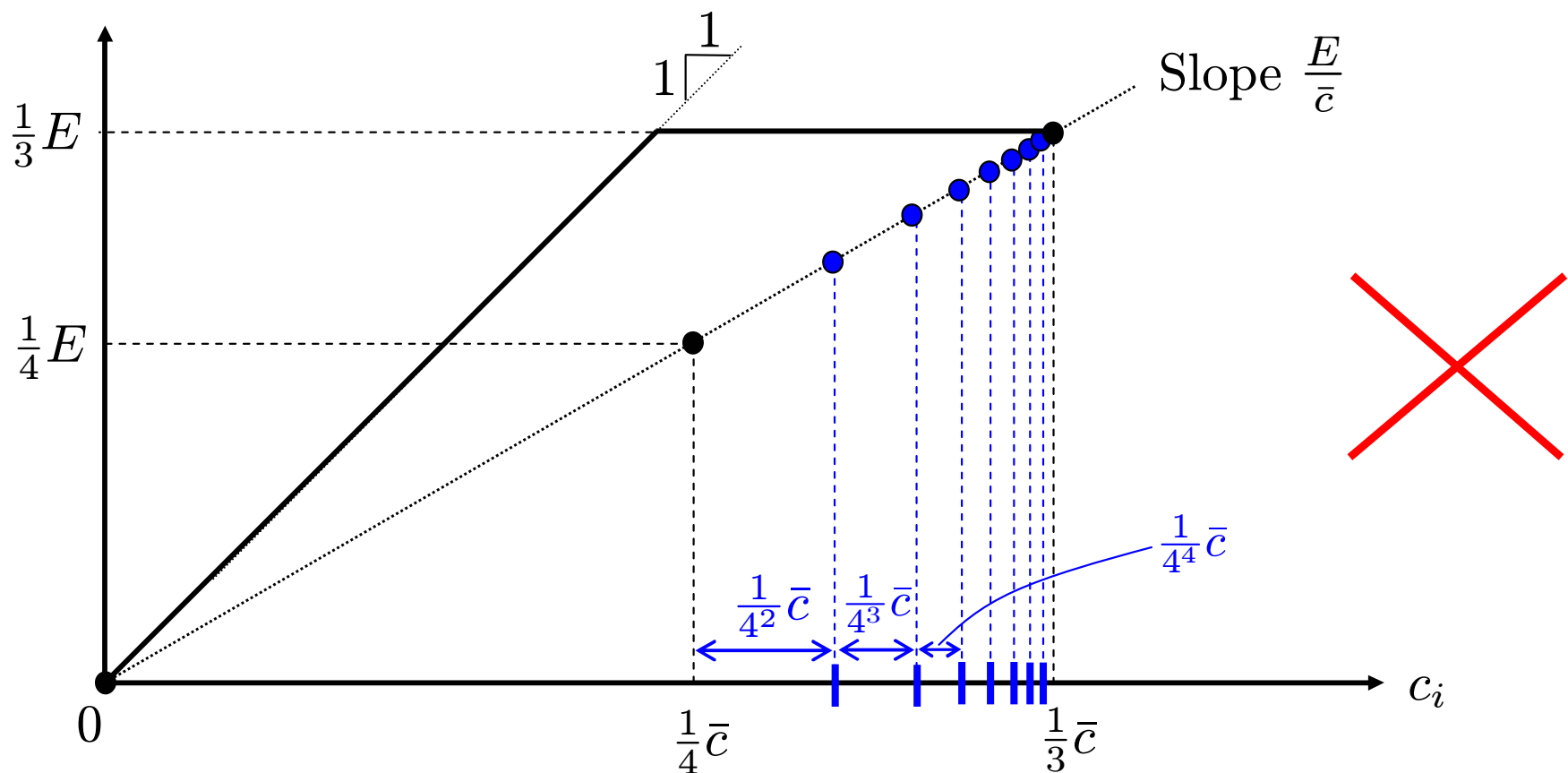


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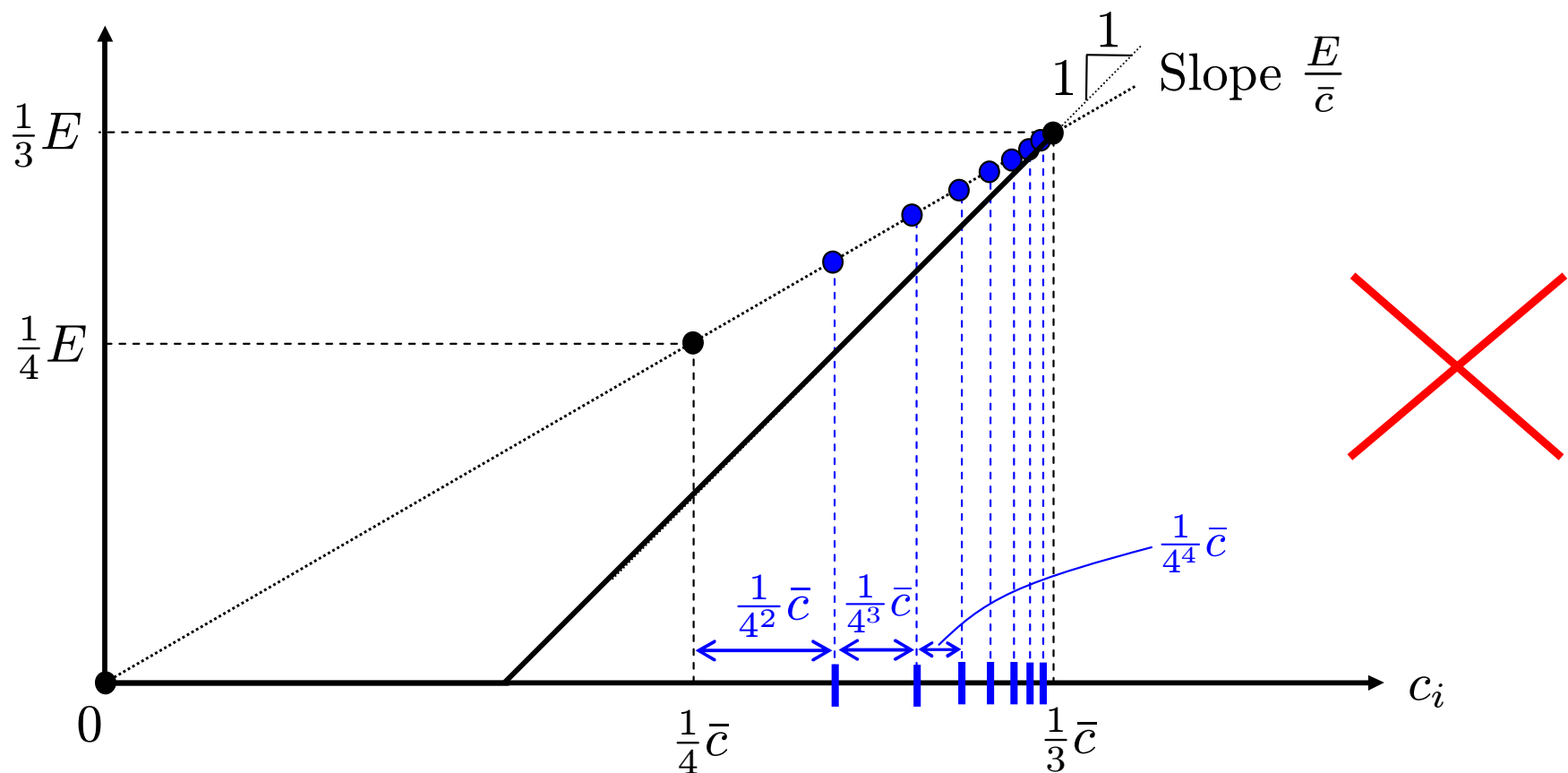


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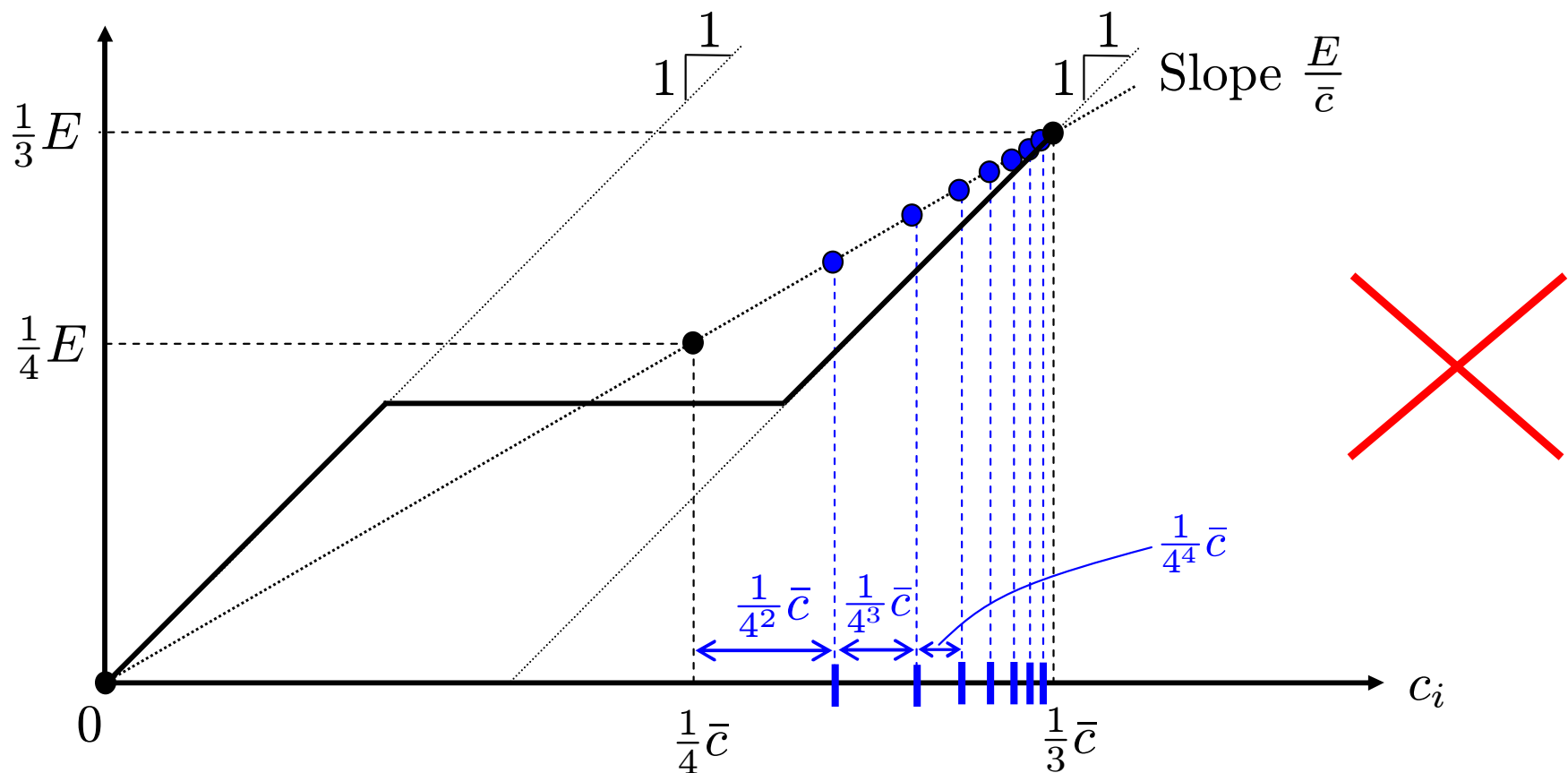


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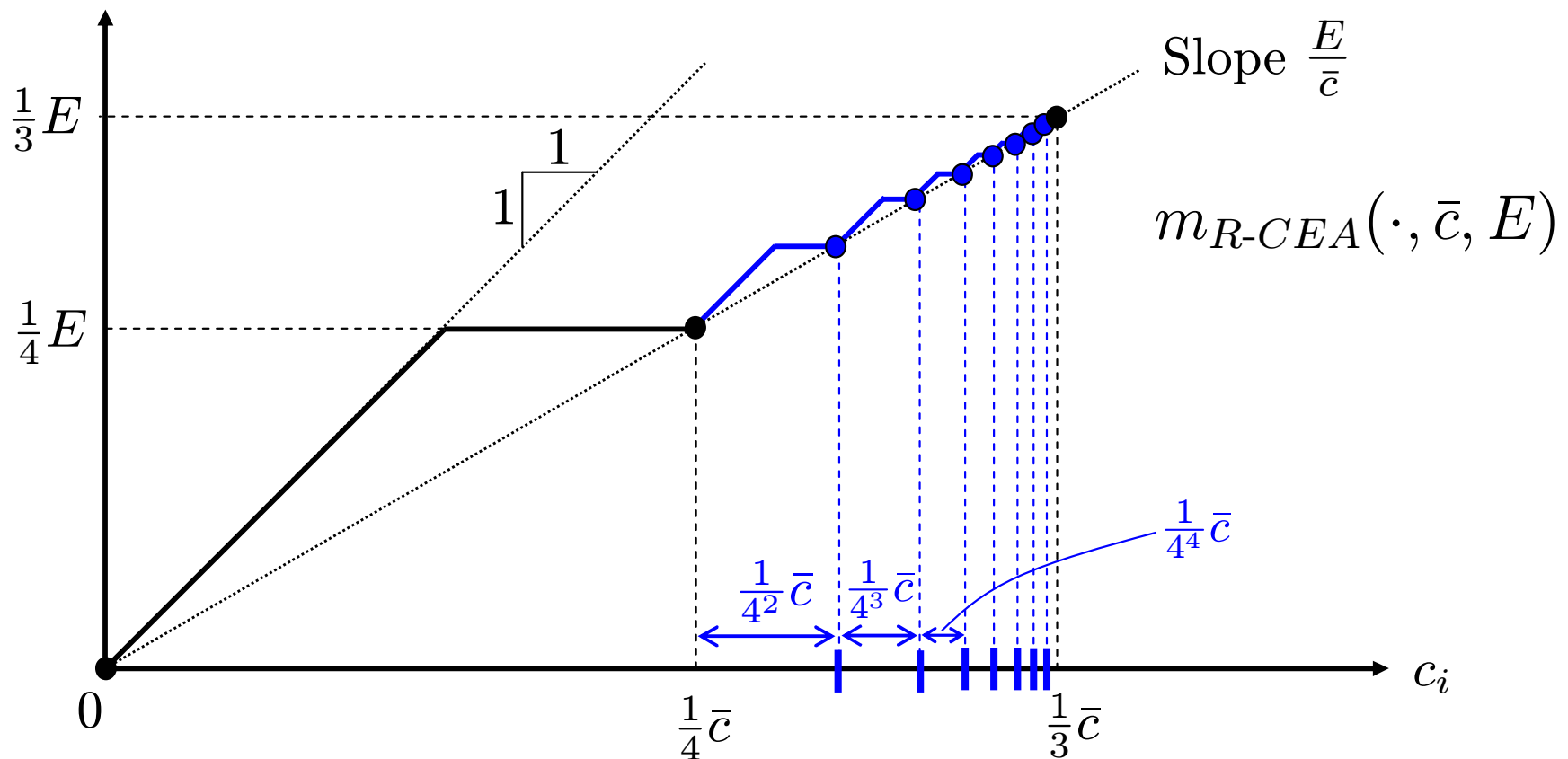


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# Results (n=3)

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- $S$  Lorenz dominates  $S'$ ,  $S \succeq_L S'$ ,  
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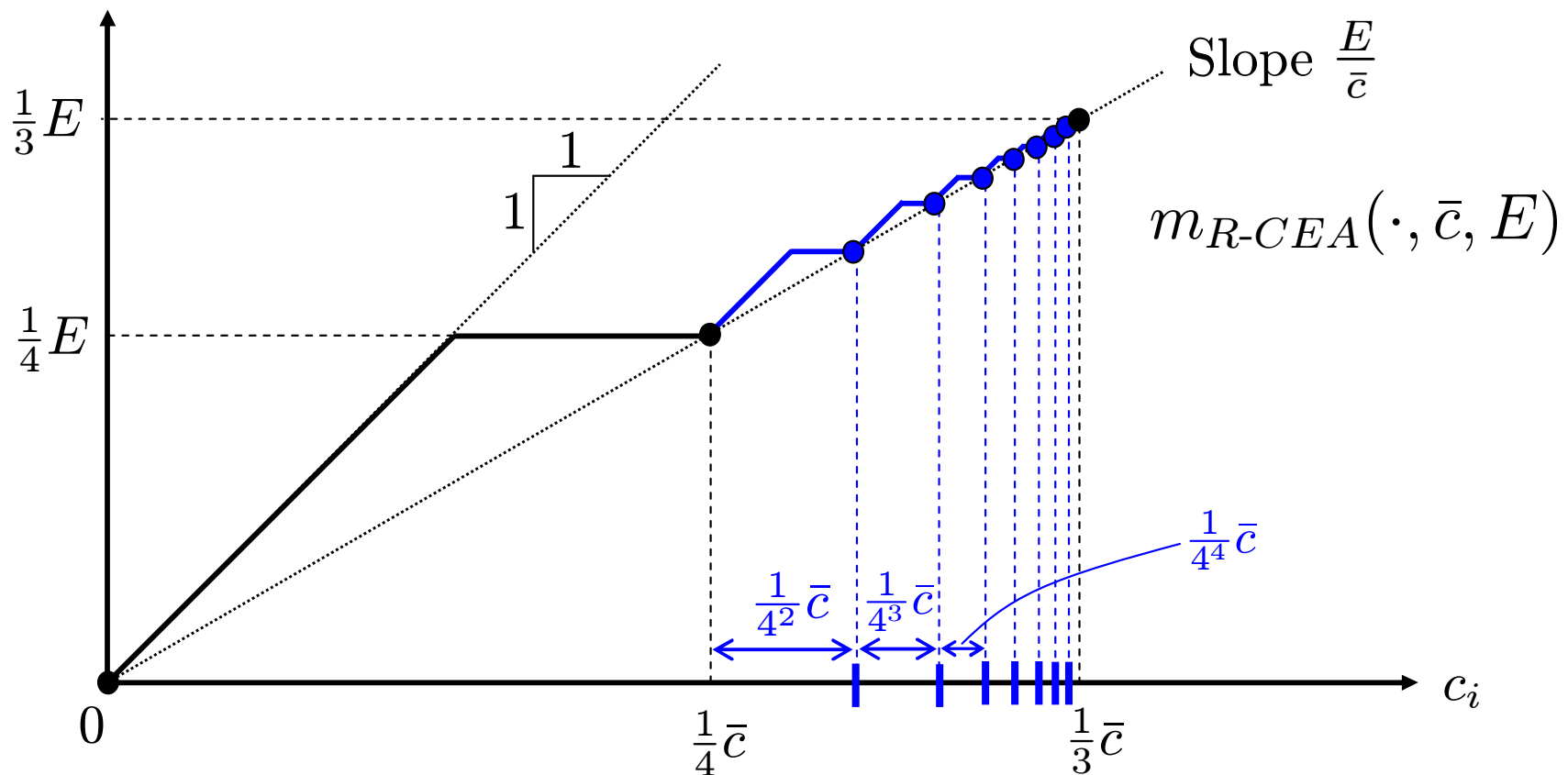
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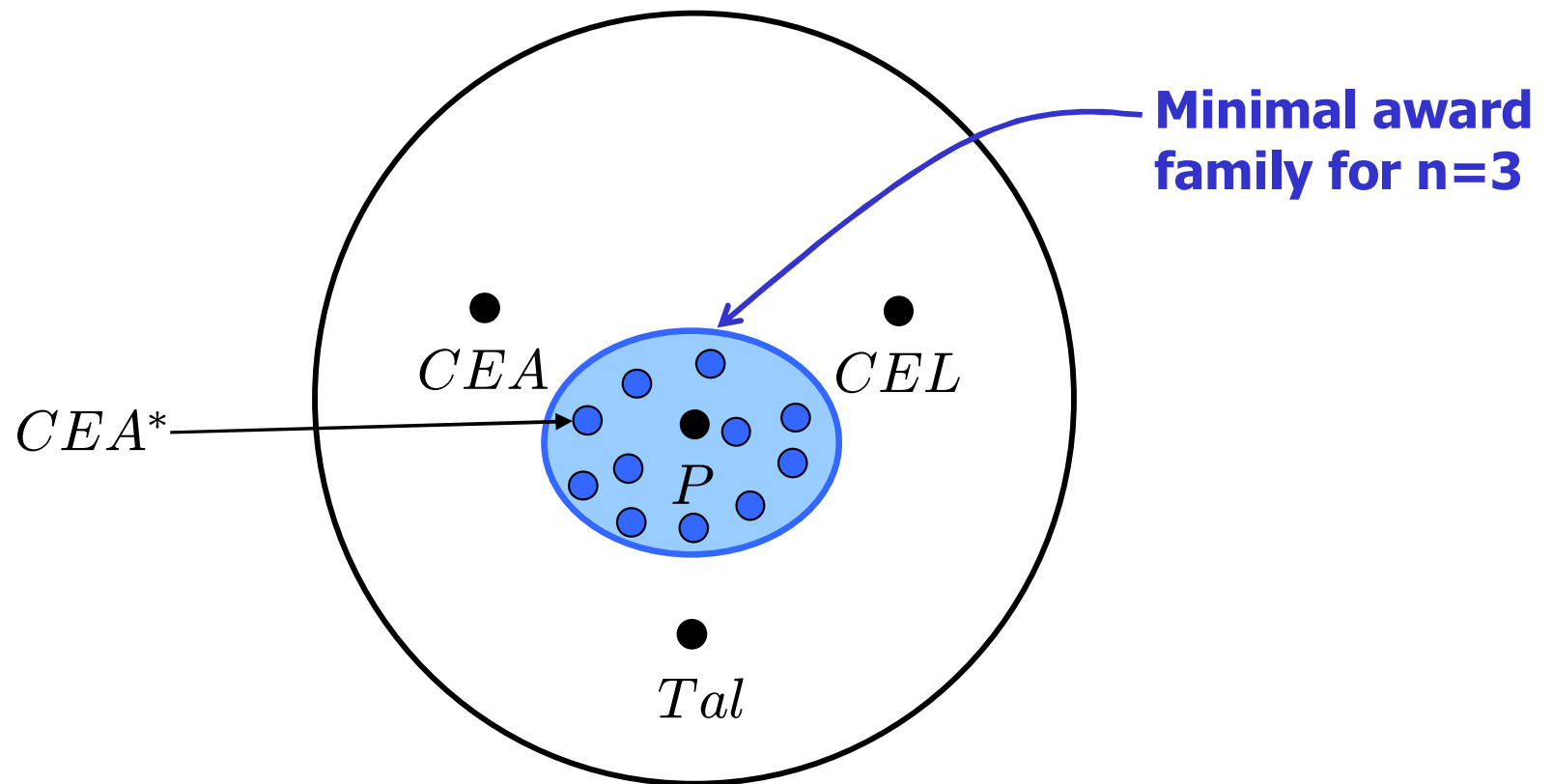
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# Results ( $n=3$ )

**Proposition 1:** Assume  $n = 3$ .

For each  $m \in \mathcal{M}$ ,  $CEA^* \succeq_L M^m$ ; moreover it is the unique one.



# Results (n=3)

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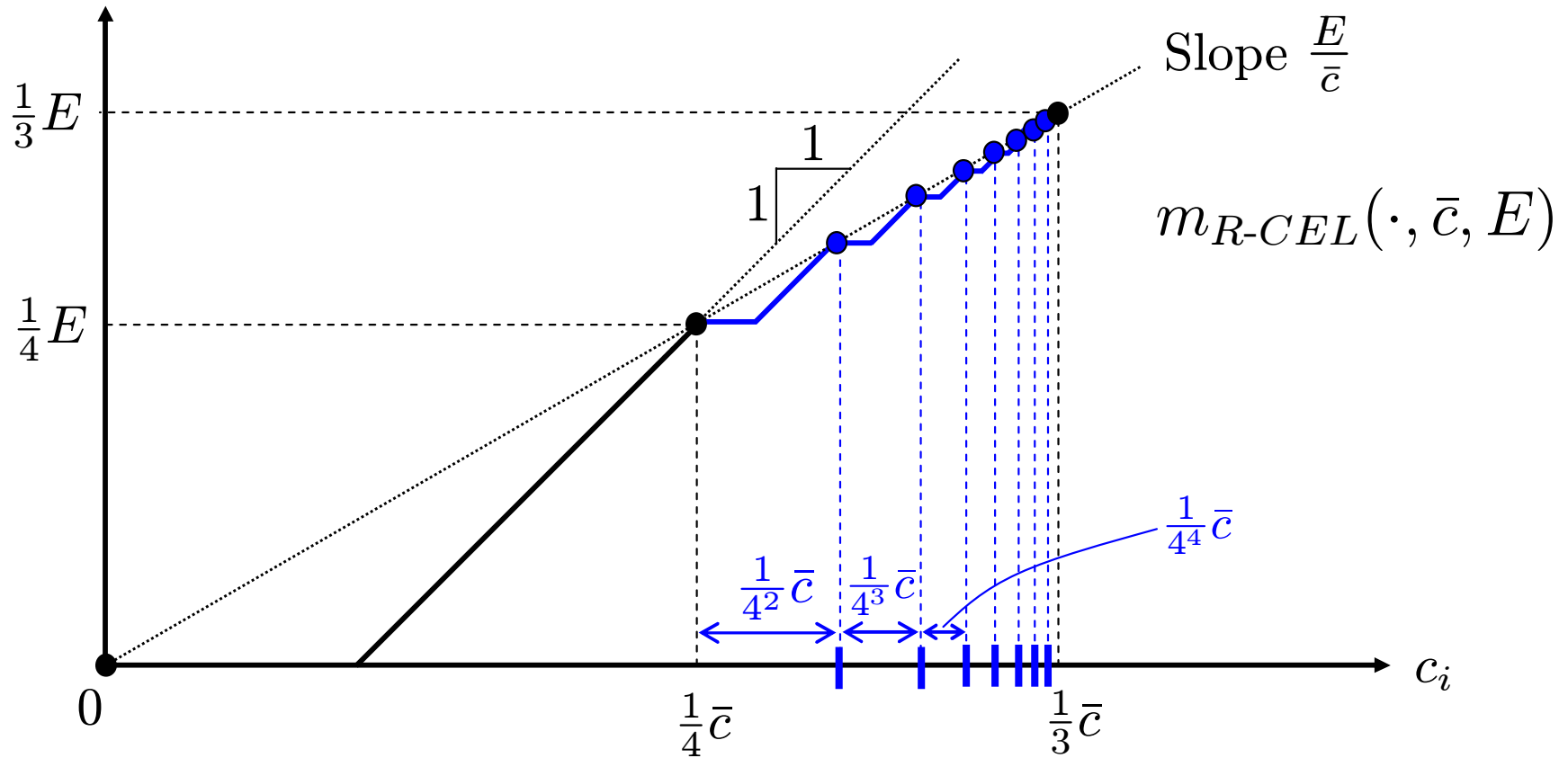
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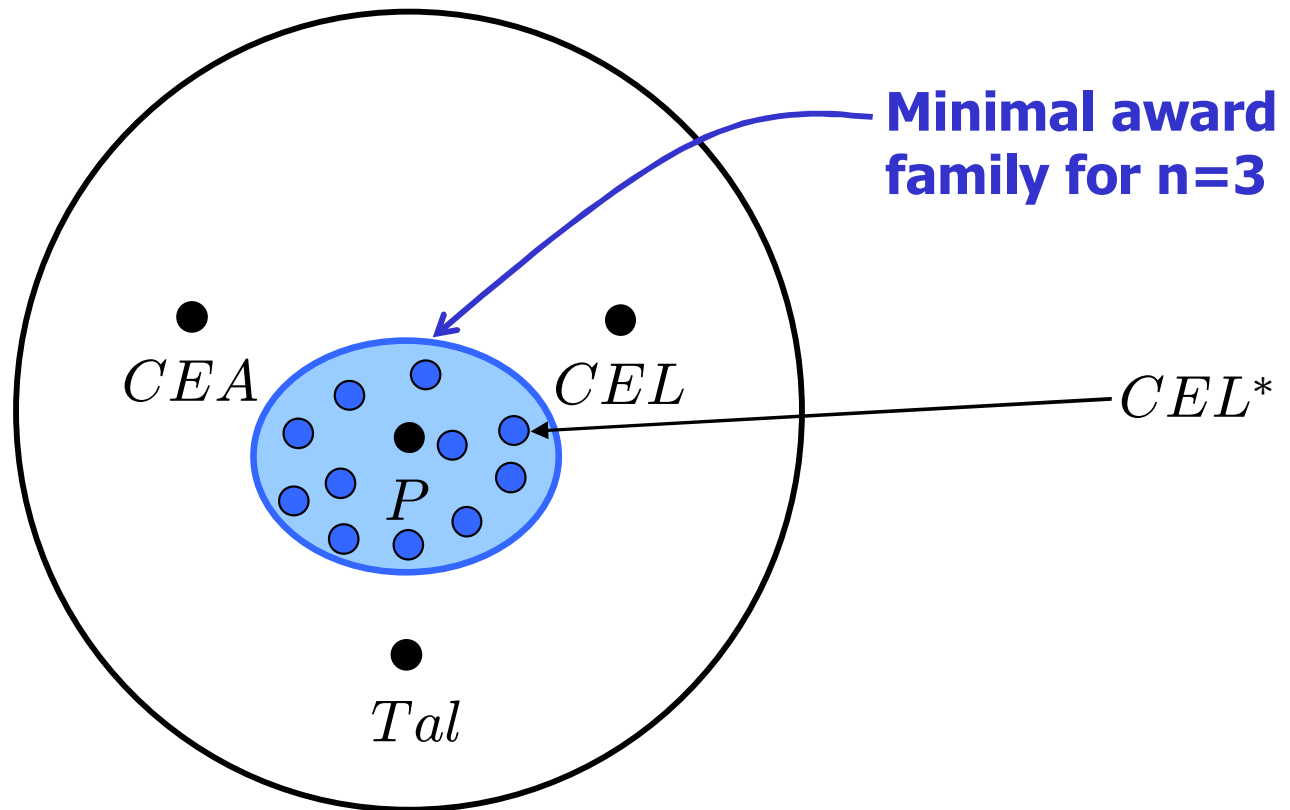
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**Proposition 2:** Assume  $n = 3$ .

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# Results ( $n > 3$ )

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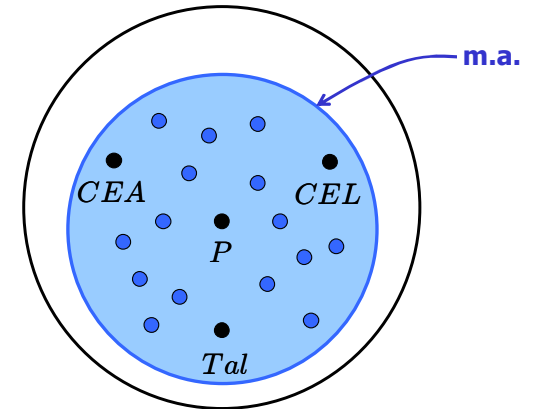
**Theorem 3:** Assume  $n > 3$ .

$$S: \begin{array}{ll} \textit{ord-pres-g} & \textit{ineq-pres-g} \\ \textit{ord-pres-l} & \textit{ineq-pres-l} \end{array} \iff S = \textit{Proportional rule}.$$

# Overview of results

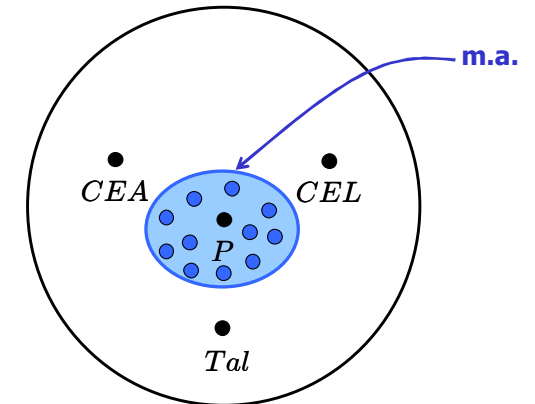
- $n = 2$ :

$S$ :  $cont$ ,  $ord-pres-g$ ,  $ineq-pres-g$   
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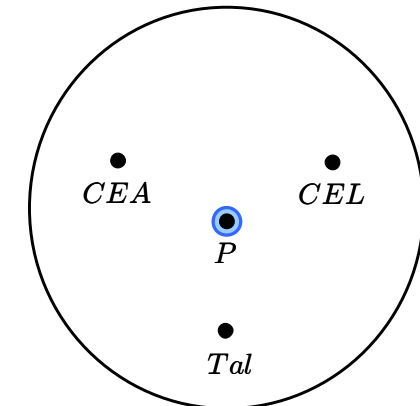
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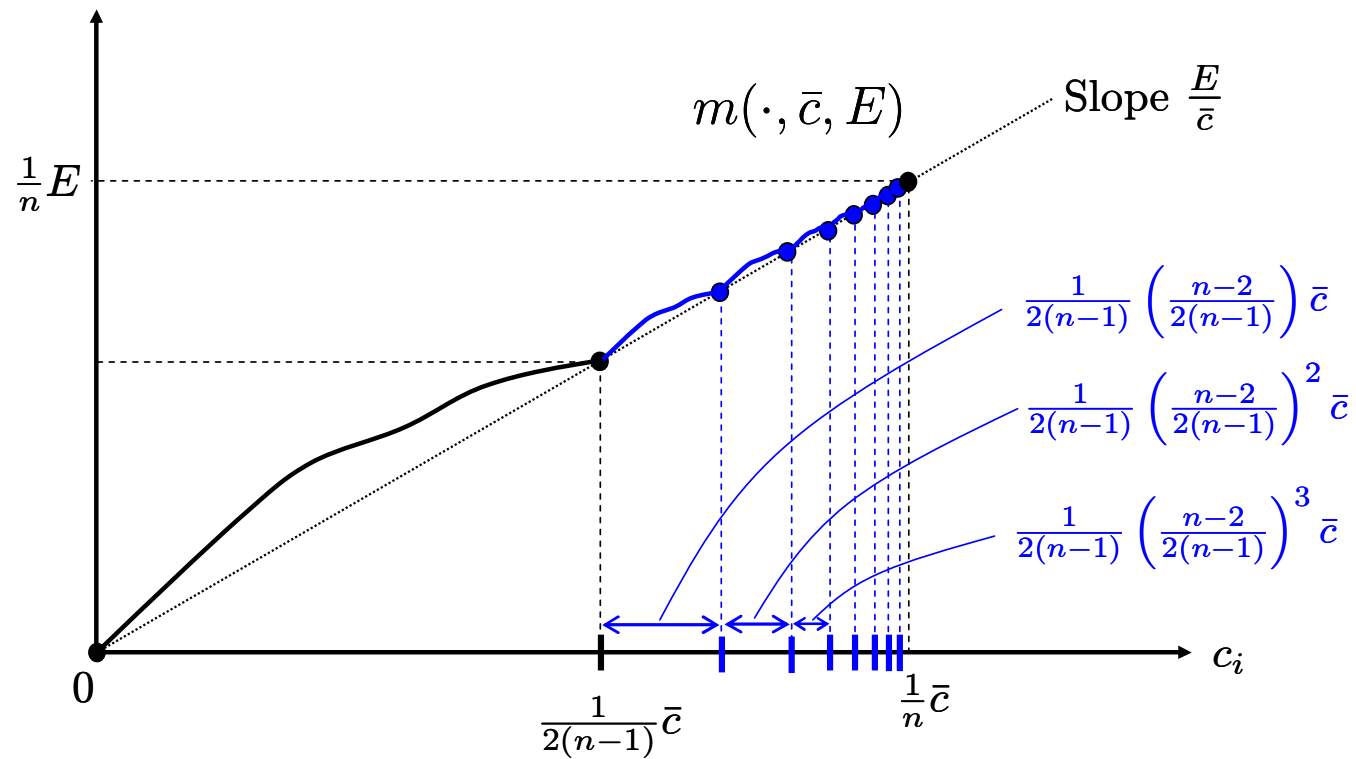


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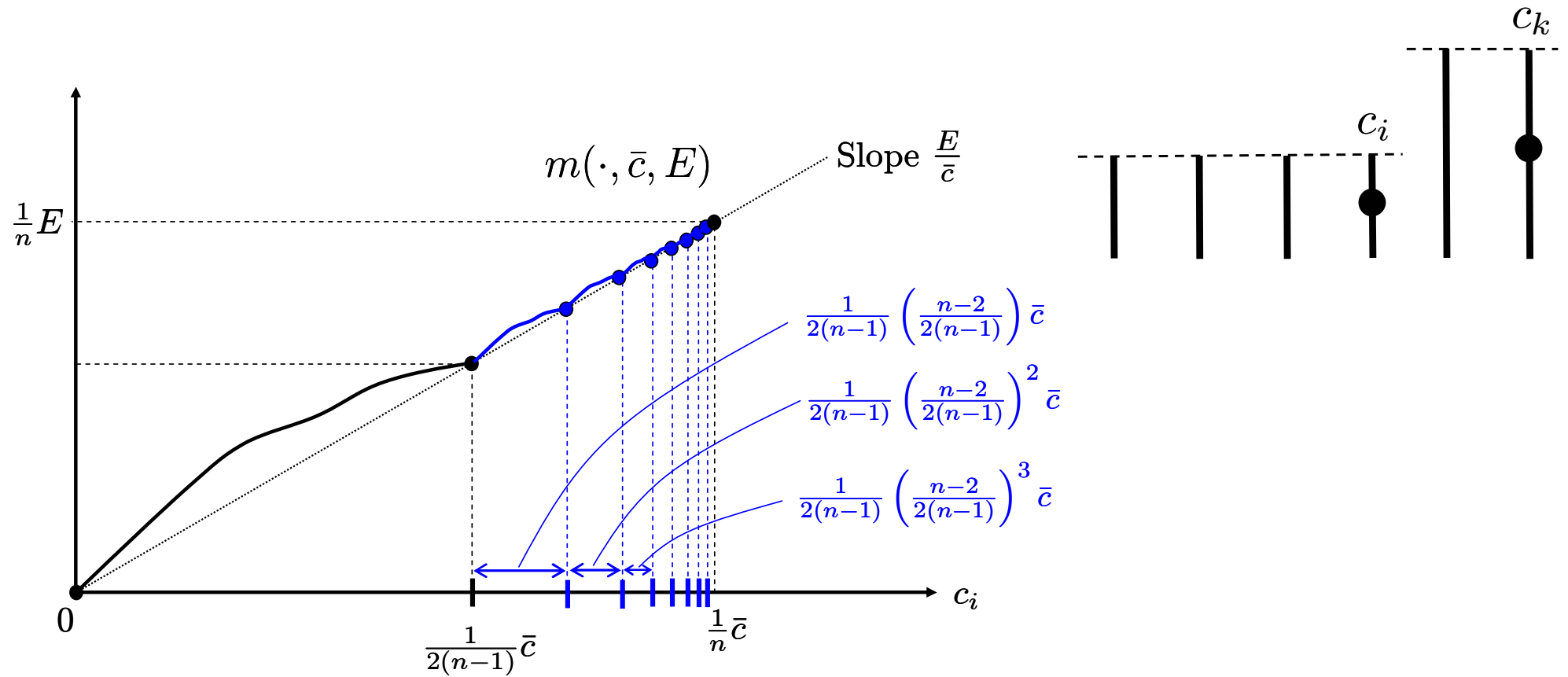
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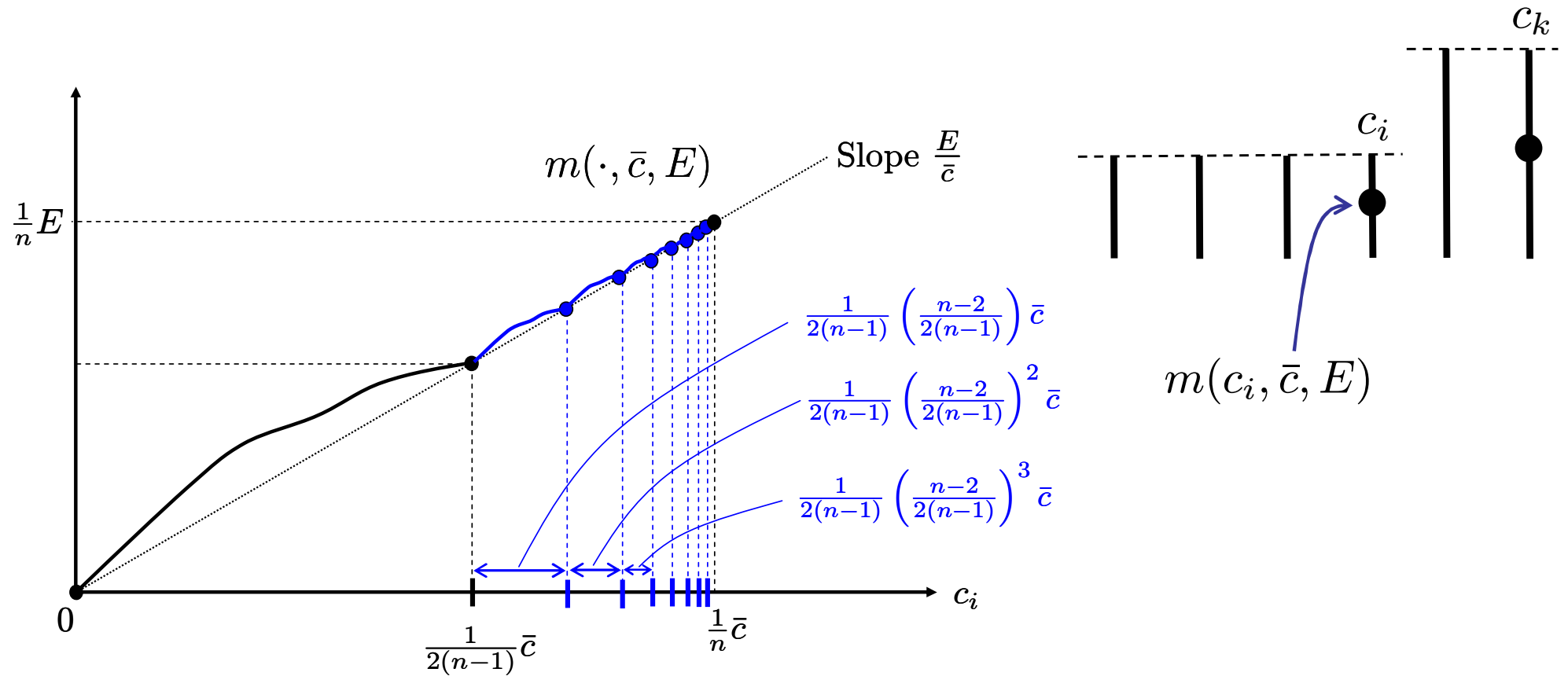
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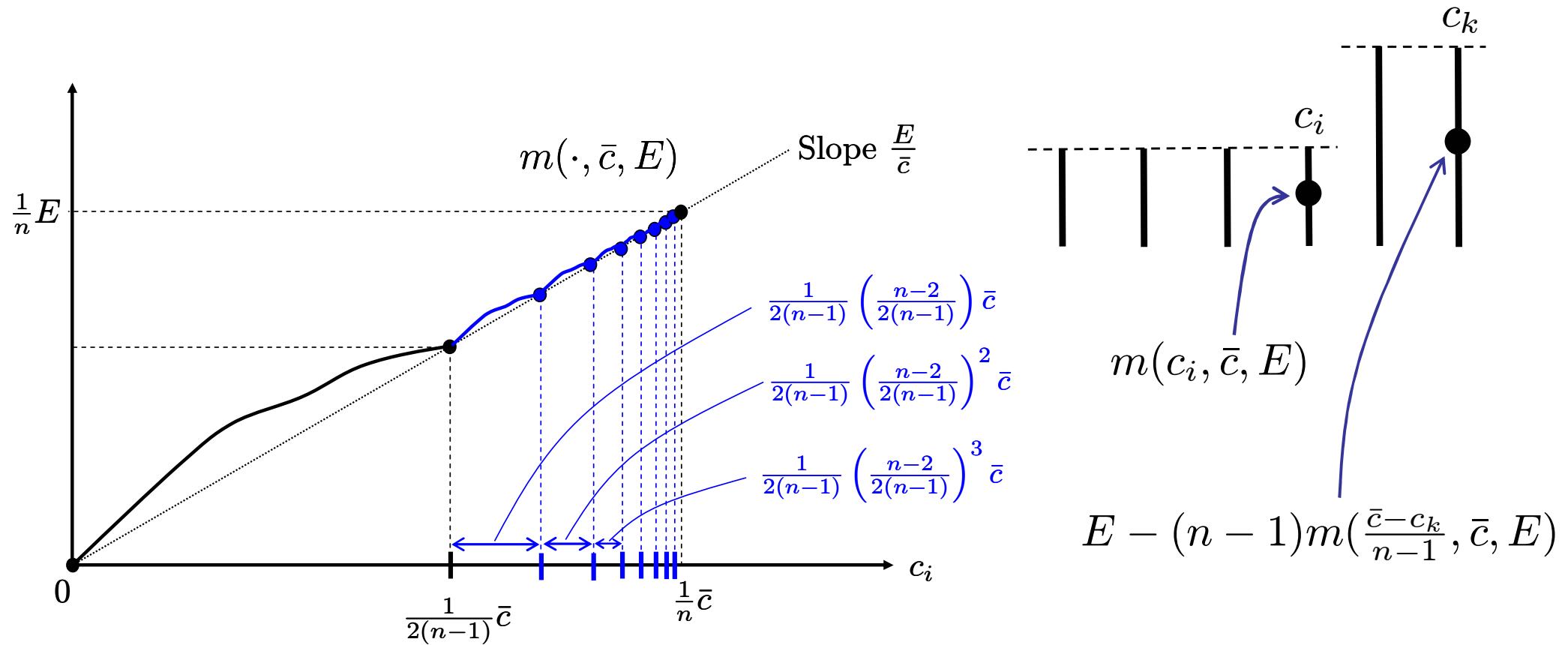
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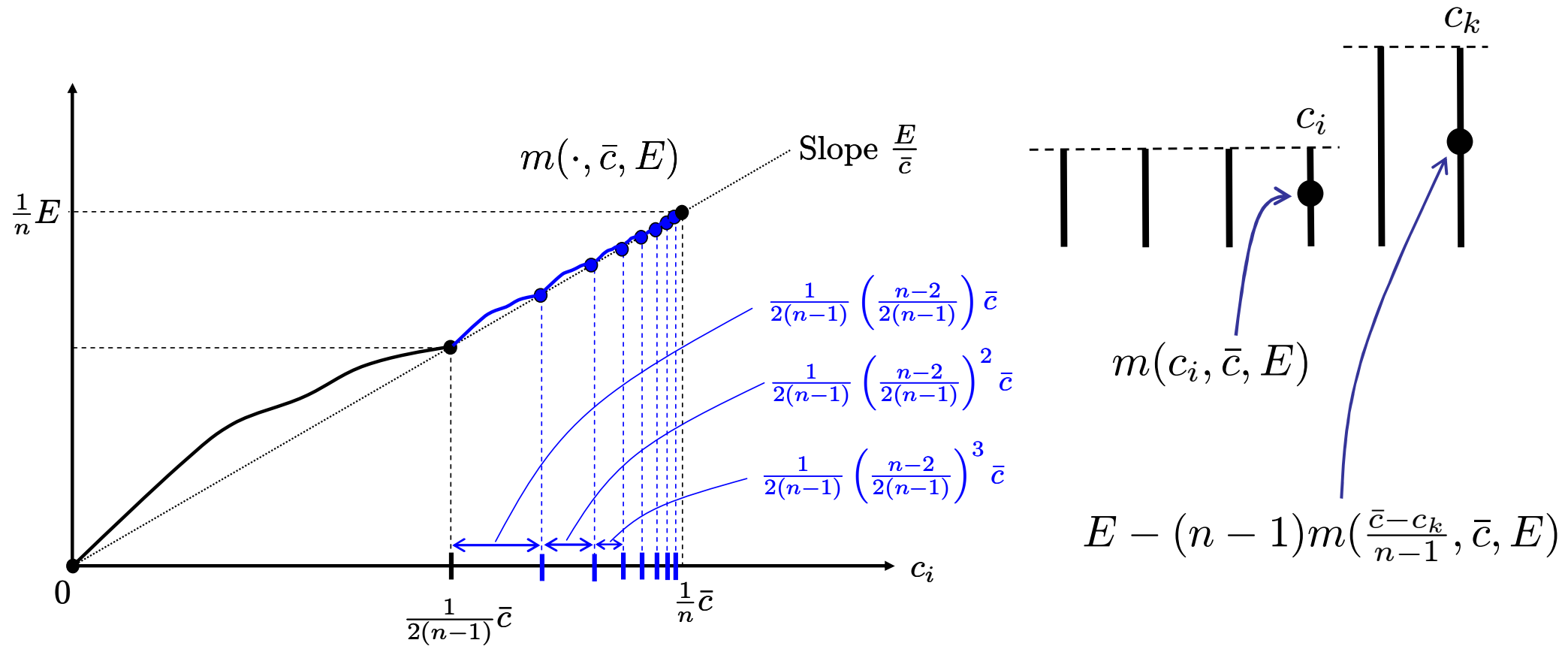
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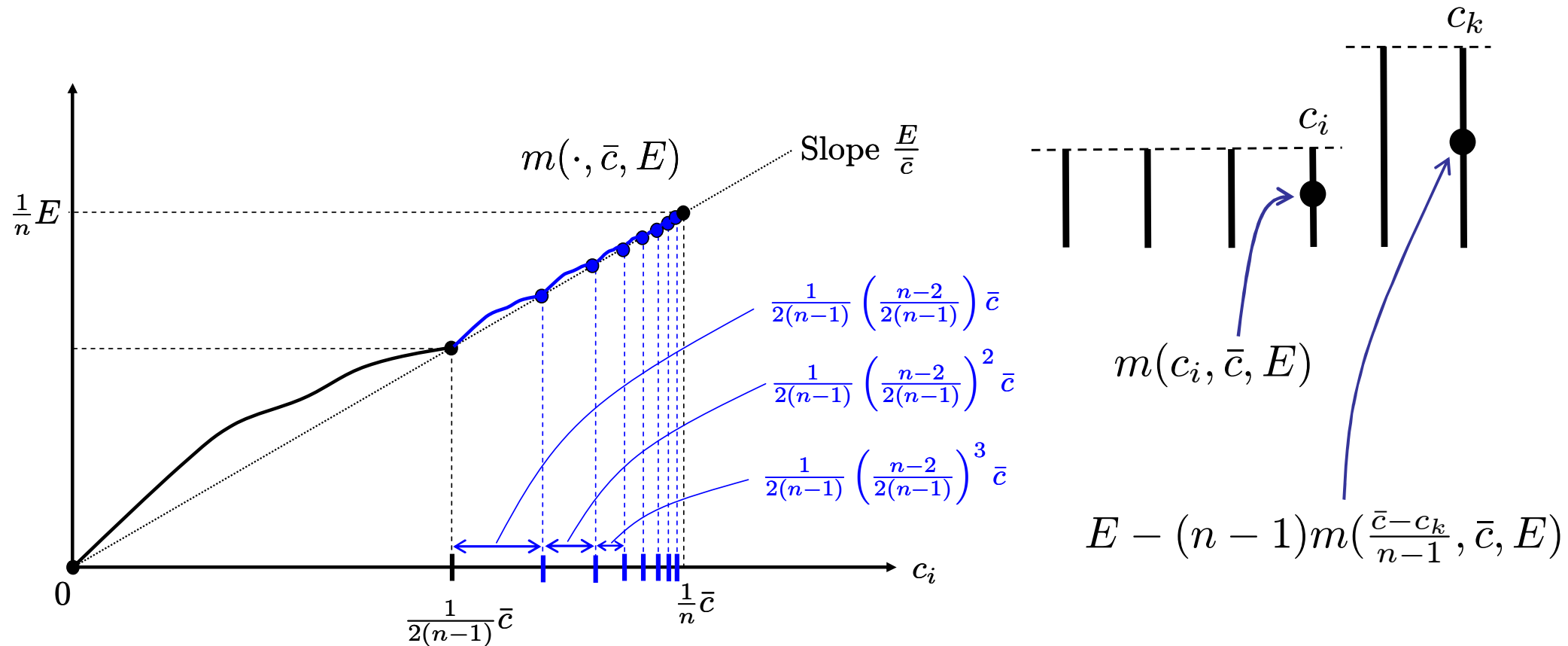


# Results ( $n > 3$ )



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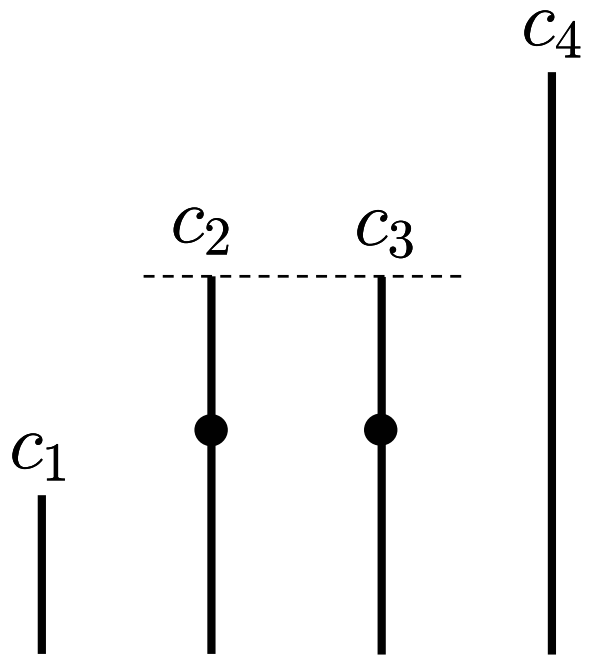


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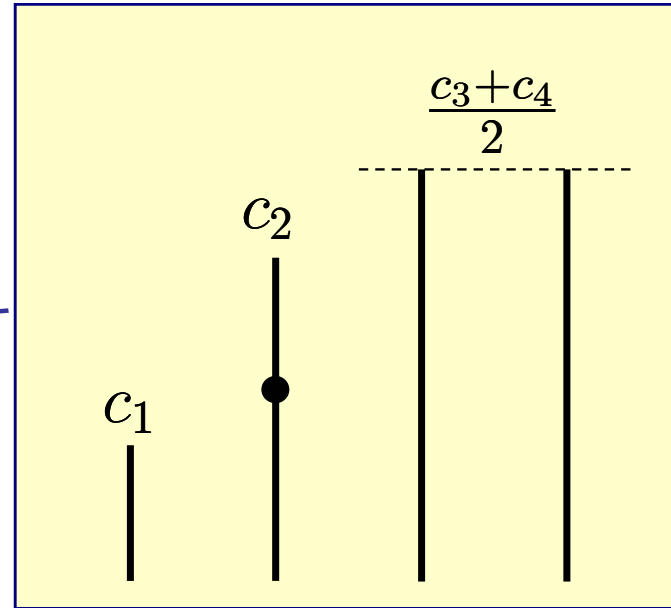
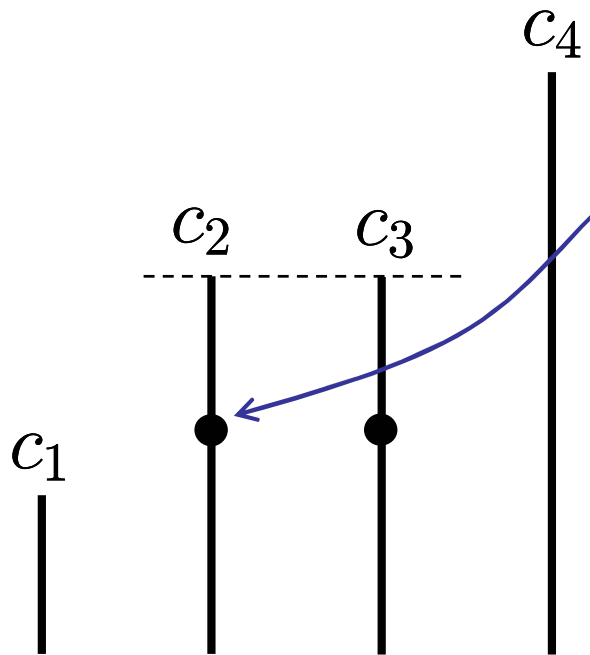
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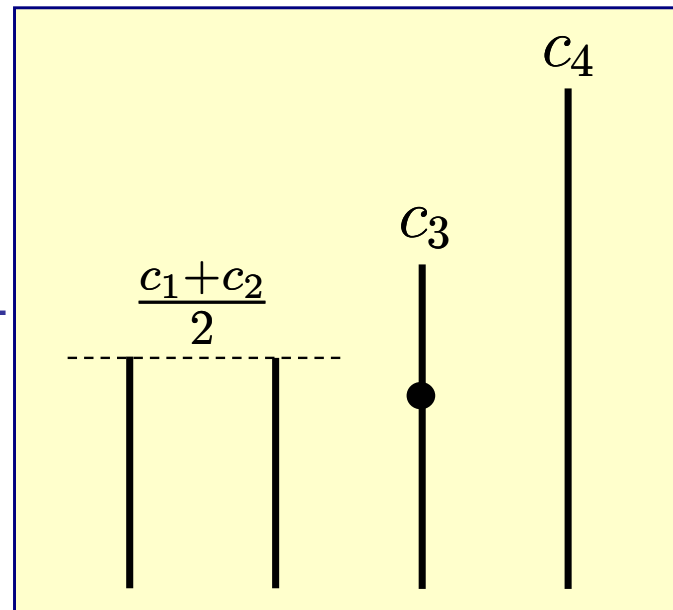
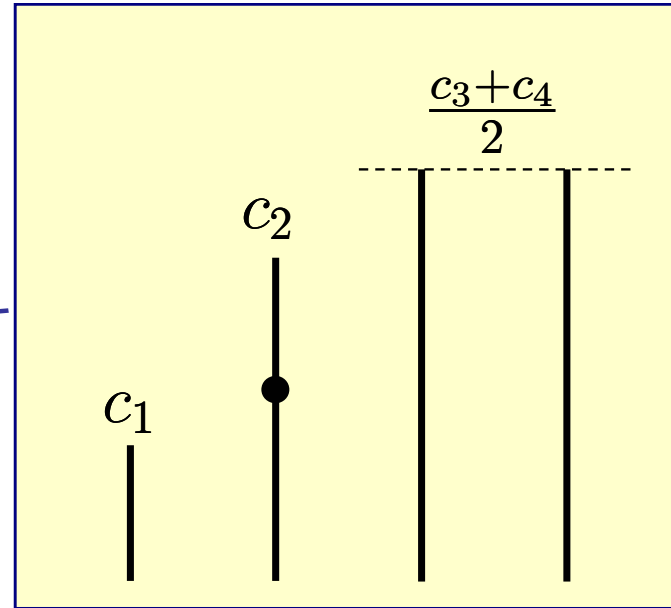
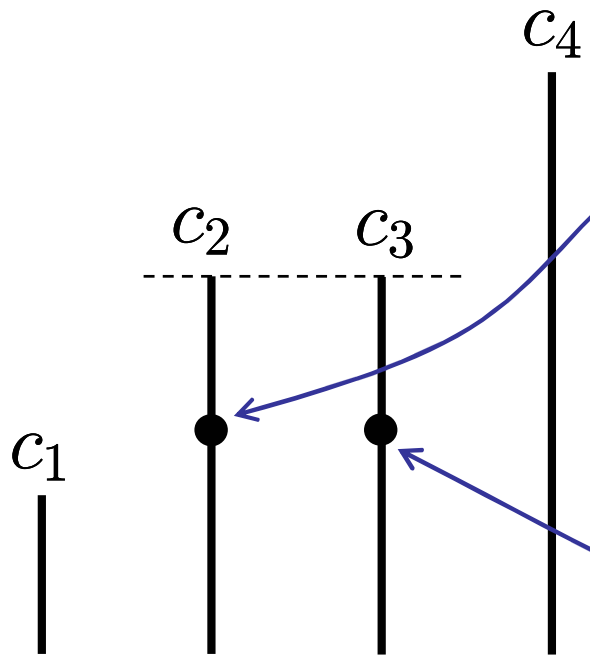
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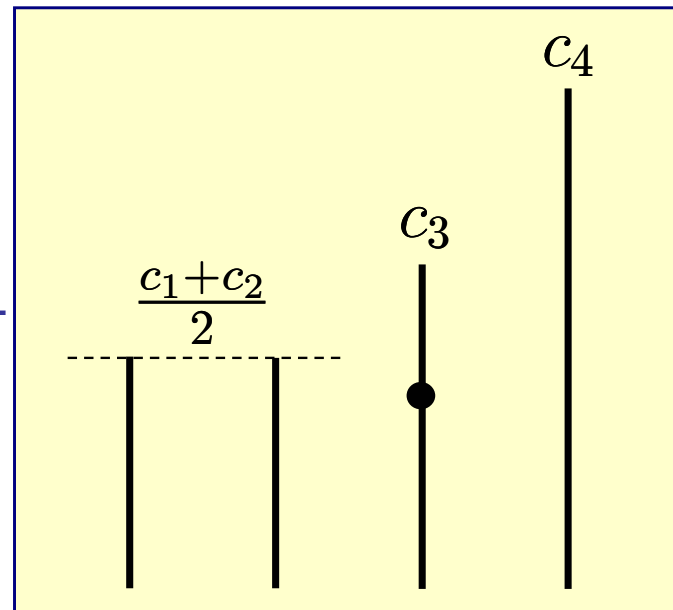
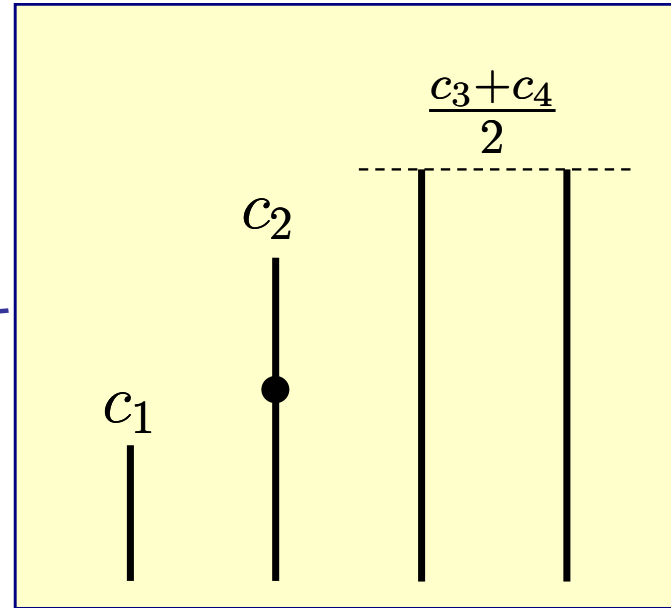
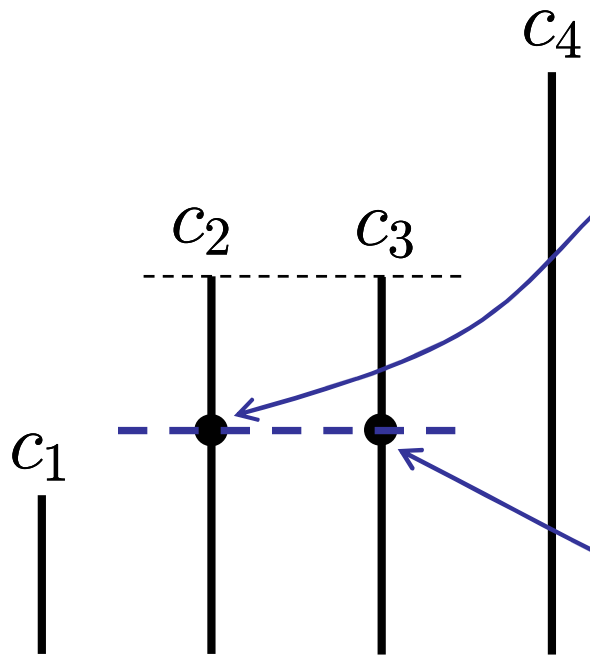
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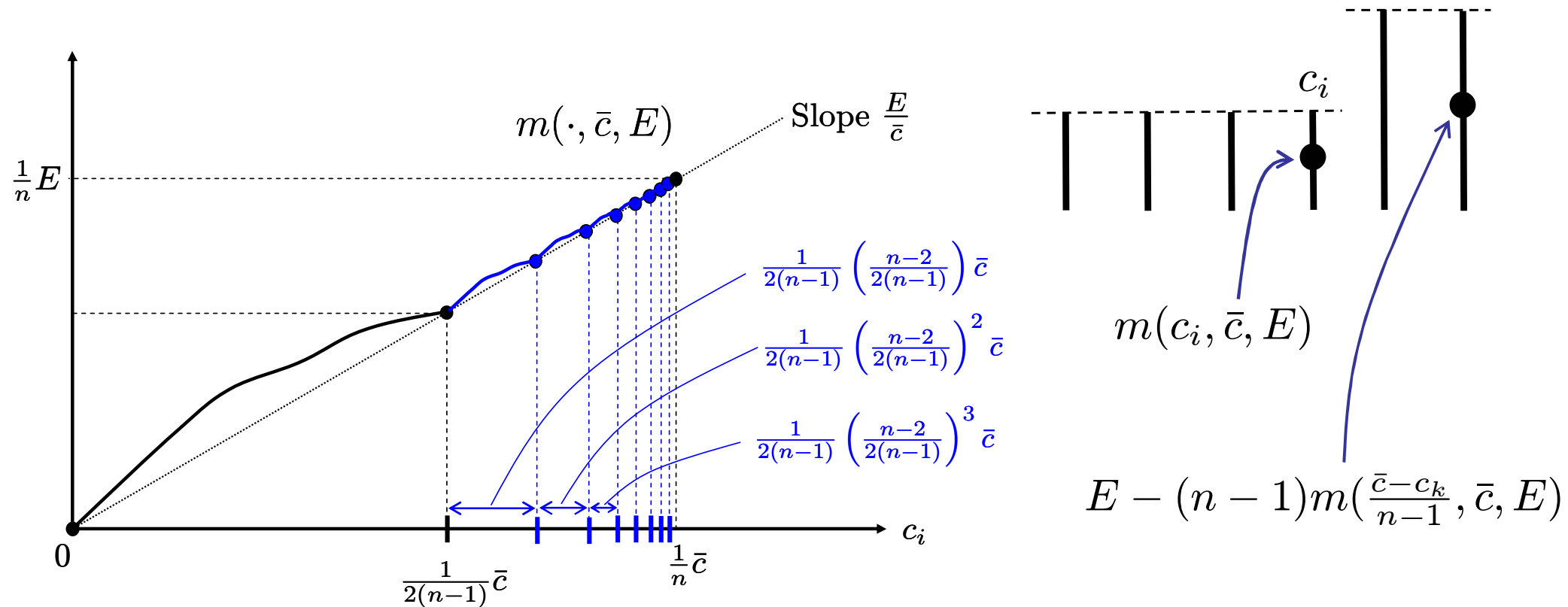
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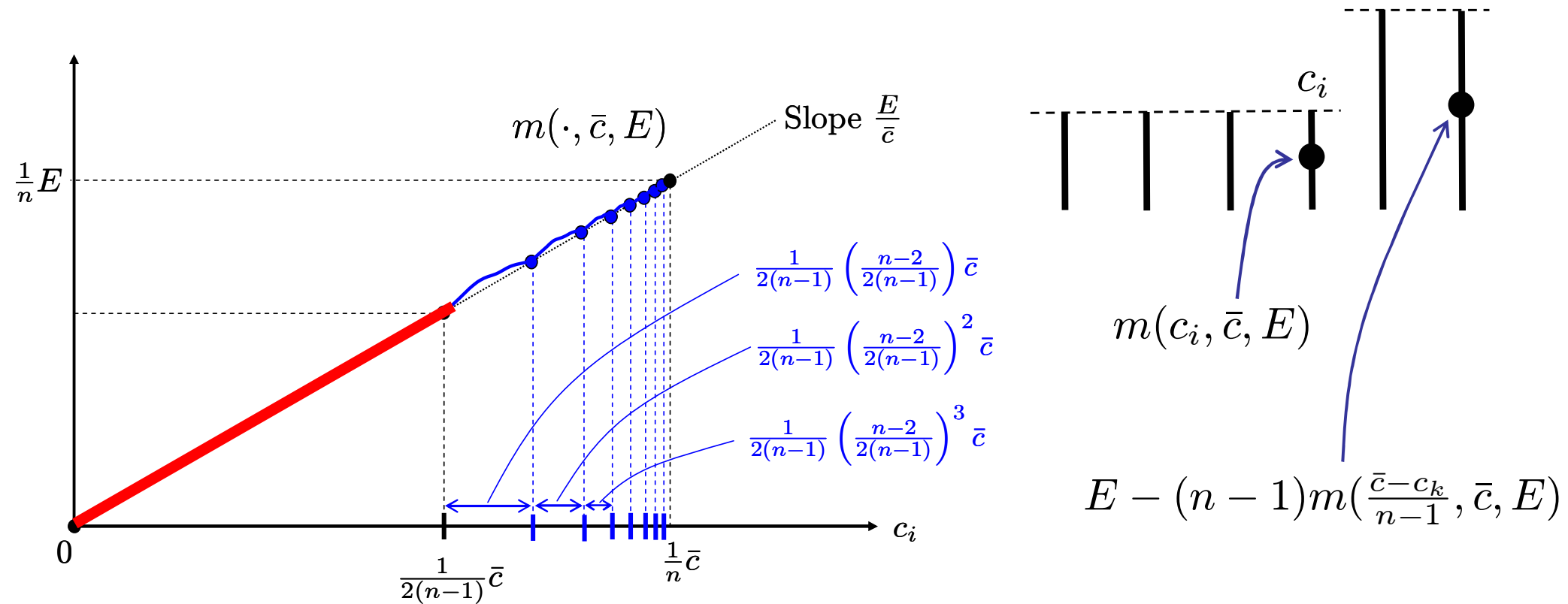
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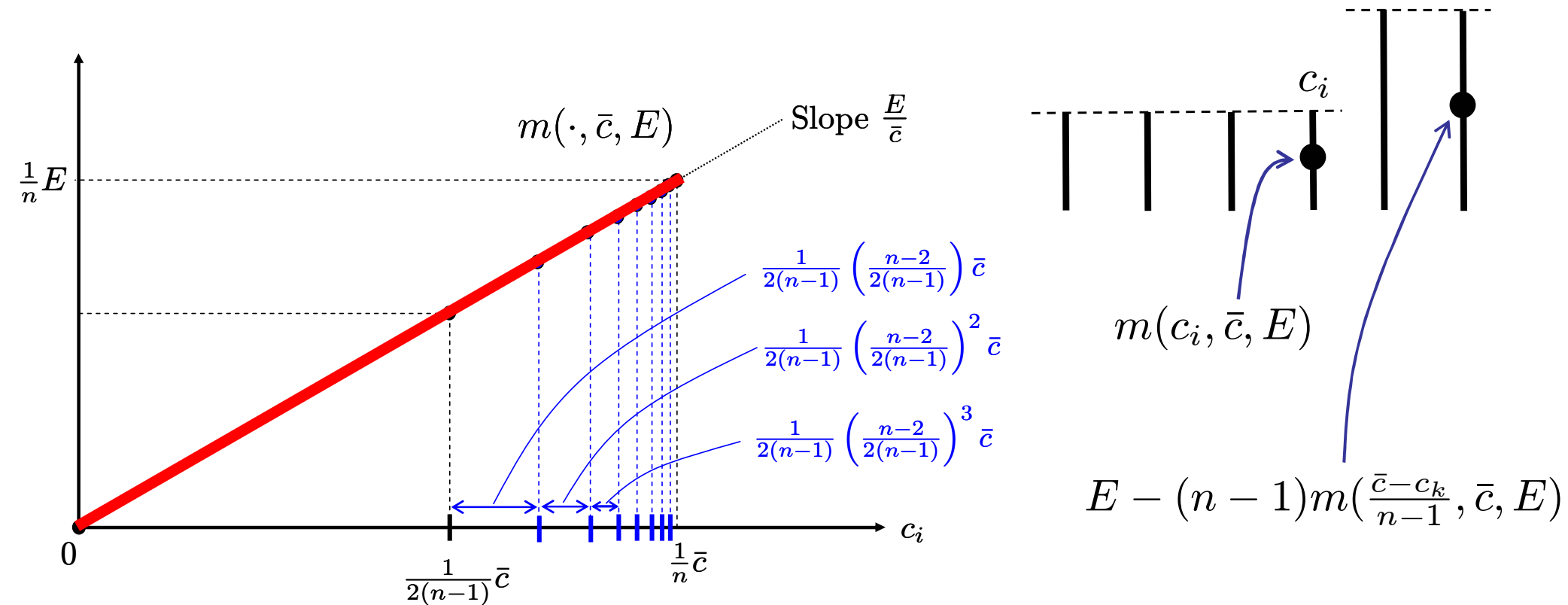
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# Additional remark

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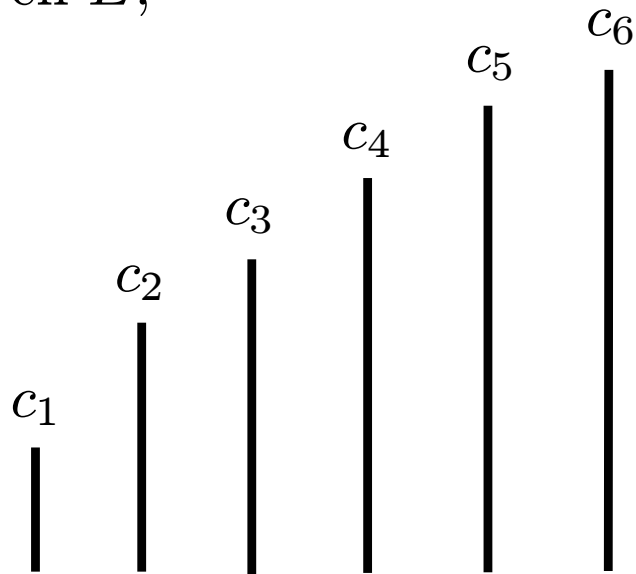
- No advantage transfer (*no-ad-trans*)

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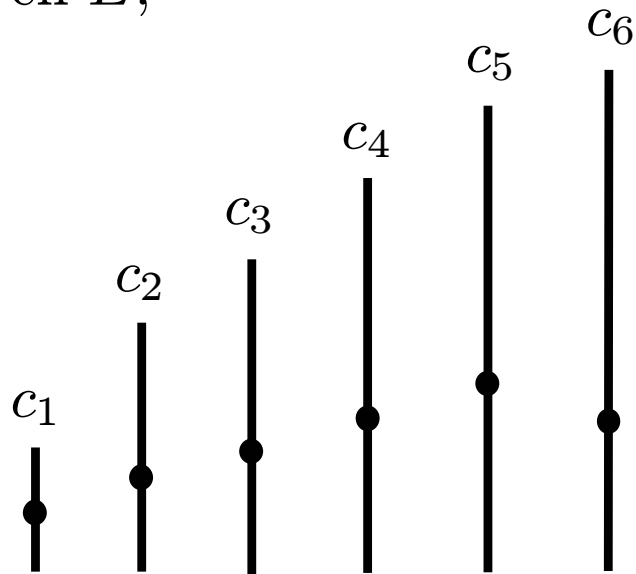


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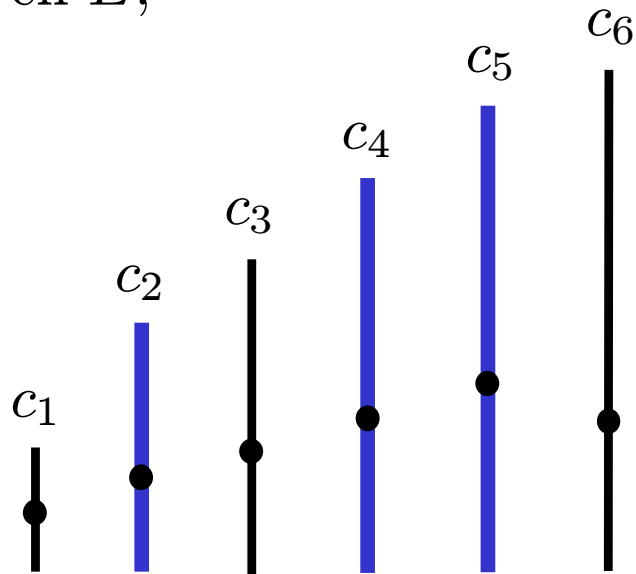


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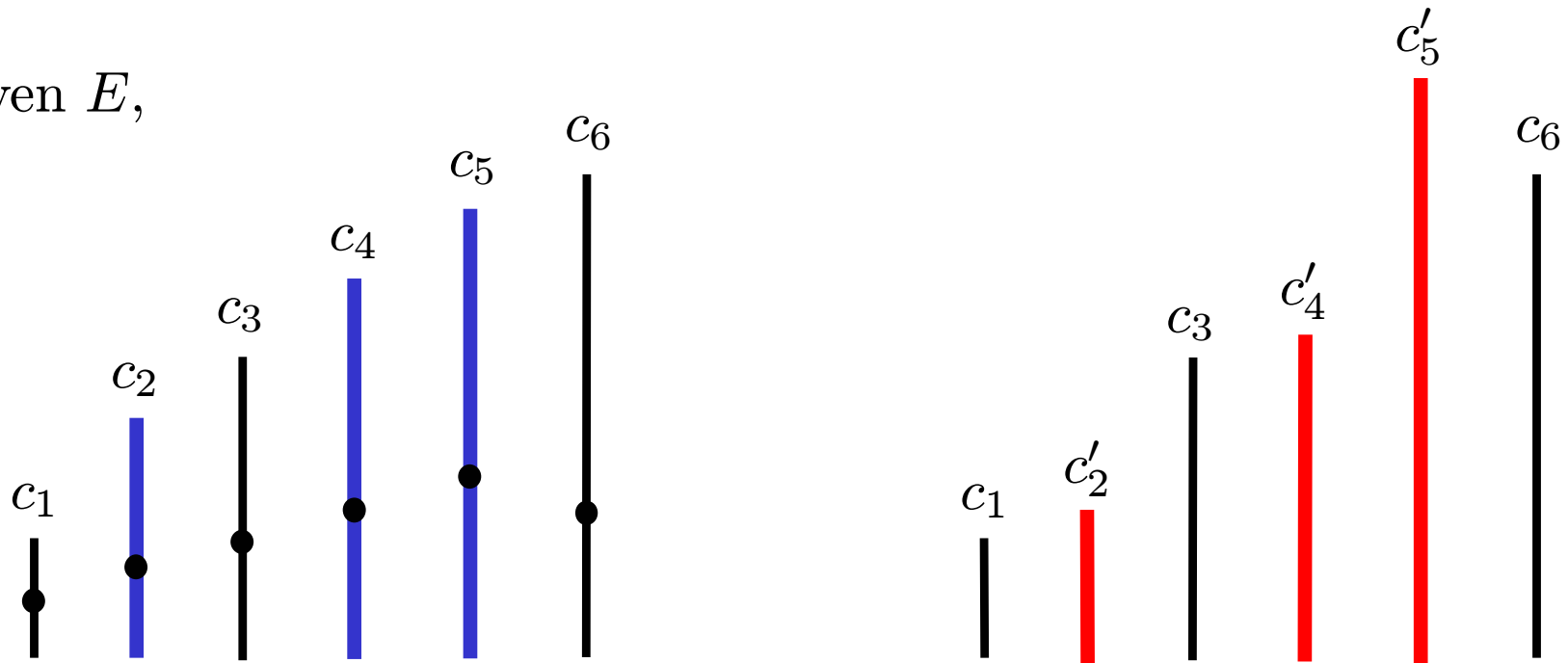


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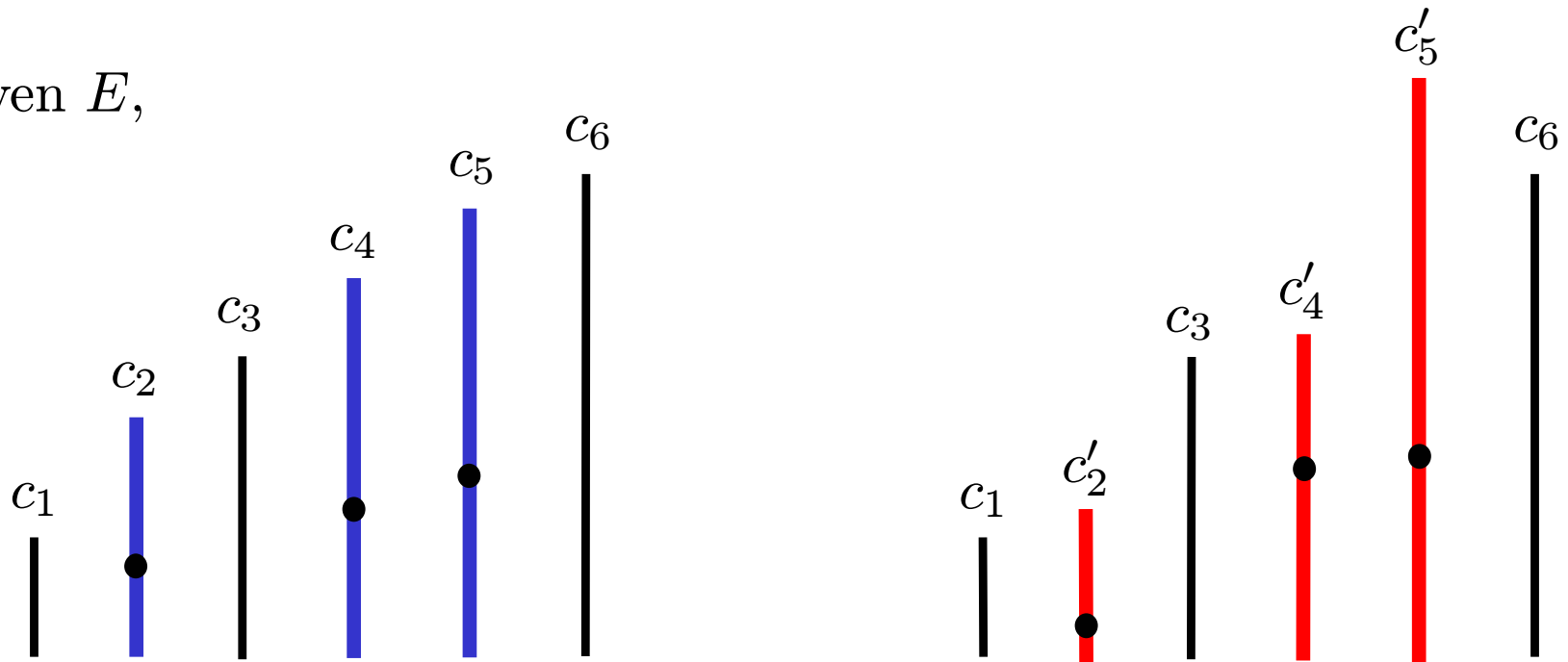


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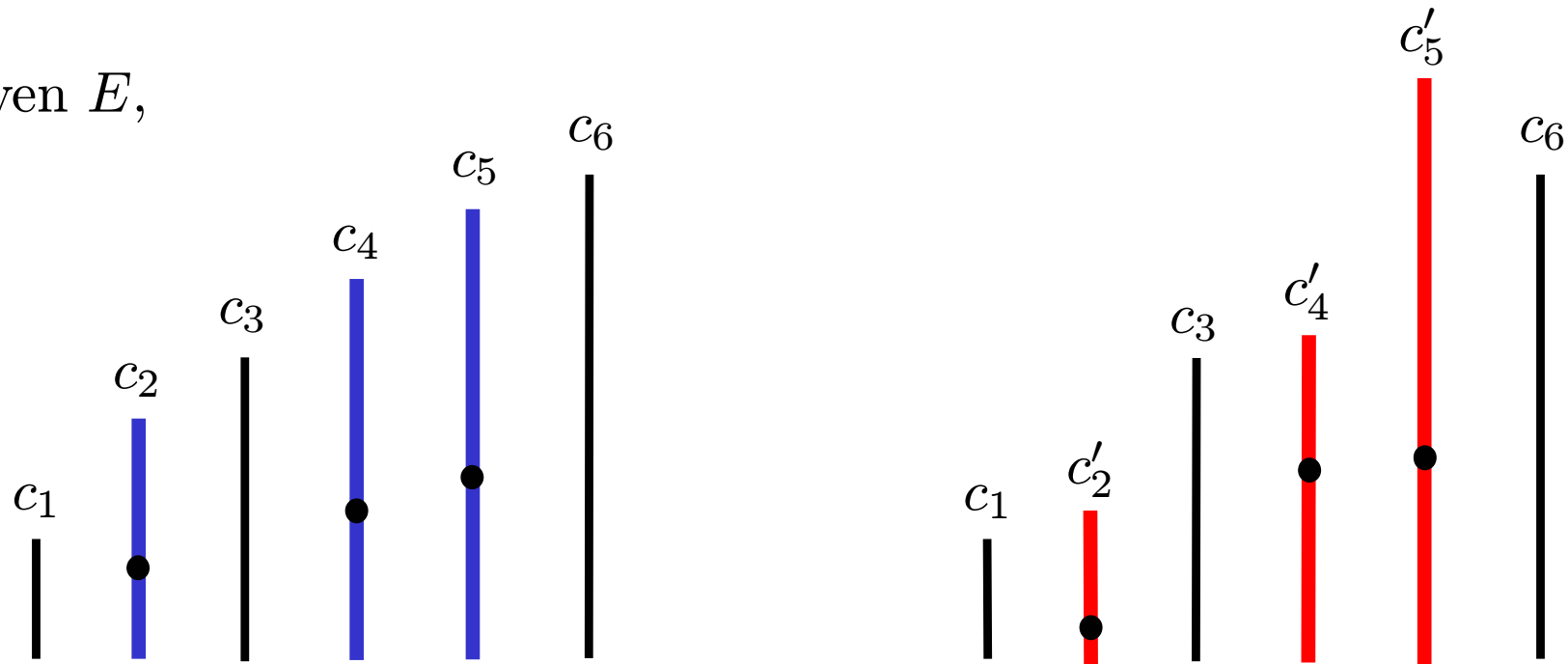
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**Theorem** (Moulin, 1985; Chun, 1988; Ju et al., 2007) :

Assume  $n > 2$ .

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# Additional remark

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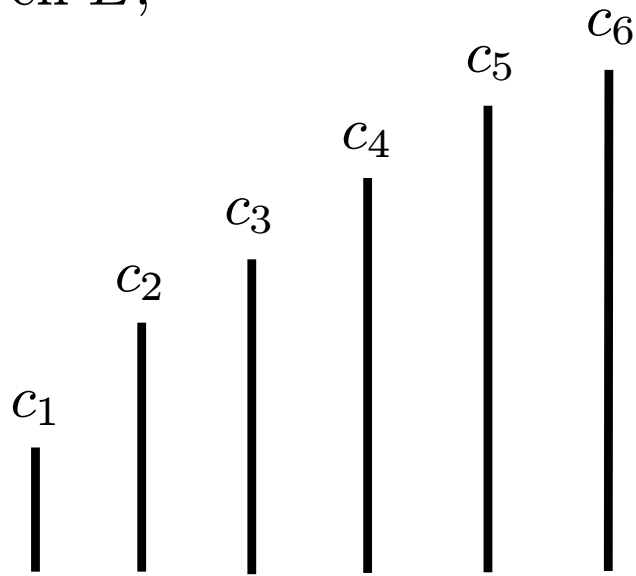
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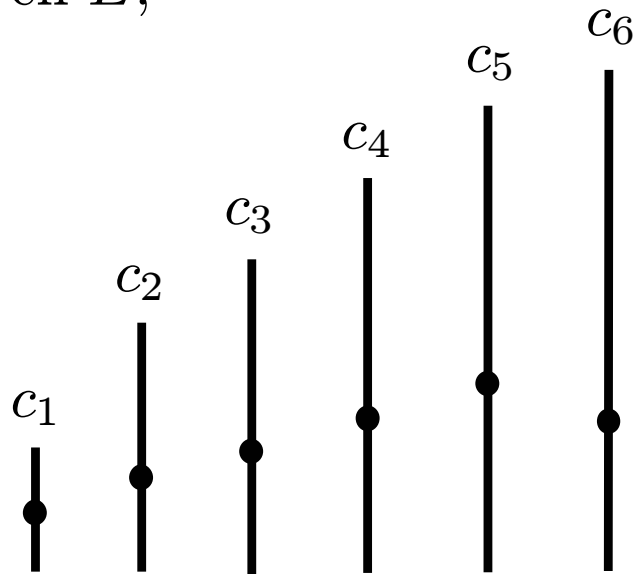


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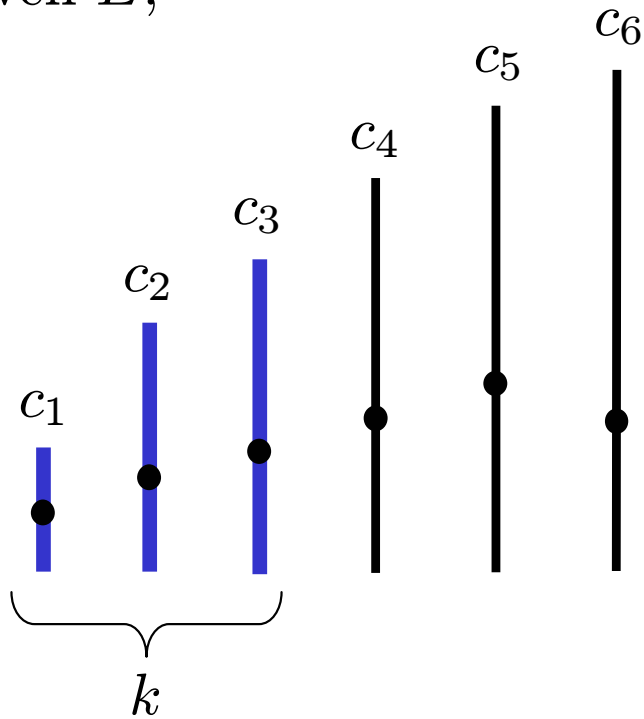


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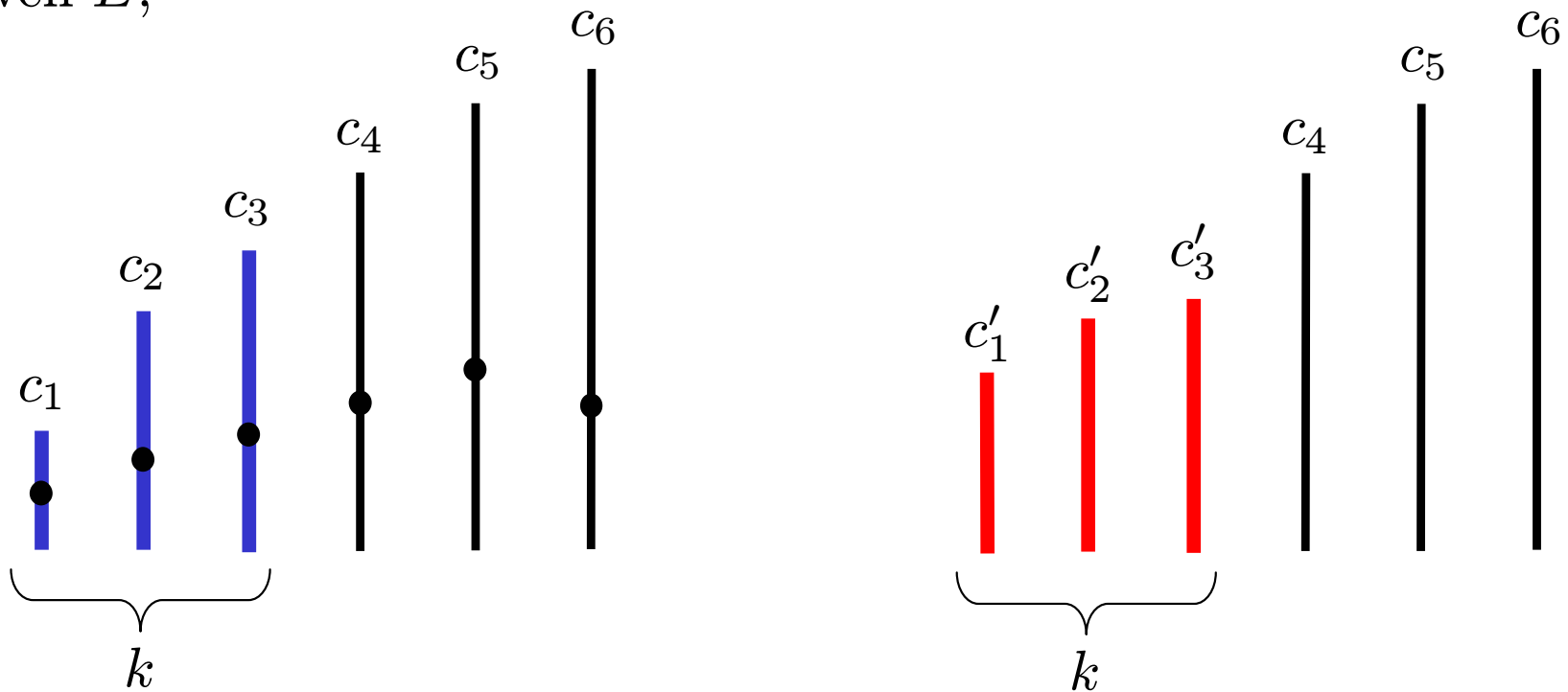
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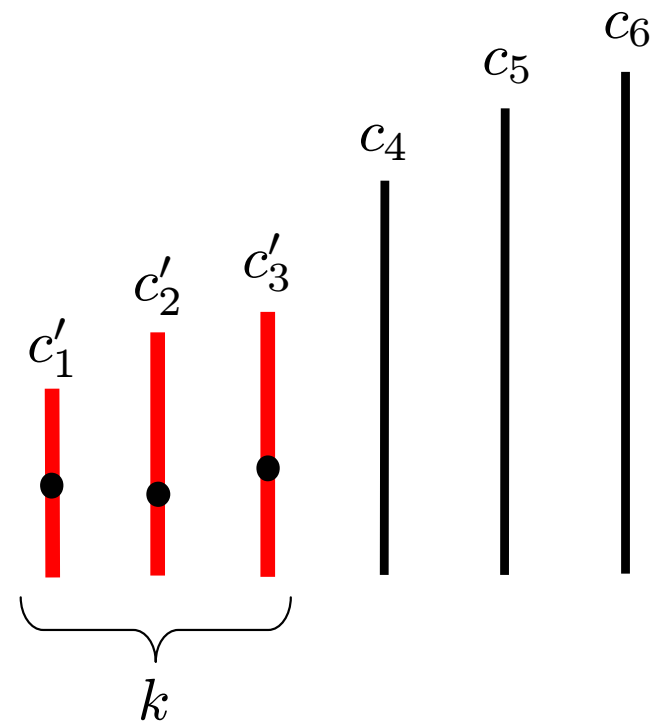
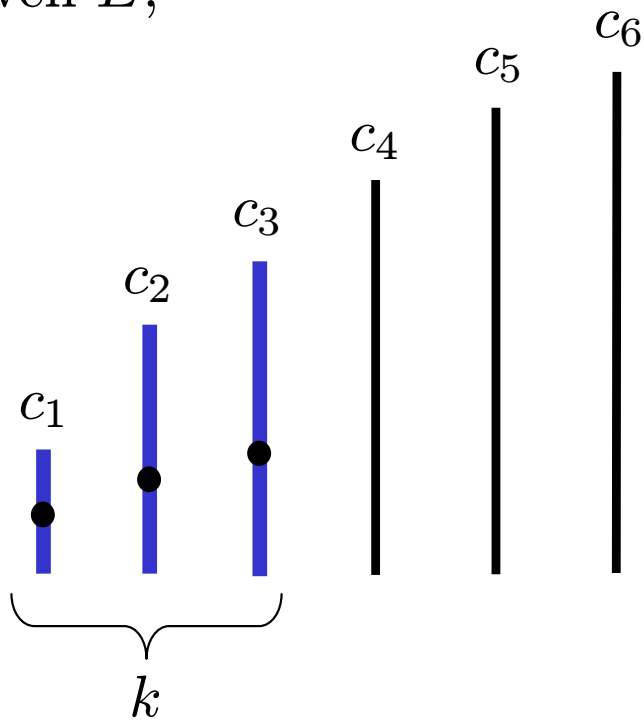


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- Weak no advantage transfer (*weak no-ad-trans*)

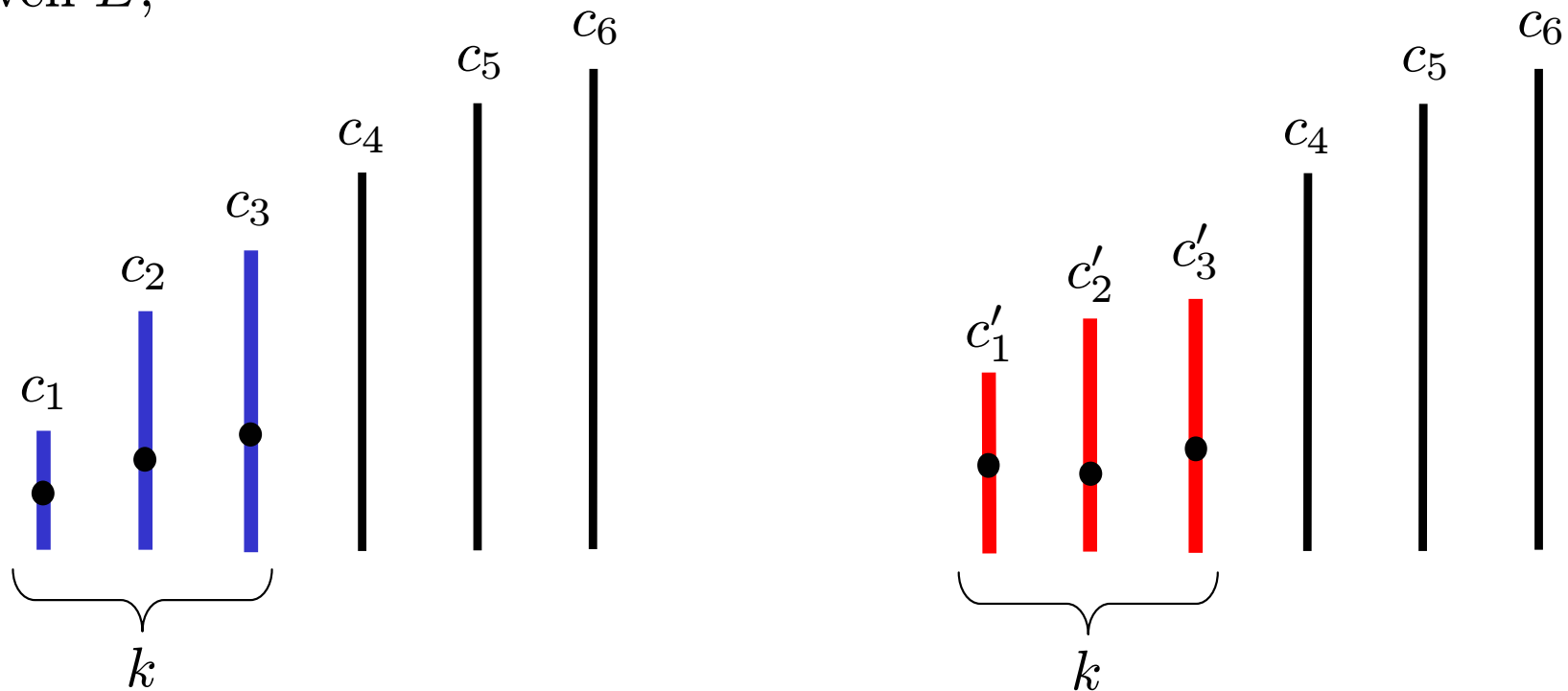
Given  $E$ ,



# Additional remark

- Weak no advantage transfer (*weak no-ad-trans*)

Given  $E$ ,



**Proposition 3 :** Assume  $n > 3$ .

$S$ : *cont, e-t-e, weak no-ad-trans*  $\iff S =$  *Proportional rule*.

# Related literature

**Claims problems:  $(c, E) \in \mathbb{R}_+^N \times \mathbb{R}_+$**

- O'Neill (1982)
- Aumann and Maschler (1985)
- Young (1987)

**<Distributional properties>**

- Hougaard and Thorlund-Petersen (2001)
- Hougaard and Østerdal (2005)
- Moreno-Tertero and Villar (2006a, 2006b)
- Ju and Moreno-Tertero (2006, 2008)
- Bosmans and Lauwers (2007)
- Thomson (2007)

**<Immunity of manipulation properties>**

- Moulin (1985)
- Chun (1988)
- Ju et al. (2007)
- Ju (2007)

**Income transformation problems:  $c \in \mathbb{R}_+^N$**

**<Distributional properties>**

- Moyes (1989, 1994)
- Arnold (1990)
- Ebert (2004)

# Summary

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# Summary

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- $S$  satisfying  $cont$ ,  $ord-pres-g$ ,  $ineq-pres-g$ ,  
 $ord-pres-l$ ,  $ineq-pres-l$

# Summary

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- $S$  satisfying  $cont$ ,  $ord-pres-g$ ,  $ord-pres-l$ ,  $ineq-pres-g$ ,  $ineq-pres-l$ 
  - $n = 2$ : Minimal award family (for the two-agent case).
  - $n = 3$ : Minimal award family (for the three-agent case).
  - $n > 3$ : Proportional rule (even without *continuity*).

# Summary

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- $S$  satisfying  $cont$ ,  $ord-pres-g$ ,  $ord-pres-l$ ,  $ineq-pres-g$ ,  $ineq-pres-l$ 
  - $n = 2$ : Minimal award family (for the two-agent case).
  - $n = 3$ : Minimal award family (for the three-agent case).
    - Unique Lorenz maximal rule in the family,  $CEA^*$
    - Unique Lorenz minimal rule in the family,  $CEL^*$
  - $n > 3$ : Proportional rule (even without *continuity*).

# Open questions

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# Open questions

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■  $S : cont + ord-pres-g + ineq-pres-g \iff S = ?$

$S : cont + ord-pres-l + ineq-pres-l \iff S = ?$

# Open questions

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■  $S : cont + ord-pres-g + ineq-pres-g \iff S = ?$

$S : cont + ord-pres-l + ineq-pres-l \iff S = ?$

■  $S : ineq-pres-g + other\ axioms \iff S = ?$

$S : ineq-pres-l + other\ axioms \iff S = ?$

# Open questions

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- $S : cont + ord-pres-g + ineq-pres-g \iff S = ?$   
 $S : cont + ord-pres-l + ineq-pres-l \iff S = ?$
- $S : ineq-pres-g + other\ axioms \iff S = ?$   
 $S : ineq-pres-l + other\ axioms \iff S = ?$
- Apply to other models.

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**Thank you for your attention.**