Fairness and Externalities

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Abstract

This paper develops a model of fair allocation of a social endowment of indivisible goods and money (e.g. tasks and salary, rooms and rent) in which preferences may exhibit consumption externalities. We extend Foley's fairness test to this environment and under general assumptions on preferences we show that: (i) for each budget, fair allocations exist; (ii) for each common lower bound on individual consumption of money (e.g. minimum salary constraint), if the budget is large enough, then fair allocations meeting the lower bound exist; and (iii) if a balanced budget fair allocation satisfies a condition we call "non-wastefulness," then the allocation is efficient. We provide a range of applications that accommodate prominent consumption externalities.

JEL classification: D61; D62; D63; D64; D70.

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1 Introduction

We study the fair allocation of a social endowment of indivisible goods, or "objects," and money (e.g. tasks and salary, rooms and rent) when preferences may exhibit consumption externalities. We generalize Foley's (1967) fairness test, also known as the no-envy test (Varian, 1974), to this environment. Then, we investigate the existence of fair allocations and the compatibility of fairness and efficiency on a

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domain of preferences defined by means of three basic axioms. Our main theorems are possibility results that apply to broad classes of economies. Moreover, all of the central results in the literature on fair allocation of objects and money when externalities are absent are corollaries of our theorems.

Fair and efficient resource allocation is a central theme of normative economics. In a seminal contribution, Foley (1967) developed an ordinal and operational test of fairness: "ask each person to imagine changing places with every other[...]If no one is willing to change, the allocation is equitable."[1] An extensive literature has investigated the existence of allocations satisfying this property and its compatibility with efficiency (see Thomson, 2010, for a survey).[2] A common feature of these studies is that each agent's welfare is assumed to be independent of the consumptions of the other agents. This assumption entails a significant loss of generality. A growing literature in experimental economics has provided evidence of human behavior that cannot be rationalized by preferences that do not exhibit consumption externalities, but can be rationalized when this possibility is recognized (see Fehr and Schmidt, 2001, for a survey). These studies –and our everyday observation of altruism, philanthropy, spite, and the unavoidable partially public nature of consumption– motivate our including consumption externalities in the study of fair allocation.

We focus on the allocation of objects and money. There are as many agents as objects and each agent must receive one object.[3] Individual consumptions of money should add up to, at most, a given amount, which we refer to as the "budget." This problem has been central in the study of fair allocation (Svensson, 1983; Maskin, 1987; Alkan et al., 1991). We depart from the assumption, universal in previous literature, that each agent's preferences are defined on her consumption space. We consider instead agents who have preferences on the set of possible allocations.

Our first contribution is the identification of the basic domain of preferences on which we later develop positive results concerning the fair allocation of objects and money when consumption externalities are present. This domain is defined by means of three ordinal axioms.

Our first axiom, the equal budget compensation assumption, intuitively requires that "no object be infinitely better than the others." This axiom has two components. Let b be the consumption of money of an agent, say i, at a given allocation. The axiom requires that if the aggregate consumption of money is large enough and each agent's consumption of money is at least b, then agent i will always prefer to swap her

[1] Kolm (1971) makes reference to an earlier formulation of this test in Tinbergen (1946).

[2] Mathematics literature on fair allocation started prior to Foley (1967), c.f., Dubins and Spanier (1961). This literature focused on the existence of fair allocations for the division of a non-homogeneous continuum and did not analyze the compatibility of fairness and efficiency. See Berliant et al. (1992) for a discussion on this matter.

[3] Section 7 discusses the extension of our results when there is a different number of agents and objects.

consumption with some other agent whose consumption of money is greater than b. The second component of the axiom requires that as the reference consumption of money b goes to minus infinity, the minimum aggregate consumption of money for which the agent exhibits this behavior also goes to minus infinity.

Our second axiom, anonymity of externalities, requires that each agent's welfare not be affected by a permutation of the other agents' assignments.[4]

Our third axiom, continuity, is standard.

In order to facilitate the identification of preferences that belong to our basic domain, we provide a stronger form of the equal budget compensation assumption that is usually satisfied in applications. The requirement is that, ceteris paribus, an agent, say i, would like to swap her consumption with another agent, say j, provided that j's consumption of money is large enough (how large the difference in consumption of money should be may depend on the objects involved in the swapping; see Section 2).

Then, we address three fundamental questions: (i) How should fairness be defined when each agent may care about the consumption of others? (ii) Do fair allocations exist? (iii) Under what conditions are fairness and efficiency compatible?

We answer the first question by extending Foley's fairness test: we require that no agent prefer the allocation obtained by swapping her consumption with that of any other agent, the other agents' consumption being kept constant. By doing this, we retain Foley's original idea of asking each agent to switch places with every other, and thus identifying allocations at which each agent sees herself at the best position in society. To avoid confusion with the multiple alternative tests of fairness that have been proposed in the literature, we refer to such an allocation as being non-contestable on fairness grounds (non-contestable for short).[5]

If there are no consumption externalities, the two basic answers to our second question, i.e., the existence of non-contestable allocations, are: (ii-i) if no lower bound on individual consumption of money is imposed, then for each budget, non-contestable allocations exist (Alkan et al., 1991); and (ii-ii) given a preference profile and a common lower bound on individual consumption of money (e.g., a minimum salary), if the budget is large enough, then non-contestable allocations meeting the lower bound exist (Svensson, 1983; Maskin, 1987; Alkan et al., 1991).

[4] Our main possibility results do not extend to the domain with non-anonymous externalities (Section 7).

[5] "Envy" and "inequity" are commonly understood as consumption externalities that can be captured in our model. Referring to the allocations that pass our test as "fair," "envy-free," or "equitable" could mislead the reader into thinking that these allocations minimize certain externality that the agents internalize in their preferences. We adopt the more neutral terminology of "non-contestability on fairness grounds," which suggests the application of an objective test of fairness (see Kolm, 1971, 1995, for a related discussion; see Rawls, 1972, for a discussion on why distributive justice is not the absence of envy).

What should we expect when each agent cares about the consumption of others? A parallel result to (ii-ii) seems unlikely. In the presence of consumption externalities, an increase in an agent's consumption of money may have a non-uniform effect on the welfare of the others. Thus, giving more money to an agent, which is the natural way to attempt preventing the possibility that the agent might be better off by swapping her consumption with some other agent, may disrupt harmony among the others. Nevertheless, contrary to this intuition, we show, in Theorems 1 and 2, that parallel statements to (ii-i) and (ii-ii) hold when preferences belong to our basic domain.

Our third question, the compatibility of fairness and efficiency, turns out to be more problematic. For the allocation of objects and money, if there are no consumption externalities and there are at least as many agents as objects, then balanced budget non-contestable allocations are efficient (Svensson, 1983).[6] By contrast, there are economies with consumption externalities for which no non-contestable allocation is efficient (Section 4). In light of this impossibility, we identify a condition we call "non-wastefulness," which, if satisfied by a balanced budget non-contestable allocation, guarantees that the allocation is efficient. This result, combined with our theorems concerning the existence of non-contestable allocations, allows us to identify domains on which fairness and efficiency are compatible.

Our model is very flexible. Thus, our positive results apply to broad classes of economies (Section 5). Indeed, we present four sub-domains of our basic domain of preferences that respectively accommodate linear externalities, inequity aversion (Fehr and Schmidt, 1999), surplus motivation (Andreoni and Miller, 2002), and agents' concern for their relative position in society (Ok and Koçkesen, 2000). In each case we identify conditions that guarantee non-contestable efficient allocations exist.

Finally, our theorems unify the theory of fair allocation of objects and money. We show that the existence results in Svensson (1983), Maskin (1987), and Alkan et al. (1991) are corollaries of our main theorems (Section 6).

The remainder of the paper is organized as follows. Section 2 introduces the model. Section 3 presents our existence results. Section 4 investigates the compatibility of fairness and efficiency. Section 5 develops applications to economies with externalities. Section 6 presents our results for economies without externalities. Section 7 discusses our results. The Appendix contains proofs omitted in the body of the paper.

[6] In the fair allocation of a bundle of infinitely divisible commodities, non-convexities of preferences may compromise the existence of non-contestable efficient allocations (Varian, 1974). This impossibility extends to convex production economies (Pazner and Schmeidler, 1974; Vohra, 1992).

2 The Model

2.1 Consumption space and preferences

Let $N \equiv \{1, \ldots, n\}$ be a set of agents and A a set of n objects (see Section 7 for the extension of our results when $|A| \neq n$). Generic objects are α and β. Each agent consumes an amount of an infinitely divisible good that we refer to as "money" and one object. An *allocation with budget* $M \in \mathbb{R}$ is a pair $(x, \mu) \in \mathbb{R}^A \times A^N$ such that $\sum_{\beta \in A} x_\beta = M$ and $\mu : N \to A$ is a bijection. Agent i's *assignment at* $z \equiv (x, \mu)$ is $z_i \equiv (x_{\mu(i)}, \mu(i))$, her *consumption of money* is $x_{\mu(i)}$, and her *assigned object* is $\mu(i)$; the *distribution of money at* z is the vector $x \in \mathbb{R}^A$. The *set of allocations with budget* M is $Z(M)$ and the *set of all allocations* is $Z \equiv \bigcup_{M \in \mathbb{R}} Z(M)$.

Agents have complete and transitive preferences on Z, i.e., they rank allocations, not individual bundles. Agent i's generic preference is R_i. As usual, I_i and P_i are the symmetric and asymmetric parts of R_i, respectively. Let $R \equiv (R_i)_{i \in N}$ denote the generic preference profile. Each profile R induces an incomplete ordering on Z, which for convenience we also denote R, as follows: for each pair $\{z, z'\} \subseteq Z$, $z' \, R \, z$ if and only if for each $i \in N$, $z' \, R_i \, z$.

We introduce six preference domains on Z. In what follows we illustrate when R_i belongs to the respective domains.

- *Basic domain,* $\mathcal{B}$: preferences satisfying the following three properties.

(i) equal budget compensation assumption: for each $b \in \mathbb{R}$ there is $M(R_i, b) \in \mathbb{R}$ such that *(i-i)* $M(R_i, b) \geq nb$; *(i-ii)* for each $M > M(R_i, b)$ and each $(x, \mu) \in Z(M)$ such that for each $\alpha \in A$, $x_\alpha \geq b$, if $x_{\mu(i)} = b$, then there is a bijection $\mu' \in A^N$ such that $x_{\mu'(i)} > b$ and $(x, \mu') \, R_i \, (x, \mu)$; and *(i-iii)* as $b \to -\infty$, $M(R_i, b) \to -\infty$.

(ii) anonymity of externalities: for each $(x, \mu) \in Z$ and each bijection $\mu' \in A^N$ such that $\mu(i) = \mu'(i)$, we have that $(x, \mu) \, I_i \, (x, \mu')$.

(iii) continuity: weak lower and weak upper contour sets are closed.

The following is a stronger form of the *equal budget compensation assumption*, which can be some times more easily verified and is usually satisfied in applications.

(i') ceteris-paribus-compensation assumption: for each pair $\{\alpha, \beta\} \subseteq A$ and each $b \in \mathbb{R}$, there is $b' \in \mathbb{R}$ such that for each $x \in \mathbb{R}^A$ such that $x_\alpha \leq b$ and $x_\beta \geq b'$ and each bijection $\mu \in A^N$ such that $\mu(i) = \alpha$, if $\mu' \in A^N$ is the bijection obtained from μ by swapping agent i's assignment and the one of the agent who receives object β at (x, μ), then $(x, \mu') \, R_i \, (x, \mu)$.

- *Ceteris-paribus-compensated domain,* $\mathcal{C}$: preferences satisfying *(i')*, *(ii)*, and *(iii)*.

The following lemma states that $\mathcal{C}$ is a sub-domain of our basic domain. The proof is in the Appendix.

Lemma 1. *$\mathcal{C} \subseteq \mathcal{B}$.*

The following are sub-domains of $\mathcal{B}$ that satisfy further requirements.

• *Own-consumption money-monotone domain,* $\mathcal{M}$: preferences in $\mathcal{B}$ satisfying the additional requirement that, ceteris paribus, the agent prefer an allocation that allots her a higher consumption of money, i.e., for each $(x, \mu) \in Z$ and each $\delta \in \mathbb{R}_{++}$, $(x + \delta 1_{\mu(i)}, \mu)\, P_i\, (x, \mu)$.[7]

• *Externality-free domain,* $\mathcal{N}$: preferences in $\mathcal{B}$ such that for each pair $\{z, z'\} \subseteq Z$ such that $z_i = z'_i$, $z\, I_i\, z'$.

• *Restricted domain,* $\mathcal{R}$: preferences in $\mathcal{N} \cap \mathcal{M}$ such that for each pair $\{\alpha, \beta\} \subseteq A$ and each $x_\alpha \in \mathbb{R}$, there is $x_\beta \in \mathbb{R}$ such that if $\{z, z'\}$ are such that $z_i \equiv (x_\alpha, \alpha)$ and $z'_i \equiv (x_\beta, \beta)$, then $z\, I_i\, z'$. This sub-domain of preferences is studied by Svensson (1983); Maskin (1987); and Alkan et al. (1991).

• *Quasi-linear domain,* $\mathcal{Q}$: preferences in $\mathcal{R}$ satisfying the additional requirement that for any pair of allocations $z \equiv (x, \mu)$ and $z' \equiv (x', \mu')$ such that $z\, R_i\, z'$ and each $\delta \in \mathbb{R}$, $(x + 1_{\mu(i)}\delta, \mu)\, R_i\, (x' + 1_{\mu'(i)}\delta, \mu')$. This sub-domain of preferences is studied by Alkan et al. (1991); Aragones (1995); and Klijn (2000).

The following proposition states that there is a proper inclusion relation among the different preference domains defined above. The proof is in the Appendix.

Proposition 1. *(i)* $\mathcal{Q} \subsetneq \mathcal{R} \subsetneq \mathcal{N} \cap \mathcal{M} \subsetneq \mathcal{C} \subsetneq \mathcal{B}$*; (ii)* $\mathcal{N} \backslash \mathcal{M} \neq \emptyset$*; and (iii)* $\mathcal{M} \backslash \mathcal{N} \neq \emptyset$*.*

2.2 Environment

An *economy* is a pair $e \equiv (R, M)$ in which $R \equiv (R_i)_{i \in N} \in \mathcal{B}^N$ and $M \in \mathbb{R}$ is an amount of money (possibly negative) to distribute among the members of N. The set of all economies is $\mathcal{E}$. For each $e \equiv (R, M) \in \mathcal{E}$ the set of *feasible allocations* is $Z(e) \equiv \bigcup_{m \leq M} Z(m)$; the set of *balanced budget* allocations for e is $Z(M)$.

We assume free disposal of money. All of our results hold if we require budget-balance instead. In some applications, budget-balance is a sensible assumption (e.g., the allocation of rooms and the division of the rent among housemates), which simplifies the presentation of the results. However, it is a strong requirement that precludes a Pareto improvement may be attained by disposing of money (this issue is of particular interest in Subsection 5.4).

2.3 Properties of allocations

We consider three properties of allocations.

[7] Let us remark that an agent with preferences in $\mathcal{M}$ would never dispose of money that she can keep for herself. However, since the stated rankings are among allocations that differ only on the agent's consumption of money, it is not implied that the agent would not benefit from donating money to another agent.

First is efficiency. It is defined as usual. Let $e \in \mathcal{E}$. An allocation $z \in Z(e)$ is *efficient for* e if there is no $z' \in Z(e)$ such that $z' \, R \, z$ and for at least one $j \in N$, $z' \, P_j \, z$. We denote the set of *efficient* allocations for e by $P(e)$.

Second is our central property, fairness. We want to test whether an allocation is fair in a distributive situation in which agents have symmetric rights over resources (see Moulin, 2003, for a discussion on distributive justice principles).[8] One may think that equal rights calls for equal division of resources. Nevertheless, equal division may not be *efficient*. Furthermore, it is not even physically possible in our environment.

We adapt Foley's (1967) fairness test to our environment and require that no agent prefer the allocation obtained by swapping her assignment with that of any other agent. Let $e \equiv (R, M) \in \mathcal{E}$. We refer to the allocations passing this test for e as *non-contestable on fairness grounds (non-contestable) for* e and denote the set of these allocations by $Nc(e)$.

Our last property imposes a lower bound on individual consumptions of money. Let $b \in \mathbb{R}$. An allocation z is *$\underline{b}$-bounded* if each agent's consumption of money at z is at least b. Let $e \in \mathcal{E}$. We denote the set of *$\underline{b}$-bounded* allocations for e by $B_b(e)$.

For each $e \in \mathcal{E}$, let $NcB_b(e) \equiv Nc(e) \cap B_b(e)$, $NcP(e) \equiv Nc(e) \cap P(e)$, and $NcPB_b(e) \equiv Nc(e) \cap P(e) \cap B_b(e)$.

3 Existence

In this section we study the existence of *non-contestable* allocations.

3.1 Existence of balanced budget non-contestable allocations

Our first theorem is a general possibility result. It states the existence of *balanced budget non-contestable* allocations when preferences belong to the basic domain.

Theorem 1. *For each* $e \equiv (R, M) \in \mathcal{E}$*,* $Nc(e) \cap Z(M) \neq \emptyset$*.*

The proof is in the Appendix. Here we sketch it. Let $e \equiv (R, M) \in \mathcal{E}$ and $b \in \mathbb{R}$. In the following subsection we develop a theorem for the related problem of existence of *$\underline{b}$-bounded balanced budget non-contestable* allocations for e. We identify an amount that depends on R and b, such that if M is at least this amount, then $NcB_b(e)$ is non-empty. It turns out that for a fixed preference profile, this "sufficient budget" goes to minus infinity as b goes to minus infinity. Thus, given R and M, one

[8] We exclude situations in which agents have to be rewarded for unequal contributions to the "production" of the resources. For instance, in the division of the surplus in an investment project, it is appropriate to award unequal rights to investors depending on the time, effort, and money each contributed. We assume away such asymmetries.

can find b low enough such that there are $\underline{b}$*-bounded balanced budget non-contestable* allocations for (R, M).

3.2 Existence of $\underline{b}$-bounded balanced budget non-contestable allocations

If there is a common lower bound in individual consumption of money, there may be no *non-contestable* allocation satisfying the lower bound.[9] Svensson (1983), Maskin (1987), and Alkan et al. (1991) show, in the no externalities case, that given a preference profile and a lower bound in individual consumptions of money, if the budget is large enough, then there are *balanced budget non-contestable* allocations satisfying the lower bound. In this section, we show that a parallel result holds when preferences belong to our basic domain: for each $R \in \mathcal{B}^N$ and each $b \in \mathbb{R}$, there is an amount that depends on R and b, which we denote by $m_{(n-1)}(R, b)$, such that if $M \geq m_{(n-1)}(R, b)$, then there are $\underline{b}$*-bounded balanced budget non-contestable* allocations for (R, M).

3.2.1 Identifying the sufficient budget $m_{(n-1)}(R, b)$

Let $R_i \in \mathcal{B}$ and $b \in \mathbb{R}$. Since R_i satisfies the *equal budget compensation assumption*, then the set

$$\left\{ M(R_i, b) \geq nb \;\middle|\; \begin{array}{l} \text{for each } M > M(R_i, b) \text{ and each } (x, \mu) \in Z(M) \text{ such that} \\ x \gneq b \text{ and } x_{\mu(i)} = b, \text{ there is a bijection } \mu' \in A^N \text{ such that} \\ x_{\mu'(i)} > b \text{ and } (x, \mu')\, R_i\, (x, \mu) \end{array} \right\},$$

is not empty. Since this set is bounded below, then it has an infimum. Since R_i is continuous, then this infimum is actually in the set. Let $m(R_i, b)$ be the minimum of this set.

If $M < m(R_i, b)$, there may be an object, say α, such that agent i, when receiving object $\beta \neq \alpha$ at a $\underline{b}$*-bounded* allocation with budget M, would prefer to swap her assignment with the agent who receives object α. Of course, only one agent receives α. Thus, if there are additional agents with similar preferences, there may be no $\underline{b}$*-bounded non-contestable* allocation with budget M. A natural candidate for a sufficient budget that guarantees the existence of $\underline{b}$*-bounded non-contestable* allocations is then $\max_{i \in N} m(R_i, b)$. It turns out that one can do significantly better than this. It is sufficient to have a budget that allows one to compensate all but one agent.

Definition 1. For each $R \in \mathcal{R}^N$ and each $b \in \mathbb{R}$, let $m_{(n-1)}(R, b)$ be the $(n-1)$-th order statistic of the vector $(m(R_i, b))_{i \in N}$.[10]

[9] A simple example is the allocation of objects with no monetary compensations, which is equivalent to the situation in which budget is zero and consumptions of money must be non-negative.

[10] For each $x \in \mathbb{R}^n$, the $(n-1)$-th order statistic of x is the value of $x_{[n-1]}$, where $(x_{[i]})_{i=1}^n$ is

3.2.2 Basic existence theorem

Our second theorem states that if M is at least $m_{(n-1)}(R, b)$, there are *$\underline{b}$-bounded balanced budget non-contestable* allocations for (R, M).

Theorem 2. *Let $R \in \mathcal{B}^N$ and $b \in \mathbb{R}$. For each $M \geq m_{(n-1)}(R, b)$,*

$$NcB_b(R, M) \cap Z(M) \neq \emptyset.$$

Our proof follows the approach to fair allocation introduced by Su (1999). We present some technical definitions and two lemmas in the Appendix.

Proof. Since preferences are continuous, we can assume without loss of generality that $M > m_{(n-1)}(R, b)$. The simplex of $\underline{b}$-bounded "divisions" of A and the amount of money M is $\Delta^{n-1}(b, M) \equiv \{x \in \mathbb{R}^A : x \geq b, \sum_{\alpha \in A} x_\alpha = M\} \subseteq \mathbb{R}^A$. A division is not an allocation. It is only a list of amounts of money indexed by elements of A. Nevertheless, a division, $x \in \Delta^{n-1}(b, M)$, induces a list of bundles $\{(x_\alpha, \alpha)\}_{\alpha \in A}$, and can be completed into a *$\underline{b}$-bounded* allocation by specifying who receives each object. We proceed in three steps.

Step 1: Triangulate the simplex of $\underline{b}$-bounded divisions, and assign a label (owner) from N to the vertices of the triangulation in such a way that: (i) all the subsimplices are fully labeled (each vertex of each subsimplex has a different owner), and (ii) all the vertices owned by n are interior, where without loss of generality, $n \in \arg\max_{i \in N} m(R_i, b)$. Figure 1 (a) shows a triangulation, and Figure 1 (b) a labeling (see the Appendix for formal definitions of triangulation and labeling, and Lemma 7, which guarantees the existence of such a labeled triangulation).

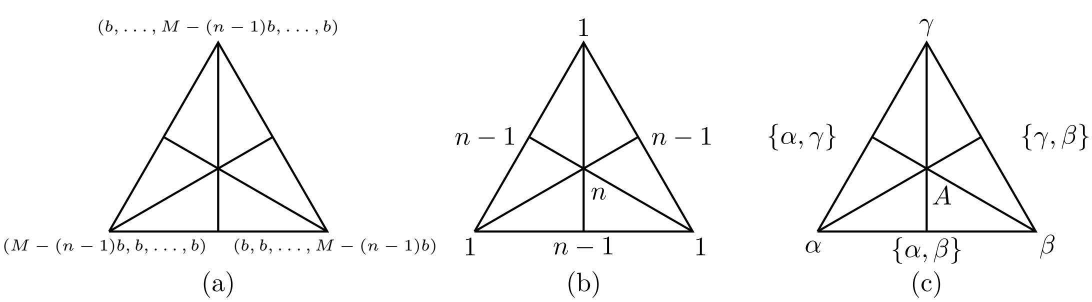

Figure 1: (a) A triangulation of $\Delta^{n-1}(b, M)$, i.e, a division of $\Delta^{n-1}(b, M)$ into subsimplices that intersect only in common faces; (b) Ownership label; (c) Preference induced label.

an ordered permutation of x, such that $x_{[1]} \leq \ldots x_{[n-1]} \leq x_{[n]}$. For instance, if $n = 3$, $x_1 = 10$, $x_2 = 15$, and $x_3 = 10$, then $x_{[n-1]} = x_{[2]} = 10$.

Step 2: Assign labels from A to the vertices of the triangulation using the ownership labeling and preferences as follows. Let x be a vertex of the triangulation. Suppose that agent i is the owner of x. The label of x is the object received by i at one of i's most preferred allocations from the ones with distribution of money x. If $M > m_{(n-1)}(R, b)$, by the definition of $m_{(n-1)}(R, b)$, the labels can be chosen from the subscripts of the bundles with money components greater than b (Figure 1 (c)). Using a result from combinatorics, Sperner's Lemma (Border, 1999; McLennan and Tourky, 2008), such a labeled triangulation has, at least, one fully labeled subsimplex. Details of this step are contained in Lemma 8 in the Appendix.[11]

Step 3: Construct a sequence of triangulations and labelings, with the properties required in Steps 1 and 2, and whose mesh size converges to zero. Extract a sequence of subsimplices, fully labeled with labels taken from the second labeling, and whose vertices form convergent sequences. Since the number of objects is finite, then this sequence can be further required to have each object's name as the label of a vertex owned by the same agent in each subsimplex of the sequence. The existence of such a sequence is guaranteed by Lemmas 7 and 8 in the Appendix. Since preferences are continuous and externalities are anonymous, the limit of such a sequence, which is a point in the simplex of partitions, induces a *non-contestable* allocation where each agent receives the object associated to her vertices in the sequence. □

A feature of $m_{(n-1)}(R, b)$ is that it does not increase under a change in the preferences of an agent with highest $m(R_i, b)$. Equivalently, one can say that our theorem provides a sufficient budget that can be calculated from the preferences of $n - 1$ agents.[12] This feature cannot be generalized. Indeed, the following proposition states that given any amount of money M and any profile of preferences for $n - 2$ agents, the preferences of the two remaining agents can be specified in such a way that the minimal amount of money necessary for the existence of a $\underline{b}$-*bounded non-contestable* allocation is greater than M. The proof is in the Appendix.

Proposition 2. *Let $M \in \mathbb{R}$ and $R_{-\{i,j\}} \in \mathcal{B}^{N \setminus \{i,j\}}$. There are $\{R_i, R_j\} \subseteq \mathcal{R}$ such that $NcB_b(R_{-\{i,j\}}, R_i, R_j, M) = \emptyset$.*

[11] Let us emphasize that a fully labeled subsimplex (with the second labeling) has a different owner in each vertex; moreover each agent prefers, out the allocations induced by "her vertex," one in which she receives an object different from the the ones received by the owners of the other vertices at their preferred allocations. If this simplex were a point, and consequently all the vertices would coincide, each agent would prefer an allocation at which she receives a different component from the list of bundles induced by the common vertex. Thus, this common vertex would induce a *non-contestable* allocation.

[12] This does not imply that a *non-contestable* allocation can be constructed with total ignorance of an agent's preferences. In the proof of Theorem 2, the labeling constructed in Step 2, satisfies the desired properties independently of the preferences of the agent who owns vertices only in the interior of the simplex. Nevertheless, her preferences determine the induced labels for her interior vertices, and thus determine the *non-contestable* allocation obtained in the limit.

4 Efficiency

In this section we study the existence of *non-contestable efficient* allocations.

In economies in which preferences belong to $\mathcal{R}$ and in which there are at least as many agents as objects, each *balanced budget non-contestable* allocation is *efficient* (Svensson, 1983). However, there are economies in our basic domain for which no *non-contestable* allocation is *efficient*. Example 4, in the Appendix, exhibits an economy in which: (i) preferences belong to $\mathcal{M}$, (ii) the set of *efficient* allocations is non-empty, and (iii) no *non-contestable* allocation is *efficient*.

In light of this impossibility, one would like to identify conditions that restore the existence of *non-contestable efficient* allocations. The following proposition advances in this direction.

Let us first make a definition. Let $e \equiv (R, M) \in \mathcal{E}$. An allocation z is *non-wasteful* for e if it can be weakly Pareto dominated only by a feasible allocation whose distribution of money (object-wise) weakly dominates the distribution of money at z. That is, $z \equiv (x, \mu) \in Z(e)$ is *non-wasteful for* e if for each $z' \equiv (x', \mu') \in Z(e)$, $z' \, R \, z$ only if $x' \geq x$.

Proposition 3. *Let $e \equiv (R, M) \in \mathcal{E}$ and $z \in Nc(e) \cap Z(M)$. If z is non-wasteful for e, then $z \in P(e)$.*

Proof. Let z be as in the statement of the lemma. We claim that $z \in P(e)$. Suppose by means of contradiction that there is $z' \equiv (x', \mu') \in Z(e)$ such that $z' \, R \, z$ and for at least one $j \in N$, $z' \, P_j \, z$. Since z is *non-wasteful*, then $x' \geq x$. Moreover, since $z \in Z(M)$ then $x' = x$. Since $z \in F(e)$, then $z \, R \, z'$. This is a contradiction. $\square$

Proposition 3 allows us to identify preference domains on which *non-contestability* and *efficiency* are compatible. The task is reduced to finding restrictions on preferences guaranteeing that *balanced budget non-contestable* allocations are *non-wasteful*. For instance, one can easily see that this is the case if preferences belong to $\mathcal{R}$. Likewise, if there is added structure on preferences, one can identify domains where this inclusion generally holds. The applications developed in the following section illustrate this point. The following corollary formalizes this result. We omit the straightforward proof.

Corollary 1. *Let $\mathcal{D} \subseteq \mathcal{B}$. Assume that for each economy with preferences in $\mathcal{D}$ each balanced budget non-contestable allocation is non-wasteful. Let $R \in \mathcal{D}^N$, $M \in \mathbb{R}$, and $b \in \mathbb{R}$. Then,*

(i) $NcP(R, M) \cap Z(M) \neq \emptyset$.

(ii) If $M \geq m_{(n-1)}(R, b)$, then $NcPB_b(R, M) \cap Z(M) \neq \emptyset$.

5 Applications to economies with externalities

In this section we present four applications to classes of economies with externalities.

5.1 Linear externalities

Consider the preference represented by a utility function of the following form:

$$u_i(z) \equiv v^i(\mu(i)) + x_{\mu(i)} + \sum_{\alpha \in A \setminus \{\mu(i)\}} c^i_{\mu(i)}(\alpha) x_\alpha,$$

where $v^i : A \to \mathbb{R}$ and for each pair of different objects $\{\alpha, \beta\}$, $0 \leq c^i_\alpha(\beta) < \frac{1}{\varphi(n)}$ where $\varphi(n) \equiv \max\left\{\left(2 - \frac{3}{n}\right)(n-1), n-1\right\}$. We refer to this domain as the linear externality domain and denote it by $\mathcal{L}$.

The restrictions on the coefficients in a linear externality preference uniformly bound the contribution of an agent's consumption of money to the welfare of the others. In particular, for agent i and allocation $(x, \mu) \in Z$,

$$\sum_{j \in N \setminus \{i\}} c^j_{\mu(j)}(\mu(i)) \leq \frac{1}{2} + O(n).$$

That is, the contribution of agent i's consumption of money to the aggregate welfare of the others is essentially never more than half the contribution to her own welfare.

Preferences in $\mathcal{L}$ are a natural representation in a network environment with consumption externalities in which agents are allotted an amount of money and a node. Here, one can interpret coefficient $c^i_\alpha(\beta)$ as the degree to which the welfare of agent i in node α is affected by the consumption of money of the agent in node β. Depending on the particular application one may impose additional restrictions on these coefficients. For instance, one may require them to be symmetric, or that they coincide with an objective measure and then be uniform across agents.

We investigate the existence of *non-contestable* and *efficient* allocations in linear externality economies. The following lemma states that $\mathcal{L}$ belongs to our basic domain of preferences (recall that in this domain our general possibility result holds).[13] Moreover, preferences in $\mathcal{L}$ extend the *quasi-linear* domain.

Lemma 2. *$\mathcal{Q} \subsetneq \mathcal{L} \subsetneq \mathcal{B}$.*

The following proposition states that in linear externality economies *balanced budget non-contestable* allocations are *non-wasteful.*

[13] In general, linear externality preferences do not satisfy the *ceteris-paribus-compensation assumption* (see the proof of Proposition 1 in the Appendix).

Proposition 4. *Let $R \in \mathcal{L}^N$ and $e \equiv (R, M) \in \mathcal{E}$. If $z \in Nc(e) \cap Z(M)$, then z is non-wasteful for e.*

A consequence of this result and our general possibility theorem is that fairness and *efficiency* are compatible in linear externality economies.

The following proposition summarizes the joint implications of Corollary 1, Lemma 2, and Proposition 4. We omit the straightforward proof.

Proposition 5. *Let $R \in \mathcal{L}^N$, $M \in \mathbb{R}$, and $b \in \mathbb{R}$.*
(i) $NcP(R, M) \cap Z(M) \neq \emptyset$.
(ii) If $M \geq m_{(n-1)}(R, b)$, then $NcPB_b(R, M) \cap Z(M) \neq \emptyset$.

5.2 Inequity aversion

In this section we study fair allocation among inequity-averse agents. Consider the preference R_i represented by a utility function of the following form:

$$u_i(z) \equiv \; v^i_{\mu(i)}(\mu(i)) + x_{\mu(i)} - \tfrac{a_i}{n-1} \textstyle\sum_{\beta \in A} \max\{v^i_{\mu(i)}(\beta) + x_\beta - [v^i_{\mu(i)}(\mu(i)) + x_{\mu(i)}], 0\}$$

$$-\tfrac{c_i}{n-1} \textstyle\sum_{\beta \in A} \max\{v^i_{\mu(i)}(\mu(i)) + x_{\mu(i)} - [v^i_{\mu(i)}(\beta) + x_\beta], 0\},$$

where a_i and c_i are non-negative constants such that $c_i < 1$ and $c_i \leq a_i$, and $\{v^i_\alpha\}_{\alpha \in A}$ is a family of real-valued functions defined on A. We refer to this domain as the inequity-averse domain and denote it by $\mathcal{F}$.

These preferences generalize a family introduced by Fehr and Schmidt (1999) for the allocation of a single good.[14] Intuitively, they capture aversion to inequity in the following way. For each pair of objects $\{\alpha, \beta\} \subseteq A$, $v^i_\alpha(\beta)$ is the "monetary value" that the agent assigns to object β when receiving object α. Thus, the monetary value of her consumption when receiving object α is $v^i_\alpha(\alpha) + x_\alpha$. The agent cares not only about the value of her consumption, but also about the equity of the allocation. Her ideal would be equality of consumption values for all agents. She loses some welfare from deviations from this standard. The deviations are treated differently depending on whether the value of her consumption is lower or greater than the values of the consumption of the others. The coefficient $\frac{a_i}{n-1}$ is applied to the summation of the excess of consumption value of the other agents relative to the agent's own consumption. Analogously, the coefficient $\frac{c_i}{n-1}$ is applied to the summation of the agent's excess of consumption value relative to the consumption of the others. The requirement $c_i \leq a_i$ captures the idea that the agent loses at least as much welfare from inequity against her than from the inequity against the others.

[14] Neilson (2006) and Sandbu (2008) characterize two sub-domains of preferences, for the distribution of a single good, containing Fehr-Schmidt preferences. Rohde (2009) characterizes the exact Fehr-Schmidt domain.

Are these subjective perceptions of fairness compatible with our generalization of Foley's test? The answer is positive. The following lemma states that preferences in $\mathcal{F}$ belong to our basic domain. Moreover, preferences in $\mathcal{F}$ extend the *quasi-linear* domain, satisfy the *ceteris-paribus-compensation assumption*, and are *money-monotone*. The proof is in the Appendix.

Lemma 3. *$\mathcal{Q} \subsetneq \mathcal{F} \subsetneq \mathcal{M} \cap \mathcal{C}$. Moreover, for each $R_i \in \mathcal{F}$ that is represented by a utility function with parameter $a_i > 0$, $R_i \in \mathcal{M} \setminus \mathcal{N}$.*

The following theorem states sufficient conditions that guarantee that in an inequity-averse economy, a *balanced budget non-contestable* allocation is *non-wasteful*. The proof is in the Appendix.

Proposition 6. *Let $R \in \mathcal{F}^N$. If for each $i \in N$, R_i is represented by a utility function with parameter $c_i < \frac{n-1}{n}$, then for each $M \in \mathbb{R}$, each $z \in Nc(R, M) \cap Z(M)$ is non-wasteful for (R, M).*

A consequence of this result is that as n grows to infinity, the measure of inequity-averse economies in which no *non-contestable* allocation is *efficient* vanishes.

The following proposition summarizes the joint implications of Corollary 1, Lemma 3, and Proposition 6. It states sufficient conditions for the existence of *balanced budget non-contestable efficient* allocations in inequity-averse economies. We omit the straightforward proof.

Proposition 7. *Let $R \in \mathcal{F}^N$, $M \in \mathbb{R}$, and $b \in \mathbb{R}$.*

(i) $Nc(R, M) \cap Z(M) \neq \emptyset$.

(ii) If $M \geq m_{(n-1)}(R, b)$, then $NcB_b(R, M) \cap Z(M) \neq \emptyset$.

Moreover, if for each $i \in N$, R_i is represented by a utility function with parameter $c_i < \frac{n-1}{n}$, then $Nc(R, M) \cap Z(M) \subseteq P(R, M)$, and thus, the allocations in (i) and (ii) are efficient.

5.3 Surplus motivation

In this section we study the fair allocation of objects and money among "surplus motivated" agents. Such an agent is willing to give up her money only in exchange for an increase in the aggregate consumption of money among the consumptions that affect her welfare.

Anecdotic evidence suggests that fund raising activities that promise matching donations from a third party induce people to give. One can also relate this phenomenon to the willingness to give for education when the returns to it are thought of as higher than market returns. Besides these everyday observations, experimental evidence from modified versions of Dictator Games suggests that there are agents who are willing to give up resources in order to increase the aggregate consumption of money "extracted" from the experimenter (Andreoni and Miller, 2002; see Section 4.1 in Fehr and Schmidt, 2001, for a survey).[15]

We introduce a domain of preferences that captures surplus-motivation. Let $i \in N$ and $\Gamma_i \equiv \{\Gamma_i(\alpha)\}_{\alpha \in A}$ be a family of subsets of A such that for each $\alpha \in A$, $\alpha \in \Gamma_i(\alpha)$. Agent i's preference relation $R_i \in \mathcal{B}$ is *surplus motivated with respect to* Γ_i if for each $(x, \mu) \in Z$ and each (x', μ') such that $\mu'(i) = \mu(i)$ and $x'_{\mu'(i)} < x_{\mu(i)}$, we have that $\sum_{\alpha \in \Gamma_i(\mu'(i))} x'_\alpha > \sum_{\alpha \in \Gamma_i(\mu(i))} x_\alpha$ whenever $(x', \mu')\, R_i\, (x, \mu)$. Agent i's preference relation $R_i \in \mathcal{B}$ is *surplus motivated* if there is a family of subsets of A, Γ_i, such that R_i is surplus motivated with respect to Γ_i. We denote the domain of surplus motivated preferences by $\mathcal{S}$.

Let R_i be surplus motivated with respect to a family of sets Γ_i. The following lemma allows us to interpret Γ_i as follows: if agent i receives object α, then her welfare is only affected by the consumption of the agents who receive objects in $\Gamma_i(\alpha)$. The proof is in the Appendix.

Lemma 4. *Let $R_i \in \mathcal{S}$ be surplus motivated with respect to a family of sets Γ_i. Then for each pair of allocations $z \equiv (x, \mu)$ and $z' \equiv (x', \mu)$ such that for each $\beta \in \Gamma_i(\mu(i))$, $x_\beta = x'_\beta$, we have that $z\, I_i\, z'$.*

We can interpret then a surplus motivated agent as one who would give up money (keeping the same object) only if the agents whose consumptions affect her welfare receive more money in the aggregate.

The following lemma states the inclusion relations that hold between surplus motivated preferences and the subdomains of preferences defined earlier. Let us remark that surplus motivated preferences are own-consumption money-monotone.[16] The proof is in the Appendix.

Lemma 5. $\mathcal{N} \subsetneq \mathcal{S} \subsetneq \mathcal{M}$.

We now identify a condition under which *balanced budget non-contestable* allocations are *non-wasteful* in surplus motivated economies.

Definition 2. Let $\Gamma \equiv \{\Gamma_i\}_{i \in N}$ be such that for each $i \in N$, $\Gamma_i \equiv \{\Gamma_i(\alpha)\}_{\alpha \in A}$ is a family of subsets of A. Then Γ satisfies the *interests-overlap property (IOP)* if for each

[15] This phenomenon is also referred to as surplus maximization (Fehr and Schmidt, 2001) and preference for efficiency (Cox et al., 2008).

[16] Surplus motivation also captures a form of altruism. Let $i \in N$ and $\{R_i, R'_i\} \subseteq \mathcal{B}$. Then R'_i *is more altruistic than (MAT)* R_i if for each $z \equiv (x, \mu) \in Z$ and each $\Delta \in \mathbb{R}^A$ such that for each $\alpha \in A \setminus \{\mu(i)\}$, $\Delta_\alpha \geq 0$, we have that $(x + \Delta, \mu)\, R'_i\, z$ whenever $(x + \Delta, \mu)\, R_i\, z$ (Cox et al., 2008). We define altruism as the combination of two conditions: (i) not being externality-free and (ii) being more altruistic than each externality-free preference. One can prove that surplus motivated preferences are altruistic whenever they are not externality-free.

pair of different objects $\{\alpha, \beta\} \subseteq A$ and each pair of different agents $\{i, j\} \subseteq N$, if $(\Gamma_i(\alpha)\backslash\{\alpha\}) \cap (\Gamma_j(\beta)\backslash\{\beta\})$ is non-empty, then either $\Gamma_i(\alpha) \subseteq \Gamma_j(\beta)$ or $\Gamma_j(\beta) \subseteq \Gamma_i(\alpha)$.

The intuition of this property is as follows. Let $R \in \mathcal{S}^N$ be surplus motivated with respect to a family Γ satisfying *IOP*. Suppose that at a given allocation agent i receives object α, agent j receives object β, and that their welfare is affected by the consumption of a third agent. Then, either (i) all consumptions that affect the welfare of agent i when receiving α, affect the welfare of agent j when receiving β, or (ii) all consumptions that affect the welfare of agent j when receiving β affect the welfare of agent i when receiving α.

The following are examples of families $\Gamma \equiv \{\Gamma_i\}_{i\in N}$ satisfying *IOP*. In each case, one can easily construct utility representations for preferences that are surplus-dominated with respect to Γ.

Example 1. Upstream externalities: let $\succeq$ be a linear order on A. We interpret this order as the location of the objects along a river, i.e., for each pair of objects $\{\alpha, \beta\}$, $\alpha \succ \beta$ if and only if β is located downstream from α. Suppose that for each $i \in N$, $\Gamma_i(\alpha) = \{\beta : \beta \succeq \alpha\}$, i.e., each agent cares about the consumption of the agents who are upstream. Clearly, for each pair of objects $\{\alpha, \beta\}$ and each pair of agents $\{i, j\}$, either $\Gamma_i(\alpha) \subseteq \Gamma_j(\beta)$ or $\Gamma_i(\beta) \subseteq \Gamma_j(\alpha)$.[17] $\square$

Example 2. Partitioned structure: Let $\Omega \equiv \{\Omega_i\}_{i\in N}$ be a family of partitions of A. For each $i \in N$ and each $\alpha \in A$, let $\Omega_i(\alpha)$ be the component of Ω_i that contains α. Suppose that for each $\alpha \in A$ and each pair of agents $\{i, j\}$, either $\Omega_j(\alpha) \subseteq \Omega_i(\alpha)$ or $\Omega_i(\alpha) \subseteq \Omega_j(\alpha)$. Suppose that for each $i \in N$ and each $\alpha \in A$, $\Gamma_i(\alpha) = \Omega_i(\alpha)$.[18]

We claim that Γ satisfies *IOP*. Let $\{\alpha, \beta\}$ and $\{i, j\}$ be pairs of different objects and agents such that $\Gamma_i(\alpha) \cap \Gamma_j(\beta)$ is non-empty. Let $\gamma \in \Gamma_i(\alpha) \cap \Gamma_j(\beta)$. Then, $\Gamma_i(\alpha) = \Omega_i(\gamma)$ and $\Gamma_j(\beta) = \Omega_j(\gamma)$. Since either $\Omega_j(\gamma) \subseteq \Omega_i(\gamma)$ or $\Omega_i(\gamma) \subseteq \Omega_j(\gamma)$, then either $\Gamma_i(\alpha) \subseteq \Gamma_j(\beta)$ or $\Gamma_j(\beta) \subseteq \Gamma_i(\alpha)$.

The partitioned structure in this example is a sensible assumption when objects are associated with geographical locations. For instance, houses may be partitioned into school districts, neighborhoods, cities, etc. An agent is surplus motivated with respect to one of these partitions when the consumptions that affect her welfare are the ones in the component to which her house belongs. $\square$

[17] When $n \leq 4$ the following partition also satisfies *IOP*. For each $i \in \mathbb{N}$ and each $\alpha \in A$, $\Gamma_i(\alpha)$ is one of the following four sets: $\{\alpha\}$, $\{\beta : \beta \succeq \alpha\}$, $\{\beta : \alpha \succeq \beta\}$, and A.

[18] The following functional form represents preferences $R_i \in \mathcal{B}$ that are surplus motivated w.r.t. Γ_i: for each $z \equiv (x, \mu) \in Z$, $u_i(z) \equiv v_i(\mu(i)) + x_{\mu(i)} - f_i(\max\{x_{\mu(i)} - \overline{x}_{\Omega_i(\mu(i))}, 0\})$, where $v_i : A \to \mathbb{R}$; $\overline{x}_{\Omega_i(\mu(i))}$ is the average consumption of money in $\Omega_i(\mu(i))$; and $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, monotone increasing, and bounded function such that $f_i(0) = 0$ and $f' < 1$.

The following proposition states that *balanced budget non-contestable* allocations are *non-wasteful* in surplus motivated economies satisfying *IOP*. The proof is in the Appendix.

Proposition 8. *Let $R \in \mathcal{S}^N$ and $\Gamma \equiv \{\Gamma_i\}_{i \in N}$ be such that for each $i \in N$, R_i is surplus motivated with respect to Γ_i. If Γ satisfies IOP, then for each $M \in \mathbb{R}$, each $z \in Nc(R, M) \cap Z(M)$ is non-wasteful for e.*

The following proposition summarizes the joint implications of Corollary 1 and Proposition 8. It states conditions that guarantee the existence of *balanced budget non-contestable efficient* allocations in a surplus motivated economy. Let us emphasize that the conditions in the theorem are satisfied by preferences that are surplus motivated with respect to the families in Examples 1 and 2. We omit the straightforward proof.

Theorem 3. *Let $R \in \mathcal{S}^N$ and $\Gamma \equiv \{\Gamma_i\}_{i \in N}$ be such that for each $i \in N$, R_i is surplus motivated with respect to Γ_i. Let $M \in \mathbb{R}$ and $b \in \mathbb{R}$. Then,*

(i) $Nc(R, M) \cap Z(M) \neq \emptyset$.

(ii) If $M \geq m_{(n-1)}(R, b)$, $NcB_b(R, M) \cap Z(M) \neq \emptyset$.

Moreover, if Γ satisfies IOP, then $Nc(R, M) \cap Z(M) \subseteq P(R, M)$, and thus, the allocations in (i) and (ii) are efficient.

5.4 Keeping up with the Joneses

In this section we investigate fair allocation in a domain of preferences that accommodates the phenomenon referred to as "keeping up with the Joneses." Here we interpret the allocation of objects and money as the allocation of tasks and salary among workers. We assume that there is the same number of agents and tasks and that each employee has to perform a task and receive a salary in compensation for it. The set of workers is N and the set of tasks is A. Employees' preferences are represented by utility functions of the form:

$$u_i(z) \equiv v_i(\mu(i)) + x_{\mu(i)} - \frac{1}{a_i} \max\{\overline{x} - x_{\mu(i)}, 0\},$$

where $a_i \in \mathbb{R}$ is such that $a_i > \frac{n-1}{n}$, $v_i : A \to \mathbb{R}$, and $\overline{x}$ is the average salary. We denote this domain by $\mathcal{K}$.

The following lemma states the inclusion relations between the domain $\mathcal{K}$ and the preference domains defined earlier. Let us remark that these preferences extend the *quasi-linear* domain and are own-consumption money-monotone. The proof is in the Appendix.

Lemma 6. $\mathcal{Q} \subsetneq \mathcal{K} \subsetneq \mathcal{M} \setminus (\mathcal{S} \cup \mathcal{N})$.

Employees with preferences in $\mathcal{K}$ not only care about the task they perform and the salary they receive, but also about their relative position in the company. An employee who receives less than the average salary may be made better off not only by an increase in her own salary, but also by a decrease in the average salary of her workmates. One can think of these as employees who experience a loss in welfare from the necessity to keep up with the "social standards" at the office.

Preferences that accommodate the agents' struggle for trying to keep up with their peers, were studied in the single good case by Ok and Koçkesen (2000). They provide a behavioral characterization of this phenomenon, which allows us to differentiate it from the related one of "spite." Intuitively, Ok and Koçkesen identify it with the "negative interdependence" of an agent's welfare with the other agents' welfare according to her own standards. That is, what causes pain to an agent is not the others' success or well-being in itself - which would be spite-, but the perception of being worse off than the other agents.[19]

The following proposition states that if preferences belong to $\mathcal{K}$ then *balanced budget non-contestable* allocations are *non-wasteful*. The proof is in the Appendix.

Proposition 9. *Let $R \in \mathcal{K}^N$. For each $M \in \mathbb{R}$, each $z \in Nc(R, M)$ is non-wasteful for (R, M).*

A consequence of this result and our general possibility theorem is that fairness and *efficiency* are compatible when preferences belong to $\mathcal{K}$. The following proposition summarizes the joint implications of Corollary 1, Lemma 6, and Proposition 9. We omit the straightforward proof.

Proposition 10. *Let $R \in \mathcal{K}^N$. Let $M \in \mathbb{R}$ and $b \in \mathbb{R}$. Then,*
(i) $NcP(R, M) \cap Z(M) \neq \emptyset$.
(ii) If $M \geq m_{(n-1)}(R, b)$, $NcPB_b(R, M) \cap Z(M) \neq \emptyset$.

6 Economies without externalities

In this section we present applications of our results to economies without externalities. Without loss of generality, we make preference statements among consumption bundles instead of allocations. First, we show that in this domain, the sufficient budget identified in Theorem 2 can be easily calculated from preferences. Then we

[19]One can extend Ok and Koçkesen's (2000) notions of negative interdependence and spite to our environment as follows: $R_i \in \mathcal{B}$ is *weakly negatively interdependent* if for each bijection $\mu \in A^N$ and each $\{x, x'\} \subseteq R^A$, $(x, \mu)\, R_i\, (x', \mu)$ whenever: (i) $x_{\mu(i)} = x'_{\mu(i)}$ and (ii) for each bijection $\mu' \in A^N$ such that $\mu'(i) \neq \mu(i)$, $(x', \mu')\, R_i\, (x, \mu')$; $R_i \in \mathcal{B}$ is *spiteful* if for each bijection $\mu \in A^N$ and each pair $\{x, x'\} \subseteq \mathbb{R}^A$, $(x, \mu)\, P_i\, (x', \mu)$, whenever $x'|_{A\setminus\{\mu(i)\}} \gg x|_{A\setminus\{\mu(i)\}}$. One can easily prove that each preference in $\mathcal{K}$ is *weakly negatively interdependent* but not *spiteful*.

show that the existence results in Svensson (1983), Maskin (1987), and Alkan et al. (1991) are corollaries of our theorems.

6.1 Identical preferences minimal budget

Let us introduce some definitions. Let $i \in N$, $R_i \in \mathcal{R}$, and $b \in \mathbb{R}$. For each pair $\{\alpha, \beta\} \subseteq A$, let $q_\beta^{R_i}(b,\alpha)$ be the amount of money such that $(q_\beta^{R_i}(b,\alpha), \beta)\, I_i\, (b,\alpha)$.[20] Figure 2 (a) illustrates the definition. If $D \subseteq A$ is such that for each pair $\{\alpha, \alpha'\} \subseteq D$, $(b,\alpha) I_i (b,\alpha')$, then for each $\beta \in A$, $q_\beta^{R_i}(b,D)$ denotes the common value $q_\beta^{R_i}(b,\alpha)$ for $\alpha \in D$.

Let $B(R_i,b)$ be the set of best objects for R_i at b, i.e., $\{\alpha \in A : \text{for each } \beta \in A, (b,\alpha)\, R_i\, (b,\beta)\}$. Since for each triple $\{\alpha,\beta,\theta\} \subseteq A$, if $(b,\theta)\, R_i\, (b,\alpha)$, then $q_\beta^{R_i}(b,\alpha) \leq q_\beta^{R_i}(b,\theta)$, then we have that $m(R_i,b) = \sum_{\alpha \in A} q_\alpha^{R_i}(b, B(R_i,b))$. Figure 2 (b) illustrates the construction.

One can easily see that if preferences belong to $\mathcal{R}$, then the amount $m(R_i,b)$ is the minimal budget necessary and sufficient for the existence of $\underline{b}$-*bounded non-contestable efficient* allocations in an economy where all agents have preferences R_i.

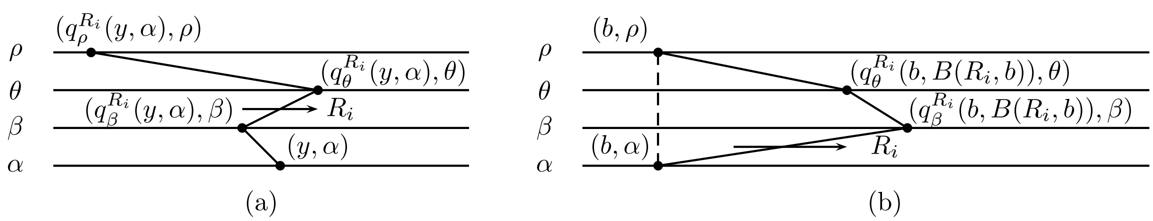

Figure 2: Each point y on the axis corresponding to object α represents bundle (y,α). Segments connect bundles that are indifferent for R_i ("indifference curve"). Panel (a) illustrates the monetary equivalent of (y,α) for R_i with respect to objects in $A \setminus \{\alpha\}$. Panel (b) illustrates $m(R_i,b)$ when $B(R_i,b) = \{\alpha,\rho\}$. Here $m(R_i,b)$ is the summation of the monetary equivalent of each object with respect to the best objects at b, i.e., $2b + q_\theta^{R_i}(b,B(R_i,b)) + q_\beta^{R_i}(b,B(R_i,b))$.

6.2 General existence and other corollaries

Our first application is the existence of *balanced budget non-contestable efficient* allocations in the restricted domain. Alkan et al. (1991) provide a proof of this result based on linear programming. They initially restrict preferences to a subdomain suitable for the application of this technique and then obtain existence in the full domain by taking a limit. A simple application of Theorem 1 avoids the construction.

Corollary 2 (Alkan et al., 1991)**.** *For each $e \equiv (R,M) \in \mathcal{E}$ such that $R \in \mathcal{R}^N$, $Nc(e) \cap Z(e) \neq \emptyset$.*

[20] $q_\beta^{R_i}(b,\alpha)$ is well defined since $R_i \in \mathcal{R}$.

The following corollaries are two applications of Theorem 2 that allow us to compare it to previous sufficient budgets in the literature. To facilitate the comparison, we deal just with non-emptiness of $NcPB_0$. Let $\Delta^{n-1}(M)$ be the simplex $\{x \in \mathbb{R}^N_+ : \sum_N x_i = M\}$.

Corollary 3 (Svensson, 1983, 1987). *Let $e = (R, M) \in \mathcal{E}$ be such that $R \in \mathcal{R}^N$. Assume that for each $i \in N$, and each $x \in \Delta^{n-1}(M)$, if there is $j \in \{1, \ldots, n\}$ such that for each $k \in \{1, \ldots, n\}$, $(x_j, \alpha_j) R_i (x_k, \alpha_k)$, then $x_j > 0$.*[21] *Then $NcB_0(e) \neq \emptyset$.*

Proof. We claim that under this assumption, $M \geq \max_{i \in N} m(R_i, 0)$. To prove this, suppose by means of contradiction that there is $i \in N$ such that $M < m(R_i, 0)$. Then, $\sum_{\alpha \in A} q_\alpha^{R_i}(b, B(R_i, b)) > M$ and there is $x \in \Delta^{n-1}(M)$ such that $x \leq (q_\alpha^{R_i}(b, B(R_i, b)))_{i \in N}$ and for each $j \in \{1, \ldots, n\}$, $(0, B(R_i, 0)) R_i (x_j, \alpha_j)$. This is a contradiction. Thus, by Theorem 2, $NcB_0(e) \neq \emptyset$. □

Corollary 4 (Maskin, 1987; Alkan et al., 1991). *Let $e = (R, M) \in \mathcal{E}$ be such that $R \in \mathcal{R}^N$. If $M \geq (n-1) \max_{(i,\alpha) \in N \times A} q_\alpha^{R_i}(0, B(R_i, 0))$, then $NcB_0(e) \neq \emptyset$.*

Proof. Under this assumption, for each $i \in N$ and each $\alpha \in A$, $\frac{M}{n-1} \geq q_\alpha^{R_i}(0, B(R_i, 0))$. Thus, $M \geq \max_{i \in N} m(R_i, 0)$. By Theorem 2, $NcB_0(e) \neq \emptyset$. □

Let us emphasize that for each economy $e \equiv (R, M) \in \mathcal{E}$ such that $\max_{i \in N} m(R_i, 0) > M \geq m_{(n-1)}(R, 0)$, Corollaries 3 and 4 do not imply non-emptyness of $NcB_0(e)$, while Theorem 2 does. Indeed, in the two-agent case, the condition in Theorem 2 is not only sufficient but also necessary for the non-emptiness of $NcPB_b$. We omit the straightforward proof.

Corollary 5. *Let $e = (R, M) \in \mathcal{E}$ be such that $R \in \mathcal{R}^N$ and $b \in \mathbb{R}$. If $n = 2$, then $NcB_b(e) \neq \emptyset$ if and only if $M \geq m_{(n-1)}(R, b)$.*

7 Discussion and Concluding Remarks

In this section we discuss the extension of our results.

Alternative extension of Foley's test. Kolm (1995) proposes an alternative formulation of equity in environments with consumption externalities. It is as follows: an allocation is equitable if no agent prefers the allocation at which all agents receive the consumption of some other agent, to the allocation at which all agents receive her own consumption. That is, $z \equiv (z_1, \ldots, z_n)$ is equitable if for each pair of agents i and j, $(z_i, \ldots, z_i)\, R_i\, (z_j, \ldots, z_j)$. Our definition of equity differs from Kolm's in that we internalize the feasibility constraint. In our environment with indivisible goods,

[21] Assumption A5'-II in Svensson (1987).

we find this feature especially appropriate. Our model accommodates situations like the allocation of the rooms and the division of the rent among housemates who collectively lease a house. In this environment Kolm's thought experiment would require, for instance, the agent who gets the basement, to compare the situation in which all agents live in basements to the situation in which all agents live in second floor rooms. This is a conceptually different exercise from the one we propose in which we ask each agent to imagine swapping her room and contribution to rent with each other housemate. At a technical level, the existence of equitable allocations under Kolm's definition is mathematically equivalent to the existence of *non-contestable* allocations in the externality-free domain $\mathcal{N}$.

Equal-budget compensation assumption. The equal budget compensation assumption is the minimum requirement on preferences for which we are able to guarantee existence of *non-contestable* allocations (Theorem 1). On the other hand, our proofs reveal that Theorem 2 holds in the larger domain of preferences satisfying continuity, anonymity of externalities, and the weaker form of the equal budget compensation assumption that does not require that as $b \to \infty$, $M(R_i, b) \to \infty$. An example of preferences that belong to this bigger domain is linear externality preferences for which some coefficients $c^i_\alpha(\beta)$ belong to the interval $\left(\frac{1}{\varphi(n)}, \frac{1}{n-1}\right)$ when $n > 3$.

Non-anonymous externalities. Our assumption of anonymity of externalities plays an important role in the proof of Theorem 2. In Step 3, it is essential to guarantee that the sequence constructed in Step 2 defines a *non-contestable* allocation in the limit. The following example shows that if externalities are non-anonymous, then *non-contestable* allocations may not exist.

Example 3. Let $A \equiv \{\alpha, \beta, \gamma\}$. Consider the economy (R, M) for the group of agents $N \equiv \{1, 2, 3\}$ in which preferences are respectively represented by:

$$u_1(z) \equiv \begin{cases} x_{\mu(1)} - 2 & \text{if } \mu(2) = \gamma \\ x_{\mu(1)} & \text{otherwise,} \end{cases} \quad u_2(z) \equiv \begin{cases} x_{\mu(2)} + 1 & \text{if } \mu(2) = \gamma \\ x_{\mu(2)} & \text{otherwise,} \end{cases} \quad u_3(z) \equiv x_{\mu(3)}.$$

These preferences are continuous and satisfy the ceteris-paribus-compensation assumption. Preferences of agents 2 and 3 are externality-free. In fact, they are quasi-linear. Agent 1's preferences are non-anonymous, for her welfare depends on the object received by agent 2. We claim that there is no *non-contestable* allocation for this economy. To see this, suppose that $(x, \mu) \in Nc(R, M)$. Clearly, $x_{\mu(3)} \geq \max\{x_{\mu(1)}, x_{\mu(2)}\}$. Now, since agent 1 does not prefer to swap her consumption with agent 3, then $x_{\mu(1)} \geq x_{\mu(3)}$. Thus, $x_{\mu(1)} = x_{\mu(3)} \geq x_{\mu(2)}$. Thus, $\mu(2) = \gamma$, for otherwise agent 2 would prefer to swap her consumption with the agent who receives object γ. Thus, $x_{\mu(2)} \geq x_{\mu(1)} - 1$, for otherwise agent 2 would prefer to swap her consumption with agent 1. Thus, agent 1 prefers to swap her consumption with agent 2. Consequently, (x, μ) cannot be a *non-contestable* allocation for (R, M). $\square$

Different number of agents and objects. Our results extend to the case in which there are more agents than objects. If $n > |A|$, our domain should be extended to allow the possibility that $n - |A|$ agents receive money and no object. If $|A| > n$, one can obtain the existence of *non-contestable* allocations in the basic domain by restricting to a subset of objects with cardinality n. Nevertheless, by following this approach one cannot guarantee that *balanced budget non-contestable* allocations that are *non-wasteful* are also *efficient*. For instance, in the restricted domain, if there are more objects than agents, then *non-contestable* allocations may not be *efficient* (Svensson, 1983).[22]

One can bypass this issue when preferences are externality-free. Let $b \in \mathbb{R}$ and $e \equiv (R, M)$ be such that $R \in \mathcal{N}^N$. Let $e^* \equiv (R^*, M)$ be obtained from e by including $|A| - n$ "dummy agents," i.e., an agent who is indifferent among all bundles that provide her equal consumption of money. For each dummy agent i^*, $m(R_{i^*}, b) = nb$. If $e = (R, M)$ is such that $M \geq m_{(n-1)}(R, b)$, then $M \geq m_{(n-1)}(R^*, b)$. By Theorem 2 $NcPB_b(e^*) \neq \emptyset$. Moreover, if $z = (x, \mu) \in NcP(e^*)$ and we denote N^* the set of dummy agents at e^*, then $z_N \in P(R, M - \sum_{N^*} x_i)$ (Alkan et al., 1991). In general, $e \neq (R, M - \sum_{N^*} x_i)$. However, the amounts of money received by the dummy agents can be chosen to be continuous functions of M (Velez, 2011). Thus, for each $e = (R, M)$ with $R \in \mathcal{N}$, there exist $e^* = (R^*, M^*)$ and $z \in NcP(e^*)$ such that the amount collectively received by all the dummy agents at z is exactly $M^* - M$. Thus, both Theorem 1 and 2 extend to the case $|A| > n$.[23]

If preferences exhibit externalities, the strategy of including dummy agents may not be successful anymore. The issue arises because including dummy agents implies extending preferences to a larger society. Since one needs that agents' preference statements hold after dummy agents are removed, then externalities in the economy with dummy agents become non-anonymous. Thus, existence of *non-contestable* allocations in the extended economy cannot be guaranteed. Indeed, assume that $A \equiv \{\alpha, \beta, \gamma\}$, $N \equiv \{1, 2\}$, and suppose that preferences are respectively represented by u_1 and u_2 in Example 3. Since there are only two agents, then externalities are

[22] Recall that in the restricted domain, if there is at least as many agents as objects, then *balanced budget non-contestable* allocations are *non-wasteful*.

[23] One can provide a tighter sufficient condition for the non-emptiness of $NcPB_b$ in the externality-free domain. Let $e \equiv (R, M)$ be such that $R \in \mathcal{N}^N$ and $e^* = (R^*, M)$ be obtained from e by including $|A| - n$ dummy agents. For each allocation $z \in NcPB_b(e^*)$, all dummy agents consume the same amount of money, which is, at least, the maximum consumption among the members of N. Thus, if $M \geq m_{(n-1)}(R, b)$, then each dummy agent consumes, at least, $\frac{1}{|A|} m_{(n-1)}(R, b)$, and we can guarantee non-emptiness of $NcPB_b(e)$ by assuming that $M \geq \frac{n}{|A|} m_{(n-1)}(R, b)$. Let us remark that even though the cardinality of A appears in the denominator of the expression $\frac{n}{|A|} m_{(n-1)}(R, b)$, this amount may not vanish as the proportion of agents relative to objects, $\frac{n}{|A|}$, vanishes.

anonymous for both agents. The economy obtained by including one dummy agent is precisely the economy in Example 3 for which we know no *non-contestable* allocation exists. It remains then an open question to identify conditions that guarantee the existence of *non-contestable* and *efficient* allocations when $|A| > n$.

8 Appendix

We divide the presentation of our proofs into two subsections. The first completes the proof of Theorem 2. The second contains the rest of the omitted proofs in the body of the paper. Additionally, we include Example 4, an economy in which no *non-contestable* allocation is *efficient*.

Proof of Theorem 2

We now complete the proof of the theorem. We introduce notation and prove two lemmas. Let $S^m \equiv \{\{x_1, \ldots, x_n\} \subseteq \mathbb{R}^m : x_2 - x_1, \ldots, x_n - x_1,$ are linearly indepentent$\}$. For each $X \equiv \{x_1, \ldots, x_n\} \in S^m$ the $(n-1)$-simplex generated by X, $\Delta^{(n-1)}(X)$, is the convex hull of X. The set all simplices in $\mathbb{R}^m$ is Λ^m. It can be proved that for each $X \in S^m$ and each $y \in \Delta^{(n-1)}(X)$, the representation in $\Delta^{(n-1)}(X)$ is unique, i.e., there is a unique vector of non-negative weights, $(\omega_i)_{i=1}^n$, such that $\sum_{i=1}^n \omega_i = 1$ and $y = \sum_{i=1}^n \omega_i x_i$ (Border, 1999; Vick, 1994). Thus, for each pair $\{X, Y\} \subseteq S^m$, if $\Delta^{(|X|-1)}(X) = \Delta^{(|Y|-1)}(Y)$, then $X = Y$. For each $\Delta \in \Lambda^m$, let $V(\Delta)$ be the set in S^m that generates Δ, i.e., the set of vertices of Δ.

Let $\Delta \in \Lambda^m$. A *triangulation of* Δ is a finite collection of simplices $T \subseteq \Lambda^m$ such that $\bigcup_{\Delta' \in T} \Delta' = \Delta$ and for each pair $\{\Delta', \Delta''\} \subseteq T$, if $\Delta' \cap \Delta''$ is not empty, then there is $Y \subseteq V(\Delta') \cap V(\Delta'')$ such that $\Delta' \cap \Delta''$ is the convex hull of Y. The *set of vertices of* T is $V(T) \equiv \bigcup_{\Delta' \in T} V(\Delta')$.

Let $\Delta \in \Lambda^m$ be an $(n-1)$-simplex. The *baricentric triangulation of* Δ, $T^B(\Delta)$, is the collection defined inductively (over n) as follows. For $n = 1$, let $T^B(\Delta) \equiv \{\Delta\}$ (in this case Δ is a singleton). Let $n \in \mathbb{N}$. Suppose that for each $(n-1)$-simplex Δ, $T^B(\Delta)$ has been defined. For each $Y \in S^m$, let $b(Y) \equiv \frac{1}{|Y|} \sum_{y \in Y} y$ be the *baricenter* of Y. Let $X \in S^m$ be such that $|X| = n+1$. The baricentric triangulation of $\Delta^n(X)$, $T^B(\Delta^n(X))$, is the collection $\{\Delta^n(V(\Delta') \cup \{b(X)\}) : Y \subseteq V(\Delta^n(X)), |Y| = n, \Delta' \in T^B(\Delta^{n-1}(Y))\}$. By construction, for each $X \in S^m$, $V(T^B(\Delta^{(|X|-1)}(X))) = \{b(Y) : Y \subseteq X\}$. Moreover, it can be proved that $T^B(\Delta^{(|X|-1)}(X))$ is a triangulation (Vick, 1994).

Let $\Delta \in \Lambda^m$. The *k-baricentric triangulation of* Δ, $T_k^B(\Delta)$, is the collection defined inductively (over k) as follows. For $k = 1$, let $T_1^B(\Delta) \equiv T^B(\Delta)$. For $k > 1$ let $T_k^B(\Delta) \equiv \bigcup_{\Delta' \in T_{k-1}^B(\Delta)} T^B(\Delta')$. It can be proved that for each $\Delta \in \Lambda^m$ and each

$k \in \mathbb{N}$, $T_k^B(\Delta)$ is a triangulation and mesh $T_k^B(\Delta) \equiv \max_{\{x,y\}\subseteq\Delta',\Delta'\in T_k^B(\Delta)} \|x-y\| \to 0$ as $k \to \infty$ (Vick, 1994, Corollary I.3, Pag. 213).[24]

Let $\Delta \in \Lambda^m$ and T be a triangulation of Δ. An *ownership labeling of* T is a function $L : V(T) \to \{1, ..., |V(\Delta)|\}$.

Lemma 7. *Let $\Delta \in \Lambda^n$. There is a sequence of triangulations of Δ, $\{T_k\}_{k\in\mathbb{N}}$, and corresponding ownership labelings, $\{L_k\}_{k\in\mathbb{N}}$, such that: (i) mesh $T_k \to 0$ as $k \to \infty$, (ii) for each $k \in \mathbb{N}$ and each $\Delta' \in T_k$, $L_k(V(\Delta')) = \{1, \ldots, n\}$, i.e., each simplex in T_k is fully labeled, and (iii) the set $L_k^{-1}(n)$ is interior, i.e., all the vertices labeled n are interior.*

Proof. For each arbitrary simplex $\Delta' \in \Lambda^n$ let $L_1^{\Delta'} : V(T^B(\Delta')) \to \{1, \ldots, n\}$ be the labeling such that for each $x \in V(T^B(\Delta'))$, $L_1^{\Delta'}(x) = |Y|$, where $Y \subseteq V(\Delta')$ is the set such that $x = b(Y)$ ($L_1^{\Delta'}$ is well defined because Y is unique). From the inductive construction of baricentric triangulations, it follows that each simplex in $T^B(\Delta')$ has as vertices the baricenters of sets with different cardinalities. Thus, for each $\Delta'' \in T^B(\Delta')$, $L_1^{\Delta'}(V(\Delta'')) = \{1, \ldots, n\}$. Moreover, the only vertex labeled n is $b(V(\Delta'))$.

Let $\Delta \in \Lambda^m$ and $\{T_k\}_{k\in\mathbb{N}} \equiv \{T_k^B(\Delta)\}_{k\in\mathbb{N}}$. We know that mesh $T_k \to 0$ as $k \to \infty$. We define a sequence of labelings $\{L_k\}_{k\in\mathbb{N}}$ with the desired properties. We proceed inductively over k. Suppose without loss of generality that $|V(\Delta)| = n$. For $k = 1$, let $L_1 \equiv L_1^{\Delta}$. L_1 inherits from L_1^{Δ} the property that for each $\Delta' \in T_1$, $L_1(V(\Delta')) = \{1, \ldots, n\}$. Moreover, the only vertex labeled n is $b(V(\Delta))$. For each $k > 1$, let $L_k^{\Delta} : V(T_k) \to \{1, \ldots, n\}$ be such that for each $x \in V(T_k)$, $L_k(x) \equiv L_1^{\Delta'}(x)$, for some $\Delta' \in T_{k-1}$ such that $x \in \Delta'$. We claim that L_k is a well-defined function. To prove this, let $x \in V(T_k)$ and $\{\Delta', \Delta''\} \subseteq T_{k-1}$ be such that $x \in \Delta' \cap \Delta''$. We prove that $L_1^{\Delta'}(x) = L_1^{\Delta''}(x)$. Since T_{k-1} is a triangulation, then $\Delta' \cap \Delta''$ is a common face of Δ' and Δ'', and since the representation of x in Δ' and Δ'' is unique then the sets $Y' \subseteq \Delta'$ and $Y'' \subseteq \Delta''$ such that $x = b(Y') = b(Y'')$, are equal. Thus, $L_1^{\Delta'}(x) = L_1^{\Delta''}(x)$. Finally, L_k inherits from $L_1^{(\cdot_{\Delta'})}$ the property that for each $\Delta' \in T_k$, $L_k(V(\Delta')) = \{1, \ldots, n\}$, and that the only vertex labeled n is $b(V(\Delta'))$. Thus, the vertices labeled n by L_k are interior. □

Let $\Delta \in \Lambda^m$ and T be a triangulation of Δ. Recall that A is a set of objects such that $|A| = |N|$. An *object labeling* of a triangulation T is a function $L : V(T) \to A$. An object labeling of T, L, is a *Sperner labeling* if: (i) $L(V(\Delta)) = A$, and (ii) for each $x \in V(T)$, $L(x) \in \{L(y) : y \in V(\Delta), x = \sum_{y\in V(\Delta)} \omega_y y, \omega_y > 0\}$.

Let $i \in N$ and $R_i \in \mathcal{B}$. For each $x \in \mathbb{R}^A$, let $\arg\max(R_i, x) \subseteq A$ be the set of objects $\alpha \in A$ such that for each bijection $\mu \in A^N$ such that $\mu(i) = \alpha$ and each bijection $\mu' \in A^N$, $(x, \mu)\, R_i\, (x, \mu')$.

[24] See also Munkres (1984, § 15, Pag. 83) for a general construction of Baricentric Triangulations.

Lemma 8. *Let $e = (R, M) \in \mathcal{E}$ and $b \in \mathbb{R}$; let $\{T_k\}_{k\in\mathbb{N}}$ be a sequence of triangulations of $\Delta^{n-1}(b, M)$ and $\{L_k\}_{k\in\mathbb{N}}$ be a sequence of labelings satisfying the properties in Lemma 7. If $M > m_{(n-1)}(R, b)$ and $n \in \arg\max m(R_i, b)$, then there is an object labeling $L_k^R : V(T_k) \to A$ such that for each $x \in V(T_k)$, $L_k^R(x) \in \{\alpha \in A : x_\alpha > b$ and $\alpha \in \arg\max(R_i, x)\}$, and such that it is a Sperner labeling.*

Proof. Let $k \in \mathbb{N}$. Since externalities are anonymous, then for each $x \in \mathbb{R}^A$, $\arg\max(R_i, x) \neq \emptyset$. We prove that for each $x \in V(T_k)$, $\{\alpha \in A : x_\alpha > b$ and $\alpha \in \arg\max(R_i, x)\} \neq \emptyset$. The claim holds trivially for interior vertices. Suppose that $x \in T$ is not interior. Then, $L_k(x) < n$. Since $m(R_{L_k(x)}, b) \leq m_{(n-1)}(R, b) < M$, then by the definition of $m(R_{L_k(x)}, b)$, there is $\alpha \in \arg\max(R_i, x)$ such that $x_\alpha > b$. Now we prove that any L_k^R satisfying the property above is a Sperner labeling. For each $x \in V(\Delta^{n-1}(b, M))$, $L_k^R(x)$ is such that $x_{L_k^R(x)} = M - (n-1)b$, and thus, $\arg\max(R_i, x) = \{L_k^R(x)\}$. Therefore, $L_k(V(\Delta^{n-1}(b, M))) = A$. Now, since for each $y \in T$, $x_{L_k^R(y)} > b$, then $\omega_{L_k^R(y)} > 0$ where $y = \sum_{x_i \in A} \omega_i x_i$. □

Proof of other results

We present the proof of results different from Theorem 2 in the order in which they are stated in the body of the paper. For convenience we make an exception. We prove Proposition 1 last. Since this proposition is not used to prove the later results, there is no circularity.

Proof of Lemma 1. Let $R_i \in \mathcal{C}$ and consider $\{\alpha, \beta\} \subseteq A$ and $b \in \mathbb{R}$. Let $V_\alpha^\beta(R_i, b) \subseteq \mathbb{R}$ be the set of real numbers $y \geq b$ such that agent i when receiving α and an amount of money at most b, would prefer to swap her consumption with an agent who receives β and an amount of money at least y. That is, $y \in V_\alpha^\beta(R_i, b)$ if (i) $y \geq b$ and (ii) for each $x \in \mathbb{R}^A$ such that $x_\beta \geq y$ and $x_\alpha \leq b$, each bijection $\mu \in A^N$ such that $\mu(i) = \alpha$, and each bijection $\mu' \in A^N$ such that $\mu'(i) = \beta$, we have $(x, \mu')\, R_i\, (x, \mu)$.

Step 1: Let $i \in N$, $\{\alpha, \beta\} \subseteq A$, and $b \in \mathbb{R}$. For each $R_i \in \mathcal{B}$, the set $V_\alpha^\beta(R_i, b)$ is non-empty and has a minimum.

We first claim that $V_\alpha^\beta(R_i, b) \neq \emptyset$. Since externalities are anonymous, then the ceteris-paribus-compensation assumption implies that there is $b' \in \mathbb{R}$ such that for each $x \in \mathbb{R}^A$ such that $x_\beta \geq b'$ and $x_\alpha \leq b$, each bijection $\mu \in A^N$ such that $\mu(i) = \alpha$, and each bijection $\mu' \in A^N$ such that $\mu'(i) = \beta$, we have $(x, \mu')\, R_i\, (x, \mu)$. If $b' \geq b$ then $b' \in V_\alpha^\beta(R_i, b)$. If $b' < b$, then $b \in V_\alpha^\beta(R_i, b)$. Thus, $V_\alpha^\beta(R_i, b) \neq \emptyset$. Let $v \equiv \inf V_\alpha^\beta(R_i, b)$. Since $V_\alpha^\beta(R_i, b)$ is bounded below by b, then $v \in \mathbb{R}$. We claim that $v \in V_\alpha^\beta(R_i, b)$. Let $x \in \mathbb{R}^A$ be such that $x_\beta \geq v$ and $x_\alpha \leq b$, $\mu \in A^N$ be a bijection such that $\mu(i) = \alpha$, and $\mu' \in A^N$ be a bijection such that $\mu'(i) = \beta$. We claim that $(x, \mu')\, R_i\, (x, \mu)$. There are two cases.

Case 1: $x_\beta > v$. Since $v = \inf V_\alpha^\beta(R_i, b)$, then there is a sequence in $V_\alpha^\beta(R_i, b)$, $\{v^k\}_{k\in\mathbb{N}}$, such that $v^k \underset{k\to\infty}{\to} v$. Since $x_\beta > v$, then there is $k \in \mathbb{N}$ such that $v^k \leq x_\beta$. Now, since $v^k \in V_\alpha^\beta(R_i, b)$, then $(x, \mu')\, R_i\, (x, \mu)$.

Case 1: $x_\beta = v$. By Case 1, for each $k \in \mathbb{N}$, $(x + \frac{1}{k}1_\beta, \mu')\, R_i\, (x + \frac{1}{k}1_\beta, \mu)$. Now, since preferences are continuous, then $(x, \mu')\, R_i\, (x, \mu)$.

Therefore, $v \in V_\alpha^\beta(R_i, b)$ and $v = \min V_\alpha^\beta(R_i, b)$.

Step 2: We now identify a minimal budget guaranteeing that agent i would prefer to swap her consumption with some other agent at each $\underline{b}$-*bounded* allocation at which her consumption of money is b. For each $i \in N$, each $b \in \mathbb{R}$, and each $R_i \in \mathcal{B}$, let

$$M(R_i, b) \equiv \max_{\alpha\in A}\left(b + \sum_{\beta\in A\setminus\{\alpha\}} \min V_\alpha^\beta(R_i, b)\right).$$

More precisely, let $x \in \mathbb{R}^A$ be such that $x \gneq b$ and $\sum_{\alpha\in A} x_\alpha > M(R_i, b)$. In the following step we prove that among the allocations with distribution x, agent i prefers one in which her consumption of money is greater than b.

Step 3: Let $i \in N$, $R_i \in \mathcal{B}$, and $b \in \mathbb{R}$. Assume that $M > M(R_i, b) \geq nb$. Then, for each $x \in \mathbb{R}^A$ such that $x \geq b$ and $\sum_{\alpha\in A} x_\alpha = M$, there is $\beta \in A$ such that $x_\beta > b$, and for each $\alpha \in A$, each bijection $\mu \in A^N$ such that $\mu(i) = \alpha$, and each bijection $\mu' \in A^N$ such that $\mu'(i) = \beta$, we have $(x, \mu')\, R_i\, (x, \mu)$.

Let $x \in \mathbb{R}^A$ be such that $x \geq b$ and $\sum_{\alpha\in A} x_\alpha = M$. Henceforth in this proof we only deal with allocations with distribution x. Since externalities are anonymous, we abuse notation and make preference statements between consumption bundles instead of allocations. Let $\arg\max(R_i, x) \subseteq A$ be the set of objects, α, such that for each $\beta \in A$, $(x_\alpha, \alpha)\, R_i\, (x_\beta, \beta)$. Since $|A| < \infty$, then $\arg\max(R_i, x)$ is non-empty. We claim that there is $\delta \in \arg\max(R_i, x)$ such that $x_\delta > b$. Suppose by means of contradiction that for each $\alpha \in \arg\max(R_i, x)$, $x_\alpha = b$. Let $\alpha \in \arg\max(R_i, x)$. Thus, if $\beta \in A$ is such that $x_\beta > b$, then $(x_\alpha, \alpha)\, P_i\, (x_\beta, \beta)$. Therefore, $]-\infty, x_\beta] \cap V_\alpha^\beta(R_i, b) = \emptyset$, for otherwise $(x_\beta, \beta)\, R_i\, (x_\alpha, \alpha)$. Thus, $x_\beta < \min V_\alpha^\beta(R_i, b)$. Now, since $M > |N|b$, then $A \setminus \arg\max(R_i, x) \neq \emptyset$. Thus,

$$x_\alpha + \sum_{\gamma\in\arg\max(R_i,x)\setminus\{\alpha\}} x_\gamma + \sum_{\beta\in A\setminus\arg\max(R_i,x)} x_\beta < b + \sum_{\beta\in A\setminus\{\alpha\}} \min V_\alpha^\beta(R_i, b) \leq M(R_i, b).$$

Thus, $M < M(R_i, b)$. This is a contradiction.

Step 4: Let $e = (M, R) \in \mathcal{E}$. Recall that for each $b \in \mathbb{R}$ and each $i \in N$,

$$M(R_i, b) \equiv \max_{\alpha\in A}\left(b + \sum_{\beta\in A\setminus\{\alpha\}} \min V_\alpha^\beta(R_i, b)\right).$$

Let $i \in N$. We claim that the function $b \mapsto \min V_\alpha^\beta(R_i, b)$ is weakly increasing. Let $\{\alpha, \beta\} \subseteq A$ and $R_i \in \mathcal{B}$. We prove that for each pair $\{b, b'\} \subseteq \mathbb{R}$ such that $b \leq b'$, $\min V_\alpha^\beta(R_i, b) \leq \min V_\alpha^\beta(R_i, b')$. It is enough to prove that $V_\alpha^\beta(R_i, b) \supseteq V_\alpha^\beta(R_i, b')$. Let $y \in V_\alpha^\beta(R_i, b')$. Let $x \in \mathbb{R}^A$ be such that $x_\alpha \leq b \leq b'$ and $x_\beta \geq y$, $\mu \in A^N$ a bijection such that $\mu(i) = \alpha$, and $\mu' \in A^N$ a bijection such that $\mu'(i) = \beta$. Since $y \in V_\alpha^\beta(R_i, b')$, then $(x, \mu')\, R_i\, (x, \mu)$. Thus, $y \in V_\alpha^\beta(R_i, b)$.

Since $b \in \mathbb{R} \mapsto \min V_\alpha^\beta(R_i, b)$ is weakly increasing, then $M(R_i, b) \to -\infty$ as $b \to -\infty$. □

Proof of Theorem 1. Since R_i satisfies *equal budget compensation assumption* then, as $b \to -\infty$, $m(R_i, b) \to \infty$. Thus, there is $b^* \in \mathbb{R}$ such that for each $i \in N$, $m(R_i, b^*) \leq M$. Thus, $m_{(n-1)}(R, b^*) \leq M$. By Theorem 2, $NcB_{b^*}(e) \cap Z(M) \neq \emptyset$. Since $Nc(e) \cap Z(M) \supseteq NcB_{b^*}(e) \cap Z(M)$, then $Nc(e) \cap Z(M) \neq \emptyset$. □

Proof of Proposition 2. To avoid trivialities, assume $M > nb$. Fix $\alpha \in A$. Let R_i and R_j be two identical preferences in $\mathcal{R}$ such that for each $\beta \in A \backslash \{\alpha\}$, $(b, \alpha)\, P_i\, (M - (n-1)b + 1, \beta)$. Since at most one agent receives a bundle containing object α, then at an allocation $z \in B_b(R_{-\{i,j\}}, R_i, R_j, M)$ such that $z \in Z(e)$, at least one agent from $\{i, j\}$ consumes an amount of money which is at most $M - (n-1)b + 1$ and an object different from α. That agent prefers (b, α) to z. Thus, $z \notin Nc(R_{-\{i,j\}}, R_i, R_j, M)$. □

Example 4. This example is an economy $(R, M) \in \mathcal{E}$ such that: (i) $P(R, M) \neq \emptyset$, and (ii) $P(R, M) \cap Nc(R, M) = \emptyset$. Let $A \equiv \{\alpha, \beta, \gamma\}$ and $N \equiv \{i, j, k\}$. For each $l \in N$, R_l is represented by the utility function, u_l, defined as follows: for each $z \equiv (x, \mu) \in Z$,

$$u_l(z) \equiv \begin{cases} x_\alpha + f(x_\beta) & \text{if } \mu(l) = \alpha, \\ f(x_\alpha) + x_\beta & \text{if } \mu(l) = \beta, \\ x_\gamma & \text{if } \mu(l) = \gamma. \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ is the function defined by: for each $a \in \mathbb{R}$,

$$f(a) \equiv \begin{cases} 0 & \text{if } a \in]-\infty, -2], \\ a + 2 & \text{if } a \in]-2, -1], \\ -a & \text{if } a \in]-1, 0], \\ 0 & \text{if } a \in]0, +\infty[. \end{cases}$$

One can easily see that $R \in \mathcal{M}^N$.

(i) $P(R, 0) = \emptyset$. Let $x \in \mathbb{R}^A$ be such that $x_\alpha = -4$, $x_\beta = -4$, and $x_\gamma = 8$. Let $\mu \in A^N$ be a bijection. We claim that $(x, \mu) \in P(R, 0)$. Let $z' = (x', \mu') \in Z(R, 0)$ be such that $z'\, R\, (x, \mu)$. We claim that $x' = x$. Suppose without loss of generality that $\mu(i) = \gamma$. Then, $x'_{\mu(i)} \geq 8$. Feasibility implies that for another agent, say j,

$x_{\mu(j)} \leq -4$. Thus, $x_{\mu(j)} = -4$. By the same argument, $x_{\mu(k)} = -4$ and $x_{\mu(i)} = 8$. Clearly, if $z' \, R \, (x, \mu)$, then $(x, \mu) \, R \, z'$.

(ii) $P(R, 0) \cap \boldsymbol{Nc}(R, 0) = \emptyset$. Since for each $\varepsilon \in \mathbb{R}_{++}$ and each $(x, \mu) \in Z$, $(z + \varepsilon 1_\gamma, \mu) \, R \, z$ and $(z + \varepsilon 1_\gamma, \mu) \, P_{\mu^{-1}(\gamma)} \, z$, then $P(R, 0) \subseteq Z(0)$. We prove that for each $(x, \mu) \in Nc(R, 0) \cap Z(0)$, $x = 0$. Let $(x, \mu) \in Nc(R, 0) \cap Z(0)$. Suppose w.l.o.g that $\mu(i) = \alpha$, $\mu(j) = \beta$, and $\mu(k) = \gamma$. First we show that $x_\alpha \geq 0$. Suppose by means of contradiction that $x_\alpha < 0$. We claim that $x_\beta \leq 0$ and $u_i(x, \mu) \leq 0$. Since $(x, \mu) \in Nc(R, 0)$, then $u_i(x, \mu) = u_j(x, \mu)$. Let $\varphi_\alpha : \mathbb{R} \to \mathbb{R}$ be the function defined by: for each $b \in \mathbb{R}$, $\varphi_\alpha(b) \equiv x_\alpha + f(b)$. Let $\varphi_\beta : \mathbb{R} \to \mathbb{R}$ be the function defined by: for each $b \in \mathbb{R}$, $\varphi_\beta(b) \equiv b + f(x_\alpha)$. Since $u_i(x, \mu) = u_j(x, \mu)$, then $\varphi_\alpha(x_\beta) = \varphi_\beta(x_\beta)$. There are three cases.

Case 1: $x_\alpha \in]-1, 0[$. Then, the functions φ_α and φ_β intersect only at $b = x_\alpha$. Thus, $x_\beta = x_\alpha$. Thus, $x_\beta < 0$ and $u_i(x, \mu) = 0$.

Case 2: $x_\alpha = -1$. Then, the functions φ_α and φ_β intersect on the segment $[-2, -1]$. $b = x_\alpha$. Thus, $x_\beta \in [-2, -1]$. Thus, $x_\beta \leq 0$ and $u_i(x, \mu) = 0$.

Case 1: $x_\alpha \in]-\infty, -2[$. Since $x_\alpha < -2$, then $u_i(x, \mu) \leq 0$. Since φ_β is monotonically increasing and $\varphi_\beta(0) \geq 0$, then $x_\beta \leq 0$.

Now, since $x_\alpha < 0$, $x_\beta < 0$, and $(x, \mu) \in Z(0)$, then $x_\gamma > 0$. Thus, $u_k(x, \mu) > 0$. Since $(x, \mu) \in Nc(R, 0)$, then $u_i(x, \mu) > 0$. This is a contradiction.

A symmetric argument shows that $x_\beta \geq 0$. Thus, $x_\gamma \leq 0$.

We claim that $x_\gamma = 0$. Suppose by means of contradiction that $x_\gamma < 0$. Thus, $u_k(x, \mu) < 0$. Now, since $x_\alpha \geq 0$ and $x_\beta \geq 0$, then $u_i(x, \mu) \geq 0$. Since $(x, \mu) \in Nc(R, 0)$, then $u_k(x, \mu) > 0$. Thus is a contradiction. Thus, $x = 0$.

Now, let $(x, \mu) \in Nc(R, 0) \cap Z(0)$. Since $x = 0$, then for each $l \in N$, $u_l(x, \mu) = 0$. Now, let $\Delta \in]-\frac{1}{2}, 0]$. Let $x' \equiv x - 1_\alpha \Delta - 1_\beta \Delta + 1_\gamma 2\Delta$. Clearly $u_i(x', \mu) = u_j(x', \mu) = 0$ and $u_k(x', \mu) > 0$. Thus, $(x, \mu) \notin P(R, 0)$. $\square$

Proof of Lemma 2. Clearly $\mathcal{Q} \subseteq \mathcal{L}$. Now, consider $R_i \in \mathcal{L}$ represented by

$$u(z) \equiv v(\mu(i)) + x_{\mu(i)} + \sum_{\alpha \in A \setminus \{\mu(i)\}} c_{\mu(i)}(\alpha) x_\alpha.$$

To simplify notation we write c_α^β instead of c_α^β. Let $b \in \mathbb{R}$ and $M \in R$. Let $z \equiv (x, \mu) \in Z(M)$ be such that $x_{\mu(i)} = b$ and for each $\alpha \in A$, $x_\alpha \geq b$. We prove that there is $M(R_i, b) \geq nb$ such that if $M > M(R_i, b)$ and μ' is a bijection such that $x_{\mu'(i)} = \max_{\alpha \in A} x_\alpha$, then $(x, \mu') \, R_i \, (x, \mu)$. Moreover, we prove that as $b \to -\infty$, $M(R_i, b) \to -\infty$.

Suppose without loss of generality that $\mu(i) = \alpha$ and $\mu'(i) = \beta$. We prove that there is $M(R_i, b)$ such that if $M > M(R_i, b)$, then

$$v(\alpha) + b + c_\alpha^\beta x_\beta + \sum_{\delta\in A\setminus\{\alpha,\beta\}} c_\alpha^\delta x_\delta \le v(\beta) + c_\beta^\alpha b + x_\beta + \sum_{\delta\in A\setminus\{\alpha,\beta\}} c_\beta^\delta x_\delta.$$

Equivalently,

$$\Phi \equiv v(\alpha) - v(\beta) + \sum_{\delta\in A\setminus\{\alpha,\beta\}} (c_\alpha^\delta - c_\beta^\delta)x_\delta \le (1 - c_\alpha^\beta)x_\beta + (c_\beta^\alpha - 1)b \equiv \Upsilon.$$

Let $\Gamma \equiv \max_{\{\alpha,\beta\}\subseteq A}(v(\alpha) - v(\beta))$. Then,

$$\begin{aligned}
\Phi &\le \Gamma + \textstyle\sum_{\delta\in A\setminus\{\alpha,\beta\}}(c_\alpha^\delta - c_\beta^\delta)x_\delta \\
&= \Gamma + \textstyle\sum_{\delta\in A\setminus\{\alpha,\beta\}}(c_\alpha^\delta - c_\beta^\delta)(x_\delta - b) + \sum_{\delta\in A\setminus\{\alpha,\beta\}}(c_\alpha^\delta - c_\beta^\delta)b \\
&\le \Gamma + \textstyle\sum_{\delta\in A\setminus\{\alpha,\beta\}} c_\alpha^\delta(x_\delta - b) + \sum_{\delta\in A\setminus\{\alpha,\beta\}}(c_\alpha^\delta - c_\beta^\delta)b \\
&\le \Gamma + \textstyle\sum_{\delta\in A\setminus\{\alpha,\beta\}} c_\alpha^\delta(x_\beta - b) + \sum_{\delta\in A\setminus\{\alpha,\beta\}}(c_\alpha^\delta - c_\beta^\delta)b \\
&\le \Gamma + (x_\beta - b)\textstyle\sum_{\delta\in A\setminus\{\alpha,\beta\}} c_\alpha^\delta + \sum_{\delta\in A\setminus\{\alpha,\beta\}}(c_\alpha^\delta - c_\beta^\delta)b
\end{aligned}$$

Suppose without loss of generality that for at least a pair $\{\alpha, \beta\} \subseteq A$, $\alpha \ne \beta$, $c_\alpha^\beta > 0$. Let $k \equiv \max_{\{\alpha,\beta\}\subseteq A, \alpha\ne\beta, c_\alpha^\beta>0} \frac{1}{c_\alpha^\beta} > \max\left\{\left(2 - \frac{3}{n}\right)(n-1), n-1\right\}$. Then,

$$\Phi \le \Gamma + (x_\beta - b)\left(\frac{n-2}{k}\right) + \sum_{\delta\in A\setminus\{\alpha,\beta\}} (c_\alpha^\delta - c_\beta^\delta)b.$$

Now,

$$\begin{aligned}
\Upsilon &\ge x_\beta\left(1 - \tfrac{1}{k}\right) + b\left(\tfrac{1}{k} - 1\right) + x_\beta\tfrac{1}{k}1_{x_\beta\le 0} - b\tfrac{1}{k}1_{b\ge 0} \\
&= (x_\beta - b)\left(1 - \tfrac{1}{k}\right) + x_\beta\tfrac{1}{k}1_{x_\beta\le 0} - b\tfrac{1}{k}1_{b\ge 0} \\
&= (x_\beta - b)\left(1 - \tfrac{1}{k}\right) + (x_\beta - b)\tfrac{1}{k}1_{x_\beta\le 0} - b\tfrac{1}{k}1_{b\ge 0} + b\tfrac{1}{k}1_{x_\beta\le 0} \\
&\ge (x_\beta - b)\left(1 - \tfrac{1}{k}\right) - b\tfrac{1}{k}1_{b\ge 0} + b\tfrac{1}{k}1_{x_\beta\le 0} \\
&\ge (x_\beta - b)\left(1 - \tfrac{1}{k}\right) - b\tfrac{1}{k}1_{b\ge 0} + b\tfrac{1}{k}1_{b\le 0} \\
&= (x_\beta - b)\left(1 - \tfrac{1}{k}\right) - |b|\tfrac{1}{k}.
\end{aligned}$$

Thus, $\Phi \le \Upsilon$ whenever

$$\Gamma + (x_\beta - b)\left(\frac{n-2}{k}\right) + \sum_{\delta\in A\setminus\{\alpha,\beta\}} (c_\alpha^\delta - c_\beta^\delta)b \le (x_\beta - b)\left(1 - \frac{1}{k}\right) - |b|\frac{1}{k}.$$

Equivalently,

$$\Gamma + \sum_{\delta\in A\setminus\{\alpha,\beta\}} (c_\alpha^\delta - c_\beta^\delta)b - |b|\frac{1}{k} \le (x_\beta - b)\left(\frac{k-(n-1)}{k}\right).$$

Since $k > n-1$ and $x_\beta - b \geq \frac{M-nb}{n-1}$, then the inequality above is satisfied whenever

$$\Gamma + \sum_{\delta \in A \setminus \{\alpha,\beta\}} (c_\alpha^\delta - c_\beta^\delta) b - |b| \frac{1}{k} \leq \frac{M-nb}{n-1} \left(\frac{k-(n-1)}{k} \right).$$

Equivalently, $M(R_i, b) \leq M$ where

$$M(R_i, b) \equiv \frac{k(n-1)}{k-(n-1)} \left(\Gamma + \sum_{\delta \in A \setminus \{\alpha,\beta\}} (c_\alpha^\delta - c_\beta^\delta) b - |b| \frac{1}{k} + \frac{nb}{n-1} \left(\frac{k-(n-1)}{k} \right) \right).$$

If $b < 0$, then the factor that multiplies b inside the parenthesis in the expression above is

$$\Psi \equiv \sum_{\delta \in A \setminus \{\alpha,\beta\}} (c_\alpha^\delta - c_\beta^\delta) + \frac{1}{k} + \frac{n}{n-1} \left(\frac{k-(n-1)}{k} \right).$$

Now,

$$\begin{aligned} \Psi &\geq \tfrac{1}{k} + \tfrac{n}{n-1} \left(\tfrac{k-(n-1)}{k} \right) - \textstyle\sum_{\delta \in A \setminus \{\alpha,\beta\}} c_\beta^\delta \\ &\geq \tfrac{1}{k} + \tfrac{n}{n-1} \left(\tfrac{k-(n-1)}{k} \right) - \tfrac{n-2}{k} \\ &= \tfrac{n}{n-1} + \tfrac{3-2n}{k} \end{aligned}.$$

Since $k > \left(2 - \frac{3}{n}\right)(n-1)$, then $\Psi > 0$. Thus, as $b \to -\infty$, $M(R_i, b) \to -\infty$. □

Proof of Proposition 4. Let $R \in \mathcal{L}^N$ and $e \equiv (R, M) \in \mathcal{E}$. We prove that if $z \equiv (x, \mu) \in Nc(e) \cap Z(M)$, then z is *non-wasteful* for e. Suppose by contradiction that there is $\hat{z} \equiv (\hat{x}, \hat{\mu}) \in Z(e)$ such that $\hat{z}\, R\, z$ and there is $\alpha \in A$ such that $\hat{x}_\alpha < x_\alpha$. Let $\alpha \in \arg\min\{\hat{x}_\beta - x_\beta : \beta \in A\}$. Let $\varepsilon \equiv \hat{x}_\alpha - x_\alpha$. Let $i \in N$ be such that $\hat{\mu}(i) = \alpha$. Since $z \in Nc(e)$, then $(x, \mu)\, R_i\, (x, \hat{\mu})$. Let u be the representation of $R_i \in \mathcal{L}$ defined by

$$u(z) \equiv v(\mu(i)) + x_{\mu(i)} + \sum_{\alpha \in A \setminus \{\mu(i)\}} c_{\mu(i)}(\alpha) x_\alpha.$$

We claim that $(x, \hat{\mu})\, P_i\, (\hat{x}, \hat{\mu})$, i.e.,

$$x_{\hat{\mu}(i)} + \sum_{\alpha \in A \setminus \{\hat{\mu}(i)\}} c_{\hat{\mu}(i)}(\alpha) x_\alpha > \hat{x}_{\hat{\mu}(i)} + \sum_{\alpha \in A \setminus \{\hat{\mu}(i)\}} c_{\hat{\mu}(i)}(\alpha) \hat{x}_\alpha.$$

Equivalently,

$$(x_{\hat{\mu}(i)} - \hat{x}_{\hat{\mu}(i)}) + \sum_{\alpha \in A \setminus \{\hat{\mu}(i)\} : x_\alpha \geq \hat{x}_\alpha} c_{\hat{\mu}(i)}(\alpha)(x_\alpha - \hat{x}_\alpha) + \sum_{\alpha \in A \setminus \{\hat{\mu}(i)\} : x_\alpha < \hat{x}_\alpha} c_{\hat{\mu}(i)}(\alpha)(x_\alpha - \hat{x}_\alpha) > 0.$$

This inequality is satisfied whenever,

$$\varepsilon = x_{\hat{\mu}(i)} - \hat{x}_{\hat{\mu}(i)} > \sum_{\alpha \in A \setminus \{\hat{\mu}(i)\}: x_\alpha < \hat{x}_\alpha} c_{\hat{\mu}(i)}(\alpha)(\hat{x}_\alpha - x_\alpha) \equiv \Upsilon.$$

Now,

$$\begin{aligned} \Upsilon &\leq \max_{\{\alpha,\beta\}} c_\alpha(\beta) \textstyle\sum_{\alpha \in A \setminus \{\hat{\mu}(i)\}: x_\alpha < \hat{x}_\alpha} (\hat{x}_\alpha - x_\alpha) \\ &= \max_{\{\alpha,\beta\}} c_\alpha(\beta) \textstyle\sum_{\alpha \in A \setminus \{\hat{\mu}(i)\}: x_\alpha \geq \hat{x}_\alpha} (x_\alpha - \hat{x}_\alpha) \\ &< \tfrac{1}{n-1}(n-1)\varepsilon = \varepsilon. \end{aligned}$$

This contradicts $\hat{z} \, R_i \, z$. □

Proof of Lemma 3. Let $i \in N$ and $R_i \in \mathcal{F}$. Thus, there are $\{a, c\} \subseteq \mathbb{R}_+$ such that $c < 1$, $c \leq a$, and a family of functions $\nu \equiv \{\nu_\alpha\}$, $\nu : A \to \mathbb{R}$ such that R_i is represented by the following utility function: for each $z \equiv (x, \mu) \in Z$,

$$\begin{aligned} u_i(z) \equiv \; & \nu_{\mu(i)}(\mu(i)) + x_{\mu(i)} - \tfrac{a}{n-1} \textstyle\sum_{\beta \in A} \max\{\nu_{\mu(i)}(\beta) + x_\beta - (\nu_{\mu(i)}(\mu(i)) + x_{\mu(i)}), 0\} \\ & - \tfrac{c}{n-1} \textstyle\sum_{\beta \in A} \max\{\nu_{\mu(i)}(\mu(i)) + x_{\mu(i)} - (\nu_{\mu(i)}(\beta) + x_\beta), 0\}. \end{aligned}$$

We claim that $R_i \in \mathcal{B}$. For each $z \equiv (x, \mu) \in Z$, $u_i(z)$ depends only on x and $\mu(i)$. Thus, externalities are anonymous for R_i. Clearly, R_i is continuous.

(i) R_i is own-consumption money-monotone. Let $z \equiv (x, \mu) \in Z$ and $\Delta \in \mathbb{R}_{++}$. We claim that $(x + 1_{\mu(i)}\Delta, \mu) \, P_i \, z$. One can easily see that

$$u_i(x + 1_{\mu(i)}\Delta, \mu) - u_i(z) \geq \Delta - |\{\beta : \beta \neq \mu(i)\}| \frac{c}{n-1}\Delta \geq (1-c)\Delta$$

Since $c < 1$, then $u_i(x + 1_{\mu(i)}\Delta, \mu) - u_i(z) > 0$.

(ii) R_i satisfies the ceteris-paribus-compensation assumption. Let $\{\alpha, \beta\} \subseteq A$ and $d \in \mathbb{R}$. Let $d' \in \mathbb{R}$ be such that

$$d' \geq d + \frac{1}{1-c} \left\{ \nu_\alpha(\alpha) - \nu_\beta(\beta) + \frac{c}{n-1} \sum_{\gamma \in A \setminus \{\beta\}} \nu_\beta(\beta) - \nu_\beta(\gamma) - \nu_\alpha(\alpha) + \nu_\alpha(\gamma) \right\}, \tag{1}$$

and for each $\gamma \in A$,

$$d' + \nu_\beta(\beta) - \nu_\beta(\gamma) \geq d + \nu_\alpha(\alpha) - \nu_\alpha(\gamma). \tag{2}$$

We claim that for each $x \in \mathbb{R}^A$ such that $x_\alpha \leq d$ and $x_\beta \geq d'$, each bijection $\mu \in A^N$ such that $\mu(i) = \alpha$, and each bijection $\mu' \in A^N$ such that $\mu'(i) = \beta$, $(x, \mu') \, R_i \, (x, \mu)$. Let x, μ, and μ' be as in the statement of our claim. For each $\gamma \in A$, let

$$\Upsilon(\gamma) \equiv \max\{\nu_\beta(\gamma) + x_\gamma - (\nu_\beta(\beta) + x_\beta), 0\} - \max\{\nu_\alpha(\gamma) + x_\gamma - (\nu_\alpha(\alpha) + x_\alpha), 0\}.$$

Let $\gamma \in A$. Since $x_\alpha \leq d$ and $x_\beta \geq d'$, from (2) we deduce that $\nu_\beta(\gamma) + x_\gamma - (\nu_\beta(\beta) + x_\beta) \leq \nu_\alpha(\gamma) + x_\gamma - (\nu_\alpha(\alpha) + x_\alpha)$. Thus,

$$\Upsilon(\gamma) \leq 0 \tag{3}$$

For each $\gamma \in A$, let

$$\Theta(\gamma) \equiv \max\{\nu_\beta(\beta) + x_\beta - (\nu_\beta(\gamma) + x_\gamma), 0\} - \max\{\nu_\alpha(\alpha) + x_\alpha - (\nu_\alpha(\gamma) + x_\gamma), 0\}.$$

We claim that for each $\gamma \in A$,

$$\Theta(\gamma) \leq x_\beta + \nu_\beta(\beta) - \nu_\beta(\gamma) - (x_\alpha + \nu_\alpha(\alpha) - \nu_\alpha(\gamma)). \tag{4}$$

There are four cases:

Case 1: $\nu_\beta(\beta) + x_\beta > \nu_\beta(\gamma) + x_\gamma$ and $\nu_\alpha(\alpha) + x_\alpha > \nu_\alpha(\gamma) + x_\gamma$. Then, (4) holds with equality.

Case 2: $\nu_\beta(\beta) + x_\beta > \nu_\beta(\gamma) + x_\gamma$ and $\nu_\alpha(\alpha) + x_\alpha \leq \nu_\alpha(\gamma) + x_\gamma$. Then, $\Theta(\gamma) = \nu_\beta(\beta) + x_\beta - \nu_\beta(\gamma) - x_\gamma$. Since $\nu_\alpha(\alpha) + x_\alpha \leq \nu_\alpha(\gamma) + x_\gamma$, then $-x_\gamma \leq -(x_\alpha + \nu_\alpha(\alpha) - \nu_\alpha(\gamma))$. Thus, $\Theta(\gamma) \leq x_\beta + \nu_\beta(\beta) - \nu_\beta(\gamma) - (x_\alpha + \nu_\alpha(\alpha) - \nu_\alpha(\gamma))$.

Case 3: $\nu_\beta(\beta) + x_\beta \leq \nu_\beta(\gamma) + x_\gamma$ and $\nu_\alpha(\alpha) + x_\alpha > \nu_\alpha(\gamma) + x_\gamma$. Thus, $x_\alpha + \nu_\alpha(\alpha) - \nu_\alpha(\gamma) > x_\gamma \geq x_\beta + \nu_\beta(\beta) - \nu_\beta(\gamma)$. Since $d \geq x_\alpha$ and $x_\beta \geq d'$, then $d + \nu_\alpha(\alpha) - \nu_\alpha(\gamma) > d' + \nu_\beta(\beta) - \nu_\beta(\gamma)$. This contradicts (2). Thus, this case cannot hold.

Case 4: $\nu_\beta(\beta) + x_\beta \leq \nu_\beta(\gamma) + x_\gamma$ and $\nu_\alpha(\alpha) + x_\alpha \leq \nu_\alpha(\gamma) + x_\gamma$. Then, $\Theta(\gamma) = 0$. Since $x_\alpha \leq d$ and $x_\beta \geq d'$, from (2) we deduce that $\nu_\beta(\gamma) + x_\gamma - (\nu_\beta(\beta) + x_\beta) \leq \nu_\alpha(\gamma) + x_\gamma - (\nu_\alpha(\alpha) + x_\alpha)$. Thus, $\Theta(\gamma) \leq x_\beta + \nu_\beta(\beta) - \nu_\beta(\gamma) - (x_\alpha + \nu_\alpha(\alpha) - \nu_\alpha(\gamma))$.

Now,

$$\begin{aligned} u_i(x, \mu') - u_i(x, \mu) = \; & \nu_\beta(\beta) - \nu_\alpha(\alpha) + x_\beta - x_\alpha \\ & -\tfrac{a}{n-1} \textstyle\sum_{\gamma \in A} \Upsilon(\gamma) - \tfrac{c}{n-1} \sum_{\gamma \in A} \Theta(\gamma). \end{aligned}$$

By (3) and (4),

$$\begin{aligned} u_i(x, \mu') - u_i(x, \mu) \geq \; & \nu_\beta(\beta) - \nu_\alpha(\alpha) + x_\beta - x_\alpha \\ & -\tfrac{a}{n-1} \Upsilon(\beta) \\ & -\tfrac{c}{n-1} \textstyle\sum_{\gamma \in A} x_\beta + \nu_\beta(\beta) - \nu_\beta(\gamma) - (x_\alpha + \nu_\alpha(\alpha) - \nu_\alpha(\gamma)). \end{aligned}$$

Since $x_\alpha \leq d$ and $x_\beta \geq d'$, from (2) we deduce that $\Upsilon(\beta) = -(\nu_\alpha(\beta) + x_\beta - (\nu_\alpha(\alpha) + x_\alpha))$. Thus,

$$\begin{aligned} u_i(x, \mu') - u_i(x, \mu) \geq \; & \nu_\beta(\beta) - \nu_\alpha(\alpha) + x_\beta - x_\alpha \\ & +\tfrac{a}{n-1}(\nu_\alpha(\beta) + x_\beta - (\nu_\alpha(\alpha) + x_\alpha)) - \tfrac{c}{n-1}(x_\beta - (x_\alpha + \nu_\alpha(\alpha) - \nu_\alpha(\beta))) \\ & -\tfrac{c}{n-1} \textstyle\sum_{\gamma \in A \setminus \{\beta\}} x_\beta + \nu_\beta(\beta) - \nu_\beta(\gamma) - (x_\alpha + \nu_\alpha(\alpha) - \nu_\alpha(\gamma)). \end{aligned}$$

Since $a \geq c$ and $\nu_\alpha(\beta) + x_\beta - (\nu_\alpha(\alpha) + x_\alpha) \geq 0$, then

$$\begin{aligned} u_i(x, \mu') - u_i(x, \mu) \geq \; & \nu_\beta(\beta) - \nu_\alpha(\alpha) + x_\beta - x_\alpha \\ & - \tfrac{c}{n-1} \textstyle\sum_{\gamma \in A \setminus \{\beta\}} x_\beta + \nu_\beta(\beta) - \nu_\beta(\gamma) - (x_\alpha + \nu_\alpha(\alpha) - \nu_\alpha(\gamma)). \end{aligned}$$

Since $c < 1$, then

$$\begin{aligned} u_i(x, \mu') - u_i(x, \mu) \geq \; & \nu_\beta(\beta) - \nu_\alpha(\alpha) + (1-c)(x_\beta - x_\alpha) \\ & - \tfrac{c}{n-1} \textstyle\sum_{\gamma \in A \setminus \{\beta\}} \nu_\beta(\beta) - \nu_\beta(\gamma) - (\nu_\alpha(\alpha) - \nu_\alpha(\gamma)). \end{aligned}$$

Since $x_\alpha \leq d \leq d' \leq x_\beta$, then $(1-c)(x_\beta - x_\alpha) \geq (1-c)(d' - d)$. Thus, from (1) we have that $u_i(x, \mu') - u_i(x, \mu) \geq 0$.

(iii) If $a > 0$ then $R_i \in \mathcal{B} \setminus \mathcal{N}$. Let $x \in \mathbb{R}^A$ and $\alpha \in A$ be such that for each $\gamma \in A \setminus \{\alpha\}$, $x_\gamma + \nu_\alpha(\gamma) \geq x_\alpha + \nu_\alpha(\alpha)$. Since $a > 0$, then for each bijection $\mu \in A^N$ such that $\mu(i) = \alpha$ and each $\Delta \in \mathbb{R}_{++}$, $(x, \mu) \, P_i \, (x + 1_{A \setminus \{\mu(i)\}} \Delta, \mu)$. Thus, $R_i \notin \mathcal{N}$. □

Proof of Proposition 6. Let $R \equiv (R_i)_{i \in N} \in \mathcal{F}^N$ be as in the statement of the proposition and $z \in Nc(R, M) \cap Z(M)$. We prove that z is *non-wasteful for* (R, M). Let $z' \equiv (x', \mu') \in Z(R, M)$ be such that $z' \, R \, z$. We claim that $x' \geq x$. Suppose by means of contradiction that there is $\alpha \in A$ such that $x'_\alpha < x_\alpha$. Suppose w.l.o.g that $\alpha \in \arg\max_{\gamma \in A} x_\gamma - x'_\gamma$. Let $\varepsilon \equiv x_\alpha - x'_\alpha > 0$. Then, for each $\gamma \in A$, $x_\gamma - x'_\gamma \leq \varepsilon$. Let $j \in N$ be such that $\mu'(j) = \alpha$. For each $y \in \mathbb{R}^A$ and each $\gamma \in A$, let

$$\Phi_j(y, \gamma) \equiv -a_j \max\{\nu^j_\alpha(\gamma) + y_\gamma - (\nu^j_\alpha(\alpha) + y_\alpha), 0\} - c_j \max\{\nu^j_\alpha(\alpha) + y_\alpha - (\nu^j_\alpha(\gamma) + y_\gamma), 0\}.$$

Since $z \in Nc(R, M)$, then $z \, R_j \, (x, \mu')$. We claim that $(x, \mu') \, P_j \, z'$. Using the notation developed above,

$$u_j(x, \mu') - u_j(z') = \varepsilon + \frac{1}{n-1} \sum_{\gamma \in A} \Phi_j(x, \gamma) - \Phi_j(x', \gamma).$$

For each $\gamma \in A$, let $\Delta_j(\gamma) \equiv \Phi_j(x, \gamma) - \Phi_j(x', \gamma)$.

Let $A_1 \equiv \{\gamma \in A : \nu^j_\alpha(\gamma) + x_\gamma \leq \nu^j_\alpha(\alpha) + x_\alpha\}$. Let $\gamma \in A_1$. We claim that $\Delta_j(\gamma) \geq -c_j(\varepsilon + x'_\gamma - x_\gamma)$. There are two cases.

Case 1: $\nu^j_\alpha(\gamma) + x'_\gamma \leq \nu^j_\alpha(\alpha) + x'_\alpha$. Then, $\Delta_j(\gamma) = -c_j(\nu^j_\alpha(\alpha) + x_\alpha - (\nu^j_\alpha(\gamma) + x_\gamma)) + c_j(\nu^j_\alpha(\alpha) + x'_\alpha - (\nu^j_\alpha(\gamma) + x'_\gamma))$. Since $x_\alpha - x'_\alpha = \varepsilon$, then $\Delta_j(\gamma) = -c_j(\varepsilon + x'_\gamma - x_\gamma)$.

Case 2: $\nu^j_\alpha(\gamma) + x'_\gamma > \nu^j_\alpha(\alpha) + x'_\alpha$. Then, $\Delta_j(\gamma) = -c_j(\nu^j_\alpha(\alpha) + x_\alpha - (\nu^j_\alpha(\gamma) + x_\gamma)) + a_j(\nu^j_\alpha(\gamma) + x'_\gamma - (\nu^j_\alpha(\alpha) + x'_\alpha)) > -c_j(\nu^j_\alpha(\alpha) + x_\alpha - (\nu^j_\alpha(\gamma) + x_\gamma))$. Now,

$$\begin{aligned} \nu^j_\alpha(\alpha) + x_\alpha - (\nu^j_\alpha(\gamma) + x_\gamma) = \; & \nu^j_\alpha(\alpha) + x_\alpha - (\nu^j_\alpha(\alpha) + x'_\alpha) \\ & + (\nu^j_\alpha(\alpha) + x'_\alpha) - (\nu^j_\alpha(\gamma) + x_\gamma) \\ = \; & \varepsilon + (\nu^j_\alpha(\alpha) + x'_\alpha) - (\nu^j_\alpha(\gamma) + x_\gamma). \end{aligned}$$

Since $\nu_\alpha^j(\gamma) + x'_\gamma > \nu_\alpha^j(\alpha) + x'_\alpha$, then

$$\nu_\alpha^j(\alpha) + x_\alpha - (\nu_\alpha^j(\gamma) + x_\gamma) < \varepsilon + (\nu_\alpha^j(\gamma) + x'_\gamma) - (\nu_\alpha^j(\gamma) + x_\gamma) = \varepsilon + x'_\gamma - x_\gamma.$$

Thus, $\Delta_j(\gamma) > -c_j(\varepsilon + x'_\gamma - x_\gamma)$.

We claim that for each $\gamma \in A \setminus A_1$, $\Delta_j(\gamma) \geq 0$. Let $\gamma \in A \setminus A_1$. Then, $\nu_\alpha^j(\gamma) + x_\gamma > \nu_\alpha^j(\alpha) + x_\alpha$. Since $x_\gamma - x'_\gamma \leq \varepsilon$, then $\nu_\alpha^j(\gamma) + x'_\gamma + \varepsilon > \nu_\alpha^j(\alpha) + x_\alpha$. Since $x'_\alpha = x_\alpha - \varepsilon$, then $\nu_\alpha^j(\gamma) + x'_\gamma > \nu_\alpha^j(\alpha) + x'_\alpha$. Thus, $\Delta_j(\gamma) = -a_j(\nu_\alpha^j(\gamma) + x_\gamma - (\nu_\alpha^j(\alpha) + x_\alpha)) + a_j(\nu_\alpha^j(\gamma) + x'_\gamma - (\nu_\alpha^j(\alpha) + x'_\alpha))$. Since $x_\gamma - x'_\gamma \leq \varepsilon$, then $\Delta_j(\gamma) \geq -a_j(\nu_\alpha^j(\gamma) + x_\gamma - (\nu_\alpha^j(\alpha) + x_\alpha)) + a_j(\nu_\alpha^j(\gamma) + x_\gamma - \varepsilon - (\nu_\alpha^j(\alpha) + x'_\alpha))$. Since $x'_\alpha + \varepsilon = x_\alpha$, then $\Delta_j(\gamma) \geq 0$.

Since for each $\gamma \in A_1$, $\Delta_j(\gamma) \geq -c_j(\varepsilon + x'_\gamma - x_\gamma)$, and for each $\gamma \in A \setminus A_1$, $\Delta_j(\gamma) \geq 0$, then $u_j(x, \mu') - u_j(z') \geq \varepsilon - \frac{c}{n-1} \sum_{\gamma \in A_1} (\varepsilon + x'_\gamma - x_\gamma)$. Now, since $z \in Z(M)$, then $\sum_{\gamma \in A} x'_\gamma \leq \sum_{\gamma \in A} x_\gamma$. Thus, $\sum_{\gamma \in A_1} x'_\gamma - x_\gamma \leq \sum_{\gamma \in A \setminus A_1} x_\gamma - x'_\gamma$. Since for each $\gamma \in A$, $x_\gamma - x'_\gamma \leq \varepsilon$, then $\sum_{\gamma \in A_1} x'_\gamma - x_\gamma \leq |A \setminus A_1| \varepsilon$. Thus,

$$u_j(x, \mu') - u_j(z') \geq \varepsilon - \frac{c}{n-1}(|A_1|\varepsilon + |A \setminus A_1|\varepsilon) = \varepsilon - \frac{c}{n-1} n\varepsilon.$$

Since $c \in [0, \frac{n-1}{n})$, then $u_j(x, \mu') - u_j(z') > 0$. Thus, $z \, P_j \, z'$. This is a contradiction. □

Proof of Lemma 4. Let $z \equiv (x, \mu)$ and $z' \equiv (x', \mu)$ be such that for each $\beta \in \Gamma(\mu(i))$, $x_\beta = x'_\beta$. We claim that $z \, I_i \, z'$. Suppose by means of contradiction (and without loss of generality) that $z \, P_i \, z'$. Since R_i is continuous, then there is $\delta \in \mathbb{R}_{++}$ such that $(x'', \mu) \equiv (x - \delta 1_{\mu(i)}, \mu) \, P_i \, z'$. Since R_i is surplus motivated w.r.t. Γ, then $\sum_{\beta \in \Gamma(\mu(i))} x''_\beta > \sum_{\beta \in \Gamma(\mu(i))} x'_\beta$. Since $\sum_{\beta \in \Gamma(\mu(i))} x''_\beta = \sum_{\beta \in \Gamma(\mu(i))} x'_\beta - \delta$, then $\delta < 0$. This is a contradiction. □

Proof of Lemma 5. We prove that $\mathcal{N} \subsetneq \mathcal{S} \subsetneq \mathcal{M}$.

(i) $\mathcal{S} \subseteq \mathcal{M}$. Let $i \in N$, $R_i \in \mathcal{S}$, and Γ be the family of sets with respect to which R_i is surplus motivated. Let $z \equiv (x, \mu) \in Z$, and $\delta \in \mathbb{R}_{++}$. Let $x' \equiv x + 1_{\mu(i)}\delta$. We want to prove that $(x', \mu) \, P_i \, z$. Suppose by means of contradiction that $z \, R_i \, (x', \mu)$. Since $R_i \in \mathcal{S}$, $x_{\mu(i)} < x'_{\mu(i)}$, and $x'|_{A \setminus \{\mu(i)\}} = x'|_{A \setminus \{\mu(i)\}}$, then $\sum_{\alpha \in \Gamma(\mu(i))} x_\alpha > \sum_{\alpha \in \Gamma(\mu(i))} x'_\alpha$. This is a contradiction.

(ii) $\mathcal{M} \setminus \mathcal{S} \neq \emptyset$. We prove that $\mathcal{F} \in \mathcal{M} \setminus \mathcal{S}$. By Lemma 3, $\mathcal{F} \subsetneq \mathcal{M}$. Let $i \in N$ and $R_i \in \mathcal{F}$ be such that $R_i \notin \mathcal{N}$. We claim that $R_i \notin \mathcal{S}$. By Lemma 4, if R_i is surplus motivated w.r.t. Γ_i, then $\Gamma_i = \{A\}$. The example that shows that $R_i \notin \mathcal{N}$ in the proof of Lemma 3, shows that R_i is not surplus motivated w.r.t. $\{A\}$.

(iii) $\mathcal{N} \subsetneq \mathcal{S}$. Let $i \in N$. Each $R_i \in \mathcal{N}$ is surplus motivated with respect to the finest partition of A, i.e., $\{\{\alpha\}\}_{\alpha \in A}$. Thus, $\mathcal{S} \supseteq \mathcal{N}$. One can easily see that $\mathcal{S} \setminus \mathcal{N} \neq \emptyset$. □

Proof of Proposition 8. Let $R \in \mathcal{S}^N$ and $\{\Gamma_i\}_{i\in N}$ be the families of subsets of A such that for each $i \in N$, R_i are surplus motivated with respect to Γ_i. Let $M \in \mathbb{R}$, $e \equiv (R, M)$, and $z \equiv (x, \mu) \in Nc(e) \cap Z(M)$. Suppose that $\{\Gamma_i\}_{i\in N}$ is as in the statement of the proposition. We prove that z is *non-wasteful for* e. Let $z' \equiv (x', \mu') \in Z(e)$ be such that $z' \, R \, z$. We claim that $x' \geq x$. Suppose by means of contradiction that there is $\alpha \in A$ such that $x'_\alpha < x_\alpha$.

Let us first make some definitions. For each $\alpha \in A$, let $\Gamma_{\mu'}(\alpha) \equiv \Gamma_{(\mu')^{-1}(\alpha)}(\alpha)$. That is, $\Gamma_{\mu'}(\alpha)$ is the set of objects that influence the welfare of the agent who receives α at μ'. Let $A^- \equiv \{\alpha \in A : x'_\alpha < x_\alpha\}$. By our hypothesis, $A^- \neq \emptyset$. For each $B \subseteq A$, let $\Gamma(B) \equiv \bigcup_{\alpha\in B} \Gamma_{\mu'}(\alpha)$.

We obtain a contradiction in three steps. First, we prove a general claim about the subsets of A^- that satisfy a minimality property. Second, we prove a general claim about A^-. Third, we conclude.

Step 1: Let $B \subseteq A^-$ be a nonempty set such that there is no non-empty subset $B' \subsetneq B$ such that $\Gamma(B') = \Gamma(B)$. Then, $\sum_{\alpha\in\Gamma(B)} x'_\alpha > \sum_{\alpha\in\Gamma(B)} x_\alpha$.

We prove this step by induction on the cardinality of B.

• $|B| = 1$. Suppose without loss of generality that $B \equiv \{\alpha_1\}$. Since $B \subseteq A^-$, then $\alpha_1 \in A^-$ and $x'_{\alpha_1} < x_{\alpha_1}$. Since agent $(\mu')^{-1}(\alpha_1)$ is surplus motivated w.r.t. $\Gamma_{(\mu')^{-1}(\alpha_1)}$, and $x'_{\alpha_1} < x_{\alpha_1}$, then $\sum_{\alpha\in\Gamma_{\mu'}(\alpha_1)} x'_\alpha > \sum_{\alpha\in\Gamma_{\mu'}(\alpha_1)} x_\alpha$. Since $\Gamma(B) = \Gamma_{\mu'}(\alpha_1)$, then $\sum_{\alpha\in\Gamma(B)} x'_\alpha > \sum_{\alpha\in\Gamma(B)} x_\alpha$.

• Let $T \in \mathbb{N}$. Suppose that $|B| = T + 1$ and our claim is true for sets with cardinality up to T. Suppose without loss of generality that $B \equiv \{\alpha_1, \ldots, \alpha_T, \alpha_{T+1}\}$. Let $C \equiv \{\alpha_1, \ldots, \alpha_T\}$. Thus, there is no non-empty $C' \subsetneq C$ such that $\Gamma(C) = \Gamma(C')$, for otherwise $C' \cup \{\alpha_{T+1}\} \subsetneq B$ is such that $\Gamma(C' \cup \{\alpha_{T+1}\}) = \Gamma(B)$. Thus, by the induction hypothesis, $\sum_{\alpha\in\Gamma(C)} x'_\alpha > \sum_{\alpha\in\Gamma(C)} x'_\alpha$.

We claim that for each $\alpha \in C$, $\Gamma_{\mu'}(\alpha) \cap \Gamma_{\mu'}(\alpha_T) \subseteq \{\alpha, \alpha_T\}$. Suppose by means of contradiction that there is $\alpha \in C$ and $\gamma \in A \setminus \{\alpha, \alpha_T\}$ such that $\gamma \in \Gamma_{\mu'}(\alpha) \cap \Gamma_{\mu'}(\alpha_T)$. Thus, $(\Gamma_{\mu'}(\alpha_T) \setminus \{\alpha_T\}) \cap (\Gamma_{\mu'}(\alpha) \setminus \{\alpha\}) \neq \emptyset$. By the hypothesis in the lemma, either $\Gamma_{\mu'}(\alpha) \subseteq \Gamma_{\mu'}(\alpha_T)$ or $\Gamma_{\mu'}(\alpha_T) \subseteq \Gamma_{\mu'}(\alpha)$. Thus, either $\Gamma(B \setminus \{\alpha\}) = \Gamma(B)$ or $\Gamma(B \setminus \{\alpha_T\}) = \Gamma(B)$. This contradicts the hypothesis of the step.

Thus,

$$\Gamma(C) \cap \Gamma_{\mu'}(\alpha_T) = \left(\bigcup_{\alpha\in C} \Gamma_{\mu'}(\alpha)\right) \cap \Gamma_{\mu'}(\alpha_T) = \bigcup_{\alpha\in C} \Gamma_{\mu'}(\alpha) \cap \Gamma_{\mu'}(\alpha_T) \subseteq \bigcup_{\alpha\in C} \{\alpha, \alpha_T\} \subseteq B \subseteq A^-.$$

Recall that $\alpha_T \in B \subseteq A^-$. Since agent $(\mu')^{-1}(\alpha_T)$ is surplus motivated w.r.t.

$\Gamma_{(\mu')^{-1}(\alpha_T)}$ and $x'_{\alpha_T} < x_{\alpha_T}$, then $\sum_{\alpha \in \Gamma_{\mu'}(\alpha_T)} x'_\alpha > \sum_{\alpha \in \Gamma_{\mu'}(\alpha_T)} x_\alpha$. Thus,

$$\sum_{\alpha \in \Gamma(C)} x'_\alpha + \sum_{\alpha \in \Gamma_{\mu'}(\alpha_T)} x'_\alpha > \sum_{\alpha \in \Gamma(C)} x_\alpha + \sum_{\alpha \in \Gamma_{\mu'}(\alpha_T)} x_\alpha.$$

Since $\Gamma(B) = \Gamma(C) \cup \Gamma_{\mu'}(\alpha_T)$, then this inequality can be rewritten as

$$\sum_{\alpha \in \Gamma(B)} x'_\alpha + \sum_{\alpha \in \Gamma(C) \cap \Gamma_{\mu'}(\alpha_T)} x'_\alpha > \sum_{\alpha \in \Gamma(B)} x_\alpha + \sum_{\alpha \in \Gamma(C) \cap \Gamma_{\mu'}(\alpha_T)} x_\alpha.$$

Since $\Gamma(C) \cap \Gamma_{\mu'}(\alpha_T) \subseteq A^-$, then $\sum_{\alpha \in \Gamma(C) \cap \Gamma_{\mu'}(\alpha_T)} x'_\alpha - \sum_{\alpha \in \Gamma(C) \cap \Gamma_{\mu'}(\alpha_T)} x_\alpha \leq 0$. Thus,

$$\sum_{\alpha \in \Gamma(B)} x'_\alpha > \sum_{\alpha \in \Gamma(B)} x_\alpha.$$

Step 2:

$$\sum_{\alpha \in \Gamma(A^-)} x'_\alpha > \sum_{\alpha \in \Gamma(A^-)} x_\alpha.$$

Since A^- is non-empty and finite, then there is $B \subseteq A^-$ be such that $\Gamma(B) = \Gamma(A^-)$ and there is no non-empty subset $B' \subsetneq B$ such that $\Gamma(B') = \Gamma(B)$ (one can construct such a set by eliminating elements from A^- until the condition is satisfied). By Step 1, $\sum_{\alpha \in \Gamma(A^-)} x'_\alpha > \sum_{\alpha \in \Gamma(A^-)} x_\alpha$.

Step 3: Conclusion. Recall that for each $\alpha \in A \setminus A^-$, $x'_\alpha \geq x_\alpha$. Now, since $A^- \subseteq \Gamma(A^-)$, then for each $\alpha \in A \setminus \Gamma(A^-)$, $x'_\alpha \geq x_\alpha$. Thus, $\sum_{\alpha \in A \setminus \Gamma(A^-)} x'_\alpha \geq \sum_{\alpha \in A \setminus \Gamma(A^-)} x_\alpha$. By Step 2,

$$\sum_{\alpha \in A} x'_\alpha = \sum_{\alpha \in \Gamma(A^-)} x'_\alpha + \sum_{\alpha \in A \setminus \Gamma(A^-)} x'_\alpha > \sum_{\alpha \in \Gamma(A^-)} x_\alpha + \sum_{\alpha \in A \setminus \Gamma(A^-)} x_\alpha = \sum_{\alpha \in A} x_\alpha.$$

Since $z \in Z(M)$, then $\sum_{\alpha \in A} x'_\alpha > M$. Thus, $z \notin Z(e)$. This is a contradiction. □

Proof of Lemma 6. We prove that $\mathcal{K} \subsetneq \mathcal{M} \setminus (\mathcal{S} \cup \mathcal{N})$. Let $i \in N$ and R_i be a preference relation on Z. Suppose that R_i is represented by the utility function, u_i, defined as follows: for each $z \equiv (x, \mu) \in Z$, $u_i(z) \equiv \nu_i(\mu(i)) + x_{\mu(i)} - \frac{1}{a} \max\{\overline{x} - x_{\mu(i)}, 0\}$, where $a \in \mathbb{R}$ is such that $a > \frac{n-1}{n}$, $\nu_i : A \to \mathbb{R}$, and $\overline{x} \equiv \frac{1}{n} \sum_{\alpha \in A} x_\alpha$.

We claim that $R_i \in \mathcal{M}$. Since u_i is the composition of continuous functions, then R_i is continuous. Thus, R_i is continuous. Since $u_i(x, \mu)$ is a function of x and $\mu(i)$, then externalities are anonymous for R_i.

(i) R_i **is own-consumption money-monotone**. Let $(x, \mu) \in Z$ and $\Delta \in \mathbb{R}_+$. Clearly, if $x_{\mu(i)} \geq \bar{x}$, then $(x + 1_{\mu(i)}\Delta, \mu)\, P_i\, (x, \mu)$. Suppose without loss of generality that $x_{\mu(i)} \leq \overline{x}$ and $x_{\mu(i)} + \Delta \leq \bar{x} + \frac{1}{n}\Delta$. Since $a > \frac{n-1}{n}$, then $a > 0$. Thus,

$$x_{\mu(i)} + \Delta - \frac{1}{a}\left(\overline{x} + \frac{1}{n}\Delta - (x_{\mu(i)} + \Delta)\right) > x_{\mu(i)} - \frac{1}{a}\left(\overline{x} - x_{\mu(i)}\right).$$

Thus, $(x + 1_{\mu(i)}\Delta, \mu)\, P_i\, (x, \mu)$.

(ii) R_i satisfies the equal budget compensation assumption. Let $b \in \mathbb{R}$ and $\{\alpha, \beta\} \subseteq A$. Let $b' = \max\{b + \nu_i(\alpha) - \nu_i(\beta), b\}$. For each $x \in \mathbb{R}^A$ such that $x_\alpha \leq b \leq b' \leq x_\beta$, $\max\{\overline{x} - x_\alpha, 0\} \geq \max\{\overline{x} - x_\beta, 0\}$. Now, since $(x_\beta - x_\alpha) \geq (\nu_i(\alpha) - \nu_i(\beta))$, then

$$-\frac{1}{a}\left(\max\{\overline{x} - x_\beta, 0\} - \max\{\overline{x} - x_\alpha, 0\}\right) + x_\beta - x_\alpha - (\nu_i(\alpha) - \nu_i(\beta)) \geq 0.$$

Thus, for each $(x, \mu) \in Z$ such that $x_\alpha \leq b$, $x_\beta \geq b'$, and $\mu(i) = \alpha$, and each bijection $\mu' \in A^N$ such that $\mu'(i) = \beta$,

$$\nu_i(\beta) + x_\beta - \frac{1}{a}\max\{\overline{x} - x_\beta, 0\} \geq \nu_i(\alpha) + x_\alpha - \frac{1}{a}\max\{\overline{x} - x_\alpha, 0\}.$$

Thus, $(x, \mu')\, R_i\, (x, \mu')$.

(iii) $R_i \notin \mathcal{N}$. Let $\alpha \in A$ and $\{x, x'\} \subseteq \mathbb{R}^A$ be such that $x_\alpha = x'_\alpha$, $\overline{x} < \overline{x}'$, and $x_\alpha < \overline{x}$. Let μ be a bijection such that $\mu(i) = \alpha$. Clearly, $(x, \mu)\, P_i\, (x', \mu)$. Thus, $R_i \notin \mathcal{N}$.

(iv) $R_i \notin \mathcal{S}$. Since u_i is a function of $\overline{x}$, then if R_i is surplus motivated w.r.t. some partition Γ, it follows that $\Gamma = \{A\}$. The example in (iii) also shows that R_i is not surplus motivated w.r.t. $\{A\}$. Thus, $R_i \notin \mathcal{S}$. □

Proof of Proposition 9. Let $R \in \mathcal{K}^N$, $M \in \mathbb{R}$, $e \equiv (R, M)$, and $z \equiv (x, \mu) \in Nc(e)$. Suppose that for each $i \in N$, R_i is represented by the following utility function: for each $z \equiv (x, \mu) \in Z$, $u_i(z) \equiv \nu_i(\mu(i)) + x_{\mu(i)} - \frac{1}{a}\max\{\overline{x} - x_{\mu(i)}, 0\}$, where $a_i \in \mathbb{R}$ is such that $a_i > \frac{n-1}{n}$, $\nu_i : A \to \mathbb{R}$, and $\bar{x} \equiv \frac{1}{n}\sum_{\alpha \in A} x_\alpha$.

We prove that z is *non-wasteful for* (R, M). Let $z' \equiv (y, \sigma) \in Z(e)$ be such that $z'\, R\, z$. We claim that $y \geq x$. Let $\alpha \in A$. We prove that $y_\alpha \geq x_\alpha$. There are two cases:

- **Case 1**: $x_\alpha \geq \bar{x}$. Let $i \in N$ be such that $\sigma(i) = \alpha$. Since $z \in Nc(e)$, then $z\, R_i\, (x, \sigma)$. Thus $u_i(z) \geq \nu_i(\alpha) + x_\alpha - \frac{1}{a_i}\max\{\bar{x} - x_\alpha, 0\} = \nu_i(\alpha) + x_\alpha$. If $z'\, R_i\, z$, then $u_i(z') \geq u_i(z)$. Thus, $\nu_i(\alpha) + y_\alpha - \frac{1}{a_i}\max\{\bar{y} - y_\alpha, 0\} \geq \nu_i(\alpha) + x_\alpha$, and $y_\alpha - x_\alpha \geq \frac{1}{a_i}\max\{\bar{y} - y_\alpha, 0\} \geq 0$.

- **Case 2**: $x_\alpha < \bar{x}$. Suppose by means of contradiction that $y_\alpha < x_\alpha$. Let $\gamma \in A$ be such that for each $\beta \in A$, $x_\gamma - y_\gamma \geq x_\beta - y_\beta$. Since $y_\alpha < x_\alpha$, then $y_\gamma < x_\gamma$.

By Case 1, $x_\gamma < \bar{x}$. Let $\Delta = x_\gamma - y_\gamma > 0$. Since for at least one $\delta \in A$, $x_\delta \geq \bar{x}$, then $\bar{y} \geq \bar{x} - \frac{n-1}{n}\Delta$. Thus, $y_\gamma - \bar{y} \leq y_\gamma - \bar{x} + \frac{n-1}{n}\Delta < y_\gamma - \bar{x} + \Delta = x_\gamma - \bar{x}$. Thus, $y_\gamma - \bar{y} < 0$ and

$$-\frac{1}{a_i}\max\{\bar{y} - y_\gamma, 0\} = -\frac{1}{a_i}(\bar{y} - y_\gamma) < -\frac{1}{a_i}(\bar{x} - x_\gamma) = -\frac{1}{a_i}\max\{\bar{x} - x_\gamma, 0\}.$$

Furthermore, since $y_\gamma < x_\gamma$, then $y_\gamma - \frac{1}{a_i}\max\{\bar{y}-y_\gamma, 0\} < x_\gamma - \frac{1}{a_i}\max\{\bar{x}-x_\gamma, 0\}$. Now, let $i \in N$ be such that $\sigma(i) = \gamma$. The preceding expression implies that $u_i(z') < u_i(x, \sigma)$. Since $z \in Nc(e)$, then $z \, R_i \, (x, \sigma)$. Thus, $z \, P_i \, z'$. This is a contradiction.

□

Proof of Proposition 1. We prove the non-trivial inclusion statements in (i).

• $\mathcal{R} \subsetneq \mathcal{N} \cap \mathcal{M}$: We construct $R_i \in \mathcal{N} \cap \mathcal{M} \setminus \mathcal{R}$. Let $\alpha \in A$ and $i \in N$. Let R_i be the preference with numerical representation defined as follows: for each $z \equiv (x, \mu) \in Z$, $u_i(z) \equiv x_\alpha 1_{\mu(i)=\alpha} + e^{x_{\mu(i)}} 1_{\mu(i)\in A\setminus\{\alpha\}}$. It is easy to see that $R_i \in \mathcal{N}$. However, $R_i \notin \mathcal{R}$. Indeed, for each $\beta \in A \setminus \{\alpha\}$ and each $x_\alpha \leq 0$, there is no $b \in \mathbb{R}$ such that $(x_\alpha, \alpha) \, I_i \, (b, \beta)$.[25]

• $\mathcal{N} \cap \mathcal{M} \subsetneq \mathcal{C}$: Let $R_i \in \mathcal{N}$. Without loss of generality, we make preference statements among consumption bundles, instead of between allocations. Recall that by definition of $\mathcal{N}$, $\mathcal{N} \subseteq \mathcal{B}$. Let $M \equiv M(R_i, b)$. Let $\{\alpha, \beta\} \subseteq A$ and $b \in \mathbb{R}$. Let $x \equiv (x_\delta)_{\delta\in A}$ be defined by: $x_\alpha \equiv M - (n-1)b$ and for each $\delta \neq \alpha$, $x_\delta \equiv b$. Then, by definition of $M(R_i, b)$, $(x_\alpha, \alpha) \, R_i \, (x_\beta, \beta)$. Since $R_i \in \mathcal{M}$, then for each $x'_\beta \leq b$ and each $x'_\alpha \geq x_\alpha$, $(x'_\alpha, \alpha) \, R_i \, (x'_\beta, \beta)$. These rankings hold independently of the consumptions associated with objects different from α and β. Thus, $R_i \in \mathcal{C}$.

• $\mathcal{C} \subsetneq \mathcal{B}$. Each preference $R_i \in \mathcal{L}$ for which there is a triple $\{\alpha, \beta, \gamma\}$ such that $c^i_\alpha(\delta) \neq c^i_\beta(\delta)$ is such that $\mathbb{R}_i \notin \mathcal{C}$: each ranking between two allocations at which the agent respectively receives α and β can be reversed by changing the consumption of money of the agent who receives object δ.

(ii) It is straightforward to see that $\mathcal{N} \setminus \mathcal{M} \neq \emptyset$.

(iii) $\mathcal{K} \cup \mathcal{F} \subseteq \mathcal{M} \setminus \mathcal{N}$. □

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