Density Forecast in Functional Space

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Part 1

Motivations

Pseudo-Panel Data Set

$$(X_{it}), i = 1, \cdots, N, t = 1, \cdots, T$$

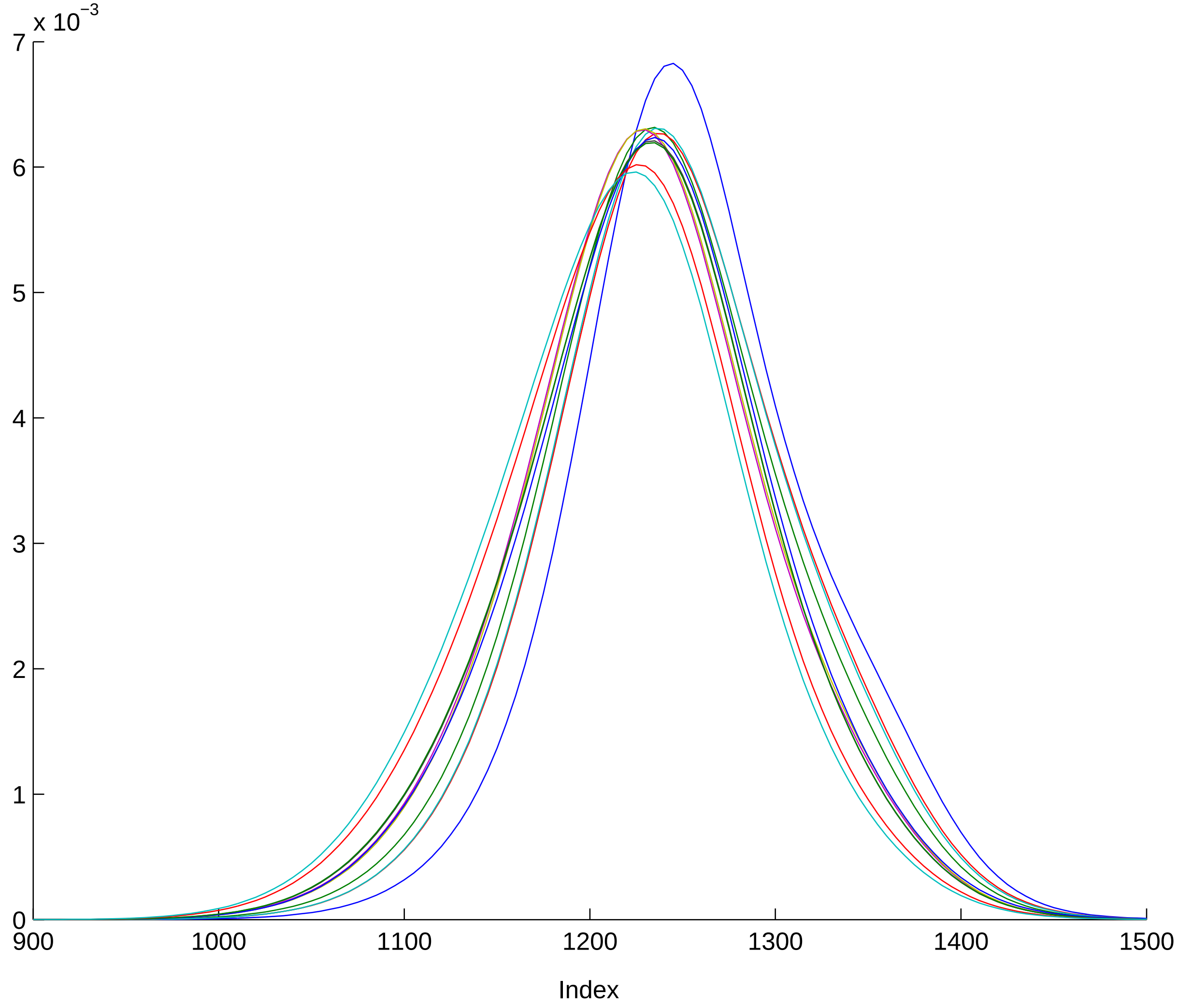
- (Example I) X_{it} can be income, energy consumption, etc., for household i at time t.
- (Example II) $X_{it} = C_t(K_i)$, where $C_t(K_i)$ are option prices with strike prices K_i at time t.

Time-Varying Distributions

$$(p_t), t = 1, \cdots, T$$

- (Example I) The pdf of $(X_{1t}, X_{2t}, \cdots, X_{N,t})$
- (Example II) Risk-neutral pdf implicit in option prices $C_t(K_i), i = 1, 2, \cdots, N$ at time t.

The Variation of Risk-Neutral PDF of S&P 500 Index



Part 2

FAR(1) Model of Time-Varying Densities

Functional AR(1) Model

Time-varying densities: $\{p_t(x)\}, t \in \mathbb{Z}, x \in [a, b]$

$$p_t - Ep_0 = A(p_{t-1} - Ep_0) + \varepsilon_t, t \in \mathbb{Z}. \tag{1}$$

AR(1) in Moments

Lemma: *If* (f, λ) *is an eigenfunction-eigenvalue pair of* A^*,

$$E_t f(X) - Ef(X) = \lambda(E_{t-1} f(X) - Ef(X)) + \eta_t.$$

AR(1) and ARCH(1)

If (f_j) is Legendre series,

$$E_t X^j - EX^j = \lambda_j (E_{t-1} X^j - EX^j) + \eta_t, j = 1, 2$$

ARCH-M

If A^* is some appropriate integral operator,

$$E_t X = \alpha E_{t-1} X + \beta E_{t-1} X^2 + \eta_t.$$

Part 3

Estimation and Forecast

Estimation of Density

$$p_{Nt} = p_t + \Delta_{Nt}, t = 1, \cdots, T,$$
$$E\|\Delta_{Nt}\|^2 = O(N^{-r})$$

- (Example I) Estimate pdf of $(X_{1t}, X_{2t}, \cdots, X_{N,t})$

$$p_{Nt}(x) = \frac{1}{Nh}\sum_{i=1}^{N} K(\frac{x - X_{it}}{h})$$

- (Example II) Estimate risk-neutral density implicit in option prices $C_t(K_i), i = 1, 2, \cdots, N$ at time t.

$$p_{Nt}(x) = e^{r\tau}\frac{\partial \hat{C}^2(x)}{\partial x^2}$$

Estimation of Autoregressive Operator: Finite-Dimensional Space

Theorem: *Let* A_{NT} *denote the estimator for the autoregressive operator* A*. If* r *is such that* $\frac{T^{1/4}/(logT)^{\beta}}{N^{r/2}} \to 0$ *for some* $\beta > 1/2$,

$$\frac{T^{1/4}}{(logT)^{\beta}} \|A_{NT} - A\|_{\mathcal{L}} \to_p 0.$$

Estimation of Autoregressive Operator: Infinite-Dimensional Space

Theorem: *Under some technical conditions, if A is Hilbert-Schmidt,*

$$\|A_{NT} - A\|_{\mathcal{L}} \to_p 0.$$

Forecast with FAR(1) Model

$$p_{T+1} = \bar{p}_{NT} + A_{NT}(p_{NT} - \bar{p}_{NT}),$$

where

$$\bar{p}_{NT} = \frac{1}{T}\sum_{t=1}^{T} p_{Nt}.$$

Part 4

A Simulation

Data Generating Process

Let $\theta_t = (\mu_t, \sigma_t^2, \beta_t)'$ be a VAR(1) process.

$$\begin{aligned} X_{it} &\sim N_t + U_t, \\ N_t &= Normal(\mu_t, \sigma_t^2) \\ U_t &= Uniform(\beta_t - \sigma_t, \beta_t + \sigma_t) \end{aligned}$$

Simulation Result

Table 1: Monte Carlo Simulation Result

	SRISE	MSE in Moments Prediction			
		Mean	Variance	Skewness	Kurtosis
$p_{N,T+1}$ (FAR(1))	0.0441	0.0658	0.112	0.103	0.167
$\bar{p}_{NT}$ (Average)	0.0476	0.0753	0.136	0.102	0.163
p_{NT} (Last)	0.0546	0.0703	0.118	0.140	0.235
Parametric	n/a	0.066	0.139	n/a	n/a

Note: SRISE denotes Square Root of Integrated Square Error, SRISE$(\hat{p}) = \sqrt{\int (\hat{p}(x) - p(x))^2 dx}$, where p is the reference density function.

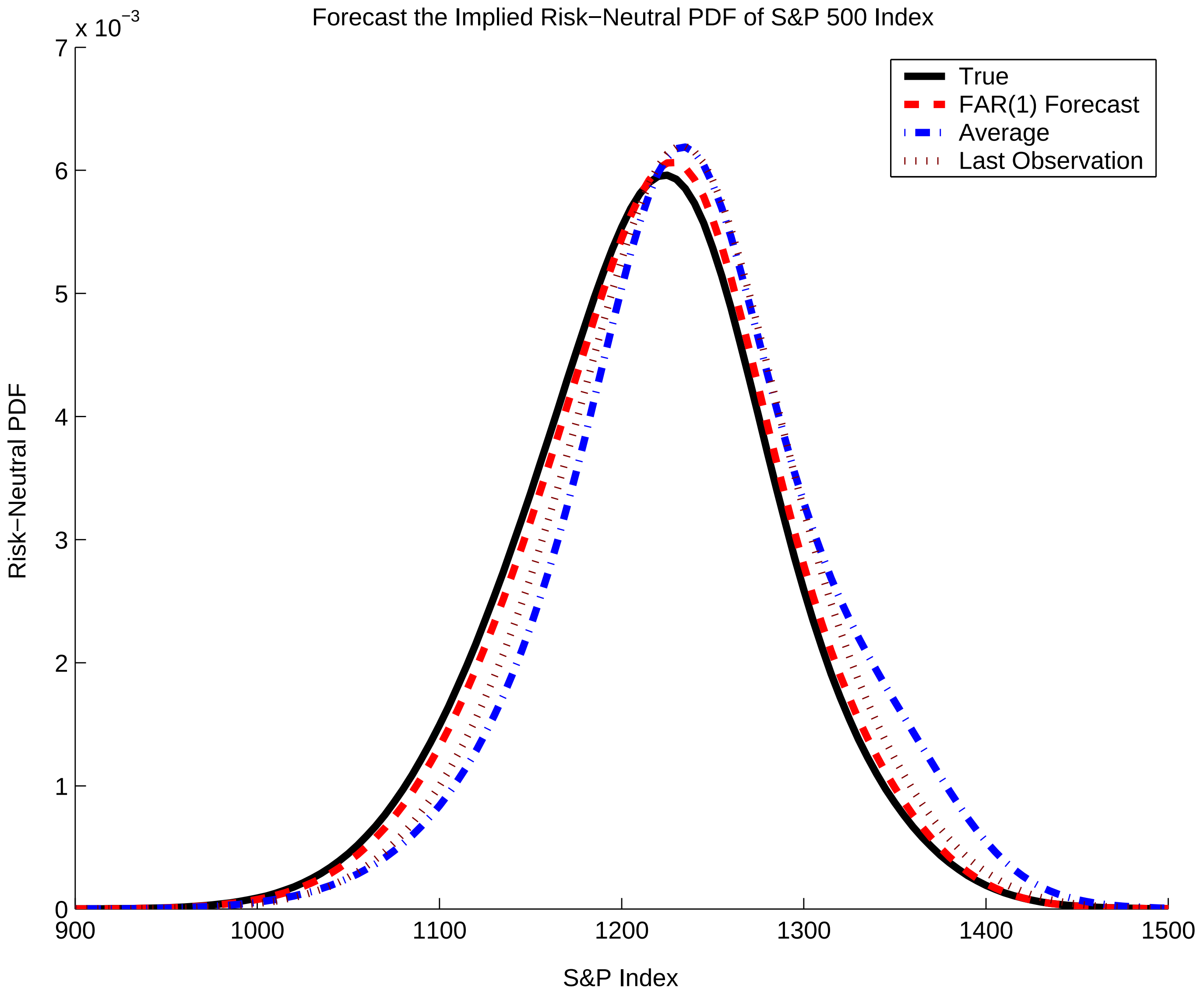
Part 5

Empirical Application in Forecasting Risk-Neutral Density

Data Description

We have daily European call prices on the S&P 500 Index that

mature on December 15 2004. We choose the prices from August 16th 2004 to November 30th 2004, estimate risk-neutral density for each day, and forecast the risk-neutral density on the next business day, December 1st 2004.



Forecast the Implied Risk−Neutral PDF of S&P 500 Index
x 10^−3
7
6
5
4
3
2
1
0
Risk−Neutral PDF
900
1000
1100
1200
1300
1400
1500
S&P Index
True
FAR(1) Forecast
Average
Last Observation

	True	FAR	Average	Last
Mean	1214.78	1219.37	1237.77	1229.50
Volatility	71.17	70.61	72.93	69.44
Skewness	-0.104	-0.121	-0.0254	-0.0747
Kurtosis	3.182	3.203	3.293	3.242
SRISE	0	0.0032	0.0132	0.0092

Table 2: Numerical Evaluation

Conclusions

- Introduction of functional AR method for modelling time varying densities. We show that it embeds models such as scalar AR, ARCH, ARCH-M, etc.
- The consistency of the estimator of autoregressive operator under conditions on the convergence of functional estimators.
- The application of FAR(1) model to Risk-Neutral PDF forecast. This provides yet another tool to risk management and option pricing.