Market Design under Distributional Constraints: Diversity in School Choice and Other Applications

Daniel Fragiadakis  
Stanford University

Peter Troyan*  
Stanford University

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Abstract

Distributional constraints are important in many market design settings. Prominent examples include the minimum manning requirements at each Army branch in military cadet matching and diversity considerations in school choice, whereby school districts impose constraints on the demographic distribution of students at each school. Standard assignment mechanisms implemented in practice are unable to accommodate these constraints. This leads policymakers to resort to ad-hoc solutions that eliminate blocks of seats ex-ante (before agents submit their preferences) to ensure that all constraints are satisfied ex-post (after the mechanism is run). We show that these current solutions ignore important information contained in the submitted preferences, resulting in avoidable inefficiency. We then introduce new dynamic quotas mechanisms that result in Pareto superior allocations while at the same time respecting all distributional constraints and satisfying important fairness and incentive properties. We expect the use of our mechanisms to improve the performance of matching markets with distributional constraints in the field.

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1 Introduction

School districts in many U.S. cities have recently begun implementing preference-based student assignment mechanisms as a way to give parents more choice over schools. Examples include New York City, Boston, Chicago, New Orleans, and Denver, among others. An important additional consideration for many school districts when implementing school choice plans is providing a diverse educational environment by ensuring demographically balanced distributions of students at all schools. Historically, race was the main factor on which diversity was based (e.g., the 1965 Racial Imbalance Act in Massachusetts), but in the present day, socioeconomic status (SES) is the most common criterion. For example, Chicago classifies students into four SES tiers and requires that all selective high schools enroll enough students from each tier. In Cambridge, MA, students are divided into high and low SES and the percentage of students at each school from each class must lie within a certain range. Louisville, KY has similar goals, but classifies students based on the area of the city in which they live. Determining the desired diversity goals is only the first step, however; next, the school district must decide on the assignment mechanism that will be used to actually achieve them in practice. This opens up a wide range of possibilities and design considerations, and it is this problem that we address in this paper. Our main contribution is the introduction of new mechanisms that can be used by school districts to improve on current mechanisms with respect to efficiency, while satisfying all distributional constraints as well as other desirable properties such as fairness and incentive compatibility.

While school choice is the main focus, our model and results have many other applications. In medical residency markets, hospitals in rural areas often suffer from doctor shortages relative to those in urban areas. To solve this problem, constraints are imposed on the geographic distribution of doctors by limiting the number of doctors who can be assigned to hospitals in urban areas. In a different context, the United States Military Academy (USMA) uses a preference-based mechanism to assign newly graduated cadets to positions in various Army branches. While the USMA attempts to respect the cadet preferences as much as possible, each Army branch has minimum staffing requirements that must be met in order for it to function properly.

The common theme in all of these applications is the presence of floors, i.e., a minimum number of agents who must be assigned to each institution (in the case of school choice, there may be multiple floors at each school, one for each socioeconomic class). While the literature thus far has developed many successful mechanisms for markets with ceiling constraints, these mechanisms are inadequate for the many institutions that also have floor constraints. For instance, the original school choice mechanisms found in the seminal paper of Abdulkadiroğlu and Sönmez (2003) would allow a school to have (for example) a total capacity of 100 seats in addition to ceiling constraints of at most 50 high SES students and at most

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1See http://www.matching-in-practice.eu for further examples in Europe.
3"Case Studies of School Choice and Open Enrollment in Four Cities,” (Cowen Institute, 2011).
4For example, in Hartford, CT, school diversity has been court-ordered (Sheff v. O’Neill 1996), but actual procedures for practical implementation are still under debate.
5The Japan Residency Matching Program follows such a procedure (Kamada and Kojima, 2013).
50 low SES students. Note, however, that an assignment of 50 high SES students would satisfy these constraints, yet would be completely segregated, and thus would not satisfy the true diversity objectives of many school districts.

Suppose the school also imposed floors of 25 high and 25 low SES students (in addition to the ceilings of 50). This will ensure a minimum level of diversity at the school. One convenient way the district can ensure that these floors will be met is to 1) lower the ceilings at other schools and then, 2) run an “off-the-shelf” mechanism designed to handle only ceiling constraints. We call this approach imposing artificial caps. It works by the simple intuitive principle that restricting someone from one school results in their being assigned to another.

Because of its intuitive nature, artificial caps is in fact a commonly used approach. It is precisely how the Japanese Residency Matching Program guarantees enough doctors will be assigned to rural hospitals, and how the USMA ensures enough cadets will be assigned to each Army branch. They first eliminate sufficiently many positions ex-ante (i.e., before preferences are submitted), and then run the well-known deferred acceptance (DA) mechanism of Gale and Shapley (1962). This procedure will satisfy all floors ex-post (i.e., when the final matching is reached). We call this algorithm artificial caps deferred acceptance (ACDA).

In this paper, we first show that artificial caps results in unavoidable inefficiency. The reason is that, in order to satisfy all of the constraints, ACDA must eliminate positions aggressively. To understand why, note that in a given matching problem, only one set of preferences, $P$, is submitted. It may be the case that the ceilings required for DA to satisfy all floors under $P$ are not as low as those needed for ACDA (which must be low enough to ensure that floors are met ex-post for any possible preferences that could have been submitted). This is problematic, since eliminating a seat at a school makes every student weakly worse off. Accordingly, we introduce the idea of a dynamic quotas (DQ) mechanism, where dynamic quotas deferred acceptance (DQDA) runs as follows: Start with the original ceilings and run DA. If the matching is feasible (i.e., meets all floors), end the algorithm. Otherwise, lower the ceiling at one school by one. This causes a rejection chain, where that school rejects a student, who then applies to her next most preferred school, which then (may) reject a student, and so forth, until a student applies to a school with an open seat. When this rejection chain ends, if the matching is feasible, end the algorithm; if not, lower the ceiling of another school by one, and so forth. We show that if we choose the order in which ceilings are lowered carefully, DQDA will (i) always produce a matching that satisfies all distributional constraints, and (ii) will Pareto dominate artificial caps.

The final ceilings implemented will depend on the submitted preferences. This is precisely how we obtain the efficiency gain over ACDA: by using information contained in the submitted preferences to determine the final ceilings, we are able to eliminate fewer seats than artificial caps. A concern that arises from this modification is that individuals now have the ability to change the ceilings by submitting different preferences. This poses a potential incentive problem: perhaps an individual can do better by misreporting her preferences than by stating them truthfully, i.e., the mechanism may not be strategyproof.\footnote{Strategyproofness is sometimes referred to as dominant strategy incentive compatibility.} Non-strategyproof mechanisms are unattractive to many school districts because they allow some parents to “game the system” and profit at the expense of less sophisticated parents who may naively be truthful.
In addition, a non-strategyproof mechanism means the collected preference data, which can be used to identify and improve undesirable schools, may be distorted and not reflective of the actual parent preferences. Thus, before implementing a dynamic quotas mechanism, it is crucial to fully understand its incentive properties. While it may seem that allowing the final ceilings to depend on the preferences will introduce obvious avenues for manipulation, one of our main results is to show that this is not the case. Indeed, we show how to construct the algorithm such that if the ceiling reductions are determined exogenously to the submitted preferences, DQDA is in fact strategyproof.

A final key concern to school districts is how to determine which students will be accepted into schools for which demand exceeds supply. To do so, many school districts create priority lists for each school, giving students higher priority for neighborhood schools, sibling attendance, languages spoken, or various other factors, depending on the district. The districts then wish to respect priorities in the following sense: if student $i$ is assigned to school $A$ but prefers the assignment of a student $j$ who is of the same socioeconomic status (e.g., school $B$), then $j$ must have a higher priority at school $B$ than $i$. Thus, $i$’s envy towards $j$ is not justified. Eliminating such justified envy concerns is often an important fairness consideration, and we show that DQDA does indeed satisfy this property.\(^8\)

Taken together, the above results show that DQDA improves on ACDA with respect to efficiency while preserving all of the latter’s incentive and fairness properties, and so replacing current mechanisms with our new approaches should lead to superior allocations. Last, in order to estimate whether the magnitude of the welfare gains would be substantial enough to overcome potential logistical barriers to making such a switch in practice, we use simulations. To keep the results close to that of a real-world market, we structure the simulations around kindergarten assignment in Cambridge, MA.\(^9\) The results show that the rank distribution from our dynamic quotas mechanisms stochastically dominate that of ACDA, with on average over 20% more students receiving their first choice school for some specifications. Thus, there do seem to be substantial welfare gains, and so we believe switching to dynamic quotas is worth consideration, especially given the theoretical properties outlined above.

Before closing, we would like to note that the paper makes a methodological contribution as well. Dubins and Freedman (1981) and Roth (1982) were the first to prove that DA is strategyproof for the proposing side in simple one-to-one matching models. Many papers have since studied what types of generalizations on school choice functions/priority structures are compatible with strategyproofness (Martinez et al., 2000; Abdulkadiroğlu, 2005; Hatfield and Milgrom, 2005; Hatfield and Kominers, 2009, 2012; Hatfield and Kojima, 2010). The presence of the floors makes our model quite different from these papers in a technical sense. In particular, they often rely on the existence of student-optimal stable matchings, a condition which fails in our setting. In addition, previous papers assume that the choice functions of the receiving side are static throughout the algorithm. We are the first to show that dynamic choice functions (quotas) are compatible with strategyproofness, and provide sufficient conditions for this to hold. Interestingly, our conditions are related to the important substitutability conditions in the (static) model of Hatfield and Milgrom (2005) (see also Kelso and Crawford, 1982 and Roth and Sotomayor, 1990). Both

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\(^8\)Both DA and ACDA satisfy this property as well. See Section 3 for details.

\(^9\)Cambridge makes a limited amount of data publicly available which we use to determine many of the simulation parameters. See Section 6.4 for details.
guarantee *monotonicity* of the corresponding cumulative offer process, which is key in proving our main results.

**Related literature**

Early papers to discuss distributional constraints in matching focused on the rural hospital problem and obtained mostly negative results. Papers such as Gale and Sotomayor (1985a,b), Roth (1984, 1986), Martinez et al. (2000), and Hatfield and Milgrom (2005) prove various versions of the “rural hospital theorem,” which says that if a doctor or a position at a hospital is unmatched at some stable matching, then they are unmatched at any stable matching. This suggests that the rural hospital problem is difficult to solve without imposing any additional structure on the market, which is what led the Japan Residency Matching Program (JRMP) to impose regional caps on the number of doctors in urban areas, an issue studied in detail by Kamada and Kojima (2013) (for further details, see Section 2).

Since it is an important goal of many school districts, diversity constraints have been discussed in the school choice literature, but most work thus far deals only with upper quotas/ceilings. Abdulkadiroğlu and Sönmez (2003) show how type specific ceilings can be easily incorporated into standard matching mechanisms. Ceiling constraints do not fully capture diversity constraints, however, since they can still result in completely segregated schools. In addition, in a model with two types of students (majority and minority), Kojima (2012) points out that simple ceiling constraints can actually make all minority students (the supposed beneficiaries) worse off. Hafalir et al. (2013) correct this by proposing deferred acceptance with minority reserves, a mechanism further generalized by Kominers and Sönmez (2013), who introduce slot-specific priorities. Abdulkadiroğlu (2005), Erdil and Kumano (2012), and Echenique and Yenmez (2012) study various generalizations of school priorities over sets of students and how they can capture certain types of diversity goals.

Differently from the aforementioned work, our model treats the ceilings and floors as *hard* constraints, i.e., constraints that must be satisfied at any feasible matching. Such hard constraints complicate the problem considerably, and lead to the incompatibility of several important properties (non-wastefulness, elimination of justified envy, and strategyproofness) that could be achieved simultaneously in prior models. Some recent literature has begun to deal with hard floor constraints. In the context of object allocation, Budish et al. (2013) study what types of constraints admit expected assignments that can be implemented as lotteries over deterministic assignments. Ehlers et al. (2012) study a school choice model with floors and ceilings similar to that studied here, but due to the above impossibility results, they advocate for a “soft” interpretation of the constraints where the floors and ceilings can be violated. While soft bounds may be an acceptable approach in some settings, there are many situations in which it is inadequate, such as medical markets suffering from the rural hospital problem, school districts with court-mandated

10 Afacan (2013) studies whether hospitals can manipulate their preferences to change the number of positions filled. Sönmez (1997) studies the complementary question of whether hospitals can manipulate their capacities to obtain a more preferred assignment of doctors.

11 See also Westkamp (2013), who proposes similar mechanisms in the context of German university admissions, and Braun et al. (2013), who conduct an experimental analysis of these mechanisms.

12 They also provide an algorithm for hard constraints that eliminates justified envy among same types and is constrained non-wasteful, but this mechanism is not strategyproof.
desegregation guidelines, or the military, where minimum manning requirements must be filled. This paper provides mechanisms that can be used in these settings. Fragiadakis et al. (2013) also study a model with floor constraints. They propose strategyproof mechanisms that satisfy all floors, but their mechanisms run differently from those given in this paper and they must introduce non-standard notions of fairness. In addition, their model is unable to handle diversity constraints, which is as an important contribution of the current paper.

Finally, the problem of distributional constraints has also garnered interest in the computer science community, where many of the results are negative. For example, Biró et al. (2010) study college admissions in Hungary, in which colleges are allowed to declare minimum quotas for their programs, and Hamada et al. (2011) study hospital-resident matching with lower bounds. Both papers focus mainly on (computational) hardness results: the former shows that the problem of determining the existence of a stable matching is NP-complete, while the latter shows that the same is true of finding a matching that minimizes the number of blocking pairs. These papers provide another perspective which says that introducing floors into matching markets complicates the problem substantially, though they do not propose specific mechanisms nor study incentive or efficiency issues, as we do here.

2 Motivating applications

School choice

While far from the only example of matching markets with distributional constraints, public school choice is the most prominent. As a concrete example, consider Cambridge, MA. Cambridge began its school choice plan in 1980, using race as the main factor. In 2001, Cambridge voluntarily shifted from classifying students by race to classifying them as either high or low socioeconomic status (SES) based on whether or not they qualify for the federal Free and Reduced Lunch program. The school district then requires that each school be within 10% of the district-wide average for each SES class. They accomplish this by imposing floors and ceilings for each SES class at each school, which translates into ranges of approximately 25-45% for low SES students and 55-75% for high SES students (these numbers may vary slightly from year to year). Many other school districts (for example, Louisville, KY and Montclair, NJ) have very similar diversity plans. Cambridge is a useful benchmark to keep in mind because they release some public data on their school choice process. We will make use of this data when conducting simulations in Section 6.

Medical residency markets

A second well-documented example of distributional concerns in matching markets is the so-called rural hospital problem often observed in matching newly graduated doctors to hospital residency programs. For example, Talbott (2007) notes that the United States as a whole has 280 doctors per 100,000 people, but the 18-county Mississippi Delta area has only 103 doctors per 100,000 people. Similar doctor shortages in rural areas are present in many countries, such as the United Kingdom and Australia, among others (Shallcross (2005); Nambiar and Bavas (2010)).
In Japan, in order to address this issue, the Japan Residency Matching Program (JRMP) imposes regional caps on the number of doctors who can be assigned to each of the country’s 47 prefectures. For example, in 2008, 860 positions were offered in Osaka (a popular urban area), but the JRMP unilaterally reduced the capacity of all hospitals in Osaka by approximately 40% before running the deferred acceptance algorithm (similar reductions occurred in most urban areas; see Kamada and Kojima (2013)). The end goal of this policy is not to limit the number of doctors in urban areas per se, but to satisfy some implicit floors in rural areas. Our results suggest that the JRMP may improve outcomes by instead modeling the floor constraints explicitly and using dynamic quotas. Doing so will satisfy the actual distributional goals (the floors) and make all doctors (weakly) better off, compared to the current approach of imposing regional caps and hoping that the resulting distribution of doctors turns out to be satisfactory.\footnote{As of now, the JRMP does not model floor constraints explicitly, and Kamada and Kojima (2013) identify flaws in their mechanism even if the regional caps are taken as the true objective. An additional distinction is that they impose caps at the regional level, while we consider floors and ceilings at the individual institutional level.}

**Military cadet matching**

As a third example, consider military cadet matching. At the end of every year, over one thousand cadets at the United States Military Academy (USMA) submit preferences over Army branches (Aviation, Infantry, etc.) in which they would like to serve. After collecting cadet preferences, the USMA then runs a matching algorithm to determine the assignment of cadets to branches (a process referred to as branching).\footnote{See Sönmez and Switzer (2013) and Sönmez (2013) for further details.}

An October 1, 2007 memorandum from the Army Deputy Chief of Staff to the USMA entitled “Branch Allocation Methodology” describes the following three phase procedure. In Phase 1, the Army provides the USMA with floors and ceilings for each branch, based on current staffing needs. In Phase 2, given these floors and ceilings, the USMA calculates “demand pegs”, which are the ceilings that will be used at each branch when the matching algorithm is run in Phase 3. These demand pegs will be lower than the true ceilings provided in Phase 1, and are analogous to what we call artificial caps. Then, in Phase 3, the matching algorithm is run using the demand pegs from Phase 2. Similar to the markets above, conducting the branching using the demand pegs is a way for the USMA to ensure that the floors given by the Army are satisfied, but it may do so inefficiently. The memo states that this is done because “there is no ex-ante closed form algorithm that optimizes program participation subject to manning requirements.” Providing such algorithms and studying their properties are precisely the goals of this paper.\footnote{Military branching can also be thought of as a special case of a firm (the Army) that must assign employees to projects (the branches), with each project having a minimum staffing requirement. For example, some technology firms in Silicon Valley use centralized mechanisms to assign new interns to positions (with managers also submitting preferences over employees). In a similar vein, for many new medical residents, the first year after medical school is a transitional year in which they rotate through various departments of a hospital. While the hospitals try to accommodate preferences as much as possible, each department has a minimum staffing requirement.}

**Hidden floor constraints**

In many of the examples above, the use of artificial caps is openly discussed and publicly known. In others, however, artificial caps may be used in such a way that it is unobserved to outsiders. That is, a
school district (for example) may set artificial caps in order to implicitly satisfy some floors, but to an outside analyst, it would appear as if the artificial caps were the “true” ceilings and there were no floors. Since the idea behind artificial caps is an intuitive one that naturally occurs to policymakers, we believe that our mechanisms may be applicable to many markets that have up to now dealt with floors in such a suboptimal manner simply for lack of a better option. Once mechanisms are available that can handle floors, they may begin to explicitly appear in many more markets beyond those listed here.

3 Model and artificial caps

For concreteness, we use the language of school choice, with the interpretation that the distributional constraints correspond to the diversity goals of a school district, though our results will apply more broadly.

There is a set $I = \{i_1, \ldots, i_n\}$ of students and a set $S = \{s_1, \ldots, s_m\}$ of schools. $\Theta = \{\theta_1, \ldots, \theta_r\}$ is a finite set of student types. Each student is of exactly one type, which, for concreteness, can be thought of as her socioeconomic class. If $|\Theta| = 1$, then all students are of one type, and the floors and ceilings will correspond to aggregate constraints on the total number of students assigned to each school (e.g., military branch assignment). The function $\tau : I \rightarrow \Theta$ gives the type of each student. $I_{\theta}$ is the set of students of type $\theta$.

Each student $i$ has a strict preference relation $P_i$ over $S$, while each school $s$ has a strict priority relation $\succ_s$ over $I$. Vectors of such relations, one for each agent, are denoted $P_I = (P_i)_{i \in I}$ and $\succ_S = (\succ_s)_{s \in S}$. Let $P$ denote the set of all individual preference relations, and $P^n$ denote the set of all preference profiles $P_I$. The student preferences are their own private information. As is standard in the school choice literature, the school priorities are fixed and known to all students. In applications, priorities are often set by law, and depend on such things as distance from a school, whether a student has a sibling attending the school, or whether a student speaks a certain language.

Each school $s$ has a type-specific floor $L_{s,\theta}$ (or lower quota) and a type-specific ceiling $U_{s,\theta}$ (or upper quota) for each type $\theta$ and a total capacity $Q_s$. We assume $0 \leq L_{s,\theta} \leq U_{s,\theta} \leq Q_s$ for all $(s, \theta)$. Let $Q = (Q_s)_{s \in S}$ be the vector of all school capacities and $L = (L_{s,\theta})_{s \in S, \theta \in \Theta}$ and $U = (U_{s,\theta})_{s \in S, \theta \in \Theta}$ be the matrices of all type-specific floors and ceilings, respectively.\(^\ast\)

A matching is a correspondence $\mu : I \cup S \rightarrow I \cup S$ that describes which students are assigned to which schools. Formally, $\mu$ must satisfy: (i) $\mu(i) \in S$ for all $i \in I$, (ii) $\mu(s) \subseteq I$ for all $s \in S$, and (iii) $\mu(i) = s$ if and only if $i \in \mu(s)$.\(^\dagger\) Let $M$ denote the set of matchings. For any $\mu \in M$, we let $\mu_{\theta}(s)$ be the set of type $\theta$ students assigned to school $s$ under matching $\mu$. Matching $\mu$ is feasible if $L_{s,\theta} \leq |\mu_{\theta}(s)| \leq U_{s,\theta}$ and

\(^\ast\)School districts sometimes state diversity constraints in terms of percentages because they are simpler to communicate, but these percentages are usually converted into absolute numbers of seats when actually running the algorithm. This is done in Cambridge, for example (see http://www3.cpsd.us/video/controlled_choice_video for a video describing the implementation of the Cambridge algorithm, targeted towards parents). From a technical perspective, the use of percentages introduces complementarities, which leads to many impossibility results (see Echenique and Yenmez (2012)). For these reasons, most formal models use absolute numbers (e.g., Ehlers et al. (2012) and Hafalir et al. (2013)). When there is only a single type, this distinction is immaterial.

\(^\dagger\)Note that we assume that every student is assigned to a school. This is consistent with school choice settings, where every student is guaranteed a seat at a public school, and military branching, where every cadet must serve in a branch. In Section 7, we discuss an extension in which students are allowed to have an outside option (e.g., a private school).
\(|\mu(s)| \leq Q_s\) for all \((s, \theta)\). In words, a feasible matching is one that satisfies all of the type-specific floors and ceilings, as well as the capacities. Let \(\mathcal{M}_f \subseteq \mathcal{M}\) denote the set of feasible matchings. We assume throughout the paper that the set of feasible matchings is nonempty; this requires that the distributional constraints be consistent with the number of students of each type actually present in the market.

A mechanism \(\psi : \mathcal{P}^n \to \mathcal{M}\) is a function that maps preference profiles to matchings. If the students submit \(P_i \in \mathcal{P}^n\), then \(\psi(P_i) \in \mathcal{M}\) is the resulting matching. We write \(\psi_i(P_i)\) for the school to which student \(i\) is assigned, and \(\psi_s(P_i)\) for the set of students assigned to school \(s\). We say that \(\psi\) is feasible if \(\psi(P_i) \in \mathcal{M}_f\) for all \(P_i \in \mathcal{P}^n\).

Given two matchings \(\mu, \nu \in \mathcal{M}_f\), \(\mu\) Pareto dominates \(\nu\) if \(\mu(i)R_i\nu(i)\) for all \(i \in I\) and \(\mu(i)P_i\nu(i)\) for some \(i \in I\).\(^{18}\) If \(\mu \in \mathcal{M}_f\) is not Pareto dominated by any other \(\nu \in \mathcal{M}_f\), then we say that \(\mu\) is Pareto efficient.\(^{19}\)

We say student \(i\) of type \(\theta\) claims an empty seat at school \(s\) if (i) \(sP_i\mu(i)\), (ii) \(|\mu(s)| < Q_s\) and \(|\mu_\theta(s)| < U_{s, \theta}\), and (iii) \(|\mu_\theta(\mu(i))| > L_{\mu(i), \theta}\). If no student claims an empty seat under matching \(\mu\), then \(\mu\) is non-wasteful. In words, non-wastefulness means that whenever a student prefers a school \(s\) to her current assignment, it is impossible to move her to \(s\) without violating feasibility.

A second property is elimination of justified envy, a fairness requirement commonly used in school choice settings.\(^{20}\) Student \(i \in \mu(s)\) justifiably envies student \(i' \in \mu(s')\) if (i) \(s'P_is\), (ii) \(i \succ_{s'} i'\), and (iii) there exists an alternative matching \(\nu \in \mathcal{M}_f\) such that \(\nu(i) = s'\), \(\nu(i') \neq s'\), and \(\nu(j) = \mu(j)\) for all \(j \neq i, i'\). If no student justifiably envies any other, then the matching eliminates justified envy. In words, student \(i\) justifiably envies \(i'\) if she prefers the school of student \(i'\), has higher priority than \(i'\) at this school, and \(i\) and \(i'\) can be reassigned without violating any constraints.\(^{21}\)

The above properties have counterparts for mechanisms. Mechanism \(\psi\) is non-wasteful if \(\psi(P_i)\) is a non-wasteful matching for all \(P_i \in \mathcal{P}^n\), and \(\psi\) eliminates justified envy if \(\psi(P_i)\) is a matching that eliminates justified envy for all \(P_i \in \mathcal{P}^n\). We say that mechanism \(\psi\) Pareto dominates mechanism \(\varphi\) if

\[
\text{for all } P_i : \psi_i(P_i)R_i\varphi_i(P_i) \text{ for all } i \in I
\]

\[
\text{for some } P_i : \psi_i(P_i)P_i\varphi_i(P_i) \text{ for some } i \in I.
\]

Mechanism \(\psi\) is strategyproof if \(\psi_i(P_i)R_i\psi_i(P'_i, P_{-i})\) for all \(i \in I\), \(P_i \in \mathcal{P}^n\), and \(P'_i \in \mathcal{P}\). In words, a mechanism is strategyproof if no student can ever gain by misreporting her preferences, no matter what

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\(^{18}\)We use \(R_i\) to denote the weak preference relation corresponding to \(P_i\), i.e., \(sR_is'\) if and only if \(sP_i s'\) or \(s = s'\).

\(^{19}\)Only the welfare of the students is considered, which is consistent with the school choice mechanism design literature in which school seats are viewed as objects to be consumed by the students (see Abdulkadiroğlu and Sönmez (2003)).

\(^{20}\)For example, this was an important criterion to administrators of the Boston school district when they were redesigning their school assignment mechanism. See Abdulkadiroğlu et al. (2005a).

\(^{21}\)In two-sided matching models without distributional constraints, non-wastefulness and elimination of justified envy are often combined into one definition called stability, which is usually then given a positive interpretation. We must separate the two definitions due to impossibility results caused by the introduction of the floors (discussed below). In addition, in many school choice settings, these properties are more usefully interpreted in a normative manner (see also Kamada and Kojima (2013), who use normative justifications for stability concepts in hospital residency matching in Japan). Balinski and Sönmez (1999), Ehlers et al. (2012), and Fragidakis et al. (2013) use similar distinctions between non-wastefulness and elimination of justified envy as we do here.
the other students report. Strategyproofness is a strong form of incentive compatibility, and is viewed as an important property for many reasons.

- First, strategyproof mechanisms advance the so-called Wilson Doctrine (Wilson (1987)), which argues that to be successful, market designs should not be sensitive to specific assumptions on agent beliefs (see also Bergemann and Morris (2005)). Strategyproof mechanisms satisfy the Wilson Doctrine in its strongest sense, since truthful reporting is optimal for any beliefs agents may have.

- Second, from a practical perspective, school districts are interested in strategyproof mechanisms because they are strategically simple for parents to play. The school district can inform the parents that all they must do is submit their true preferences, and unsophisticated parents who are unable to strategize effectively will not be disadvantaged.\(^{22}\)

- Third, school districts often lack hard data on what makes schools desirable to parents. School choice mechanisms produce rich data on parent preferences, and, if the mechanism is strategyproof, it is reasonable to assume the data is truthful. This data can then be analyzed and used to improve schools in accordance with parent preferences.\(^{23}\)

For these reasons, many cities have opted for school choice mechanisms that are strategyproof (among them, New York City, Boston, and New Orleans). Strategyproofness has been an important design consideration in other settings as well, such as hospital-resident matching (Roth (1991); Roth and Peranson (2002)).\(^{24}\)

When there are no floor constraints, variations of the deferred acceptance (DA) algorithm of Gale and Shapley (1962) are often used. The following is a simple generalization of Gale and Shapley’s algorithm for school choice with ceiling constraints.

**Deferred acceptance (DA)**

**Step 1** Each student applies to the first school on her preference list. Each school \(s\) considers all students who have applied to it, and tentatively accepts students as follows:

1. **Type-specific seats:** For each type \(\theta\), school \(s\) accepts the \(L_{s,\theta}\) highest-ranked type \(\theta\) students according to \(\succ_s\).

2. **Open seats:** For any students remaining in the applicant pool, school \(s\) admits students one-by-one from the top of its priority order, unless either some type-specific ceiling \(U_{s,\theta}\) would be violated or \(Q_s - \sum_{\theta \in \Theta} L_{s,\theta}\) open seats have already been filled. All students not accepted are rejected.


\(^{23}\)See, for example, Elissa Gootman “Lafayette Among 5 High Schools to Close,” *New York Times*, December 14, 2006, which reports that demand data from the NYC assignment algorithm was an important reason for the closing of South Shore High School.

\(^{24}\)Strategyproofness is of course not costless, as is shown in a recent strand of the school choice literature started by Abdulkadiroğlu et al. (2011) that finds that non-strategyproof mechanisms may sometimes outperform strategyproof ones on welfare grounds, at least in equilibrium (see also Featherstone and Niederle (2011), Troyan (2012), and Akyol (2013)). However, the equilibria of these mechanisms can be complex, and it may be difficult for many parents to calculate best responses.
Step \( k \): Each student who was rejected in step \( k - 1 \) applies to her most preferred school that has not yet rejected her. Each school \( s \) considers its new applicants in step \( k \) jointly with the students tentatively admitted from step \( k - 1 \), and again tentatively accepts students in its applicant pool in the same manner as above. All students not accepted are rejected.

In this version of DA, each school reserves \( L_{s,\theta} \) seats exclusively for students of type \( \theta \); the remaining \( Q_{s} - \sum_{\theta \in \Theta} L_{s,\theta} \) seats are open seats, that can go to students of any type, subject to the ceiling constraints \( U_{s,\theta} \). When the floors are set to 0 at all schools, the above algorithm reduces to that defined by Abdulkadiroğlu and Sönmez (2003), and when there is only a single type, it reduces further to the algorithm of Gale and Shapley (1962). In these simpler environments, DA is non-wasteful, eliminates justified envy, and is strategyproof. The main shortcoming is that with hard floor constraints, it can produce a matching that is not feasible. This is shown in the following example.

Example 1. Let there be three schools \( A, B, \) and \( C \), each with a capacity of 20 seats. There are 40 students, divided into two types: students \( h_1, \ldots, h_{20} \) are of high socioeconomic status (type \( h \)), and students \( \ell_1, \ldots, \ell_{20} \) are of low socioeconomic status (type \( \ell \)). The diversity goals of the school district are to have between 5-15 students of each type at each school. Let the preferences of all of the \( h \) students be \( P_{h_i} : A, B, C \) and the preferences of all of the \( \ell \) students be \( P_{\ell_i} : A, C, B \). For simplicity, let the priorities of all schools rank \( h_1 \succ_s h_2 \cdots \succ_s h_{20} \succ_s \ell_1 \cdots \succ_s \ell_{20} \). DA using the true ceilings of 15 for each type at each school produces the output shown in Figure 1.

![Figure 1: Outcome of DA with ceilings of 15 for each type (represented by the red line). The green bars represent the number of \( h \) students at each school, while the orange bars represent the number of \( \ell \) students at each school.](image)

The problem is that schools \( B \) and \( C \) are completely homogeneous. A solution to this problem is to run DA under some lower ceilings, a mechanism we call artificial caps deferred acceptance (ACDA). However, to guarantee that all of the floors will be filled for any preference profile, the artificial caps must be quite strict. For example, Figure 2 shows that under the above preferences, artificial caps of 8 will still leave some floors unfulfilled. Artificial caps of 7 will fill all floors, not just for the given preference profile, but indeed for any possible preference profile. For other preference profiles, however, artificial of 7 may be unnecessarily stringent, and hence, very wasteful. For example, consider an alternative preference profile in which the first choice of students \( h_1, \ldots, h_{12} \) is \( A \), that of \( h_{13}, \ldots, h_{16} \) is \( B \), and that of \( h_{17}, \ldots, h_{20} \)
is $C$. An assignment which gave all students their first choice would be feasible, but would violate the artificial caps of 7. Thus, if rigid artificial caps were used, the resulting assignment will be inefficient.

To define artificial caps DA formally, let $\bar{U} = (\bar{U}_s, \bar{\theta})_{s, \theta \in \Theta}$ be some alternative type-specific ceilings and $\bar{Q} = (\bar{Q}_s)_{s \in S}$ be some alternative capacities that may be different from the primitive $U$ and $Q$. We call any such $(\bar{U}, \bar{Q})$ a vector of artificial caps.

Let $M(\bar{U}, \bar{Q}) = \{\mu \in M : |\mu_\theta(s)| \leq \bar{U}_s, \bar{\theta} \text{ for all } (s, \theta) \text{ and } |\mu(s)| \leq \bar{Q}_s \text{ for all } s\}$. In words, $M(\bar{U}, \bar{Q})$ is the set of matchings that respect $(\bar{U}, \bar{Q})$. Such matchings need not be feasible, as some floors may still be violated.

**Definition 1.** Artificial caps $(\bar{U}, \bar{Q})$ ensure a feasible match if $M(\bar{U}, \bar{Q}) \subseteq M_f$.

In words, ensuring a feasible match means that any matching that satisfies the explicit artificial caps also implicitly satisfies the (true) ceilings, capacities, and floors. The first natural question to ask is whether such a feasibility ensuring $(\bar{U}, \bar{Q})$ exists.

**Theorem 1.** The set of vectors $(\bar{U}, \bar{Q})$ that ensure a feasible match is nonempty.

The proof chooses some feasible $\mu$ and sets $\bar{U}_s, \theta = |\mu_\theta(s)|$ and $\bar{Q}_s = |\mu(s)|$ for all $(s, \theta)$, which corresponds to predetermining exactly the number of students of each type $\theta$ who will be assigned to each school before students even submit their preferences. There will in general be many choices of $(\bar{U}, \bar{Q})$ that ensure a feasible match.

We then formally define the artificial caps deferred acceptance algorithm (ACDA) as the deferred acceptance algorithm using some $(\bar{U}, \bar{Q})$ that ensure a feasible matching.

**Properties of ACDA**

Without floors, it is well-known that DA satisfies non-wastefulness, elimination of justified envy, and strategyproofness. In the presence of floors, however, an impossibility result obtains: matchings that eliminate justified envy may not even exist (Ehlers et al. (2012)). This is intuitive, since school districts often use floors to give certain groups of students access to schools they would not be able to obtain based
on priority alone. This observation leads to a natural alternative fairness criterion: a matching/mechanism **eliminates justified envy among same types** if no student justifiably envies another student of her same type. This is a reasonable criterion, because any remaining priority violations are caused by the diversity constraints, which the school district finds inherently valuable.

**Theorem 2.** ACDA eliminates justified envy among same types and is strategyproof. However, ACDA may be wasteful.

The strategyproofness and (same type) envy-freeness of ACDA are immediately inherited from the fact that DA is strategyproof and eliminates justified envy among same types. ACDA is likely a popular mechanism because it satisfies these two properties, while at the same time filling all floors and, crucially, being very easy to implement. However, the cost of this approach is the potential inefficiencies that arise from the waste of seats.25

### 4 Dynamic quotas deferred acceptance (DQDA)

We now define our new dynamic quotas DA algorithm. To do so, we introduce the concept of a reduction sequence.

**Definition 2.** A reduction sequence is a sequence \( \eta = \{(U^1, Q^1), (U^2, Q^2), \ldots, (U^K, Q^K)\} \) of ceiling-capacity vectors that satisfies:

(i) For all \( k \), \( (U^{k+1}, Q^{k+1}) \leq (U^k, Q^k) \leq (U, Q) \).

(ii) \( (U^K, Q^K) \) ensures a feasible matching.

**Definition 3.** A reduction sequence is **minimal** if the following hold for all \( k \):

(i) For one \((s, \theta)\): \( U_{s, \theta}^{k+1} = U_{s, \theta}^k - 1 \) and \( Q_s^{k+1} = Q_s^k - 1 \)

(ii) For all \((s', \theta') \neq (s, \theta)\): \( U_{s', \theta'}^{k+1} = U_{s', \theta'}^k \)

(iii) For all \( s'' \neq s \): \( Q_{s''}^{k+1} = Q_{s''}^k \)

In words, a reduction sequence is simply a monotonically decreasing sequence of ceiling-capacity vectors. Minimality means that in moving from stage \( k \) to \( k + 1 \), we choose a school-type pair \((s, \theta)\) and lower the type \( \theta \) ceiling at \( s \) and the capacity of \( s \) by exactly one seat; the ceilings and capacities of the remaining schools are unchanged. This construction will be needed for both incentive and efficiency reasons (discussed in detail in the next section).

**Dynamic quotas deferred acceptance (DQDA)**

**Stage 1** Compute the outcome of standard DA under \((U^1, Q^1)\). If the resulting matching is feasible, end the algorithm and output this matching. If not, proceed to stage 2.

**Stage \( k \)** for \( k \geq 2 \)

\( k.0 \) Lower the ceilings and capacities to \((U^k, Q^k)\). Divide each school \( s \) into \( L_{s, \theta} \) type-specific seats for each type \( \theta \) and \( Q_s^k - \sum_{\theta} L_{s, \theta} \) open seats which can be assigned to any type.

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25The wastefulness of ACDA can be seen by referring back to Example 1 (or Example 2 below).
k.1 Beginning with an applicant pool equal to the set of students held at the end of stage \( k - 1 \), school \( s \) tentatively fills the type-specific seats for each type \( \theta \) with the \( L_{s,\theta} \) highest-ranked type \( \theta \) students according to \( \succ_s \). If there are students remaining in the applicant pool, school \( s \) tentatively admits students one by one from the top of its priority order to the open seats, unless either some stage \( k \) type-specific ceiling \( U_{s,\theta}^k \) would be violated or \( Q_s^k - \sum_{\theta} L_{s,\theta} \) open seats have already been filled. All students not accepted are rejected.

\( k.j \) Each student who was rejected in stage \( k.(j - 1) \) applies to her most preferred school that has not yet rejected her. Each school \( s \) considers its new applicants jointly with the students held at the end of stage \( k.(j - 1) \), and tentatively accepts students in the same manner as described in \( k.1 \). All students not accepted are rejected.

Stage \( k \) continues until the substage \( k.j \) at which no students are rejected. If the tentative matching at this point is feasible, end the algorithm and output this matching. If not, proceed to stage \( k + 1 \).

The basic idea behind DQDA is to start with high ceilings and capacities, and check whether given the submitted preferences, the output of DA satisfies the floors as well. If so, the algorithm ends with the high ceilings. If not, only then do we lower the ceilings. This initiates a rejection chain. The rejected students then apply to their next most preferred school, which rejects its lowest priority students, and so forth, continuing until no further students are rejected. We continue gradually lowering the ceilings until all floors are filled. The key is that the dynamic adjustment process of DQDA only lowers ceilings after taking the submitted preferences of the students into account and stops as soon as all floors are filled, which results in fewer seats being eliminated unnecessarily.\(^{26}\)

**Example 2.** The following example provides an illustration of the DQDA algorithm. Let \( S = \{s_1, s_2, s_3, s_4\} \), \( \Theta = \{\ell, h\} \), and \( I = \{\ell_1, h_1, h_2\} \). Consider the school quotas/priorities and student preferences given in the following table. All floors are zero except for the type \( h \) floor at \( s_4 \).

<table>
<thead>
<tr>
<th>Schools</th>
<th>( L_{s,\ell} )</th>
<th>( L_{s,h} )</th>
<th>( U_{s,\ell} )</th>
<th>( U_{s,h} )</th>
<th>( Q_s )</th>
<th>( \succ_s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_1 )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( h_1 \succ_s h_2 \succ_s \ell_1 )</td>
</tr>
<tr>
<td>( s_2 )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( \ell_1 \succ_s h_1 \succ_s h_2 )</td>
</tr>
<tr>
<td>( s_3 )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( \ell_1 \succ_s h_2 \succ_s h_1 )</td>
</tr>
<tr>
<td>( s_4 )</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>( h_1 \succ_s h_2 \succ_s \ell_1 )</td>
</tr>
</tbody>
</table>

26Note that DQDA takes the reduction sequence \( \eta \) as an input, and different choices of \( \eta \) will lead to different DQDA mechanisms. That is, \( \eta \) is part of the definition of the mechanism, rather than a primitive of the model. As an analogy, consider the serial dictatorship mechanism, which first fixes some ordering of the students, and then allows the students to pick their favorite schools according to this ordering. The formal definition of the serial dictatorship takes the fixed student ordering as an input, similar to how DQDA takes \( \eta \) as an input, and different orderings lead to different serial dictatorship mechanisms. As we show below, any choice of a minimal reduction sequence will Pareto dominate ACDA. They key feature needed to retain strategyproofness is that \( \eta \) be fixed ex-ante, i.e., the submitted student preferences cannot affect the order in which ceilings are reduced. Within these constraints, policymakers may actively choose the ceiling that is to be reduced at each stage in order to achieve some desirable policy goals; alternatively, at each stage \( k \) we can randomly choose some pair \( (s, \theta) \), subject to feasibility constraints (just as the random serial dictatorship randomly chooses a student ordering and implements the serial dictatorship using this ordering).
The last object needed is a reduction sequence. Consider the following:

$$\eta = \left\{ \left( \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, 1 \right), \left( \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, 2 \right), \left( \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, 2 \right) \right\}.$$ 

In words, this reduction sequence starts with $$(U^1, Q^1) = (U, Q)$$. At the beginning of stage 2, we lower the capacity and type $h$ ceiling at $s_1$ by 1, while at the beginning of stage 3, we lower the capacity and type $h$ ceiling at $s_2$ by 1. Note that the final entry, $(U^3, Q^3)$, ensures a feasible matching, while $(U^1, Q^1)$ and $(U^2, Q^2)$ do not.

The output of stage $k = 1$ is

$$\mu^1 = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ h_1 & \ell_1 & \ell_2 & \emptyset \end{pmatrix}.$$ 

The type $h$ floor at $s_4$ is not satisfied, and so we must move to stage 2. At the beginning of stage 2, $U_{s_1,h}$ and $Q_{s_1}$ are lowered by 1, and so student $h_1$ is rejected from $s_1$, beginning a rejection chain. When this rejection chain ends, the output at the end of stage 2 is

$$\mu^2 = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ \emptyset & \ell_1 & \ell_2 & h_1 \end{pmatrix}.$$ 

Matching $\mu^2$ satisfies all of the primitive floors and ceilings, and thus there is no need to move to stage 3. The final output is $\mu^2$, which Pareto dominates the matching that would have been implemented by ACDA under artificial caps of $(\bar{U}, \bar{Q}) = (U^3, Q^3)$:

$$\mu^{ACDA} = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ \emptyset & \emptyset & \ell_1 & h_1, h_2 \end{pmatrix}.$$ 

Student $h_1$ is indifferent between the two outcomes, but both $\ell_1$ and $h_2$ strictly prefer DQDA.

There is an alternative way to define a dynamic quotas algorithm that is very natural. At the end of each stage, rather than leaving everyone at their assigned schools and lowering the ceilings from $(U^{k-1}, Q^{k-1})$ to $(U^k, Q^k)$, we could instead remove all students from their assigned schools and run the entire deferred acceptance algorithm from the beginning under the lower ceilings $(U^k, Q^k)$. We call this version of the algorithm sequential deferred acceptance (SDA).

**Sequential deferred acceptance (SDA)**

Let $DA^{(U', Q')} : \mathcal{P}^n \to \mathcal{M}$ denote the DA mechanism (as defined Section 3) using ceilings and capacities $(U', Q')$. Given a reduction sequence $\eta = \{(U^1, Q^1), (U^2, Q^2), \ldots, (U^K, Q^K)\}$, the sequential deferred acceptance algorithm is defined as follows.

**Stage 1** Compute $DA^{(U^1, Q^1)}(P_I)$, the outcome of DA under $(U^1, Q^1)$. If $DA^{(U^1, Q^1)}(P_I)$ is a feasible matching, end the algorithm and output this matching. If not, proceed to stage 2.
In general,

**Stage k** Lower the ceilings and capacities to \((U^k, Q^k)\) and compute \(DA(U^k, Q^k)(P_I)\), the outcome of DA under \((U^k, Q^k)\). If \(DA(U^k, Q^k)(P_I)\) is a feasible matching, end the algorithm and output this matching.

If not, proceed to stage \(k + 1\).

While DQDA and SDA are similar, they are not always equivalent. However, there is an important class of reduction sequences where they will produce the same output: minimal reduction sequences.

**Theorem 3.** For any preferences \(P_I\), the matching produced by DQDA under reduction sequence \(\eta\) is equivalent to that produced by SDA under \(\eta\).

The important implication of Theorem 3 is that the final matching output by DQDA is ex-post equivalent to the matching that would have been produced by standard DA under some (fixed) ceiling-capacity vector \((U^k, Q^k)\). It is important to note, however, that ex-ante, it is not known what the final \((U^k, Q^k)\) will be, as this is determined as a function of the submitted preferences.

## 5 Analysis of DQDA: Single type case

This section and the next contain the main theoretical results. There are important distinctions between the single type case and the multiple type case. In addition, the intuition is easier to grasp when all students are of the same type. Given this, and the fact that the single type model is particularly relevant for markets which only care about the aggregate number of agents assigned to each institution (e.g., military branching, the Japanese medical match, some school districts), we first focus specifically on this case. Section 6 discusses multiple types.

We therefore assume for this section only that \(\Theta = \{\theta\}\). When there is only a single type, the distinction between the type-specific ceilings and capacities is irrelevant. To avoid redundant notation, we ignore the type-specific ceilings \(U\) and write a reduction sequence as a sequence of only capacity vectors: \(\eta = \{Q^1, Q^2, \ldots, Q^K\}\), where each \(Q^k = (Q^k_s)_{s \in S}\) is a vector of school capacities. We still assume that \(\eta\) is a minimal reduction sequence, which in this case means that in moving from \(k\) to \(k + 1\), exactly one school reduces its capacity by exactly one seat.

The following is the main result of this section.

**Theorem 4.** Let \(\eta = \{Q^1, \ldots, Q^K\}\) be a minimal reduction sequence. Then, the following hold:

1. DQDA Pareto dominates ACDA under \(Q^K\).
2. DQDA eliminates justified envy.
3. DQDA is strategyproof.

**Discussion of Theorem 4**

The intuition behind parts (1) and (2) is as follows. By Theorem 3, the final matching produced by DQDA is equivalent to DA under capacities \(Q^k\) for some \(k \leq K\). Since \(Q^k_s \geq Q^k_s\) for all \(s\), there are more seats available at all schools, which makes all students better off (part 1). Further, since within-stage DA eliminates justified envy, the final DQDA matching eliminates justified envy as well (part 2).
Parts (1) and (2) can be shown by analyzing DA within each stage \( k \) separately. Analyzing the incentive properties of DQDA is more complicated, because they cannot be understood by looking at each stage independently. The submitted preferences themselves have the potential to alter the final capacities, which may make it seem that agents have the potential to manipulate.

To understand why DQDA is strategyproof, we first fix the preferences of all students other than \( i \) at \( P_{-i} \), and then classify the potential manipulations a student \( i \) can make into three types. Let \( k \) be the final stage of DQDA if \( i \) submits her true preferences \( P_i \). The three classes of manipulations \( P'_i \) are: contractions (\( P'_i \) causes DQDA to end at some stage \( k' < k \)), extensions (\( P'_i \) causes DQDA to end at some stage \( k' > k \)), and equivalences (\( P'_i \) causes DQDA to end in stage \( k' = k \)). The last category is the easiest to rule out: if both \( P_i \) and \( P'_i \) cause DQDA to end in stage \( k \), then, since DQDA is equivalent to standard DA in stage \( k \), strategyproofness of DA implies that any such manipulation is not profitable. For extensions, if \( P'_i \) causes DQDA to continue beyond stage \( k \), then there are less seats available for all students. This intuitively makes every student worse off, and so extensions will never be profitable.

The most difficult types of manipulations to rule out are contractions. It may seem at first that contractions might be profitable because, just as less seats tend to make students worse off, more seats may have the potential to make students better off, implying that student \( i \) may want to submit a false preference that causes the algorithm to end earlier, under higher ceilings. To understand why this seemingly intuitive reasoning is incorrect, consider a student \( i \) with preferences \( P_i : s_1, s_2, s_3, \ldots \). Begin by running DA on all students other than \( i \), and assume after this is done, there is only one floor seat left to be filled, at school \( s_2 \). Now, the algorithm ends the next time any student applies to \( s_2 \). One option available to student \( i \) is to lie and list school \( s_2 \) as her true first choice, thereby ending the algorithm immediately (a contraction) with her receiving \( s_2 \). If \( i \) instead submits her true preferences and first applies to \( s_1 \), she may initiate a chain of rejections that ends with some other student applying to \( s_2 \), in which case \( i \) receives \( s_1 \), her true first choice. The key observation is the following: even if \( i \) is eventually rejected from \( s_1 \) (e.g., if \( \eta \) eliminates a seat at \( s_1 \)), she will then simply apply to \( s_2 \) and the algorithm ends. The seat at \( s_2 \) will always be available until someone applies to it, at which point the algorithm ends and all assignments are made permanent. Thus, there is no harm in \( i \) reporting her true top choices, because seats at lower-ranked schools are always available to her.

Remark 1. An implicit feature of the definition of DQDA is that the reduction sequence \( \eta \) be determined exogenously to the submitted preferences. This feature is in fact crucial in proving strategyproofness. In Section 6, we define an alternative endogenous-reduction DQDA (EDQDA) algorithm that allows the reduction sequence to be determined endogenously. EDQDA is not strategyproof.

Optimality of DQDA

To summarize, Theorem 4 shows that DQDA satisfies the same incentive and fairness properties as ACDA, while improving on ACDA with respect to efficiency in a Pareto sense. Thus, in our view, market designers would be better served by using a dynamic quotas mechanism. The next natural question is

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\[27\]The order in which students are allowed to apply is irrelevant, a result first shown without distributional constraints by McVitie and Wilson (1971).
whether DQDA is itself an optimal mechanism. The following result says that this is indeed the case, if we also require strategyproofness.

**Theorem 5.** Let $\eta = \{Q^1, \ldots, Q^K\}$ be a minimal reduction sequence that satisfies $Q^1 = Q$. If $\psi$ is a feasible mechanism that Pareto dominates DQDA under $\eta$, then $\psi$ is not strategyproof.

Another way to phrase Theorem 5 is to say that DQDA is on the Pareto frontier of strategyproof mechanisms.\(^{28}\) Thus, it is impossible to improve on DQDA without sacrificing incentives. Given the importance of strategyproofness to many institutions, this theorem gives a sense in which DQDA is an optimal choice of mechanism.

### 6 Analysis of DQDA: Multiple type case

In this section, we return to the general model and discuss the properties of DQDA when students can be of multiple types. With multiple types, each school once again has type-specific ceilings and floors $U_{s,\theta}$ and $L_{s,\theta}$, and a distinct capacity $Q_s$.

#### 6.1 Properties of DQDA with multiple types

The next result is the analogue of Theorem 4 with multiple types.

**Theorem 6.** Let $\eta = \{(U^1, Q^1), \ldots, (U^K, Q^K)\}$ be a minimal reduction sequence. Then, the following hold:

1. DQDA Pareto dominates ACDA under $(U^K, Q^K)$.
2. DQDA eliminates justified envy among same types.
3. DQDA is strategyproof.

Theorem 6 is similar to Theorem 4, but the definition of a minimal reduction sequence is more complex in the multiple type case. Also, in the single type case, DQDA eliminates all justified envy, but here, it only eliminates justified envy among same types.\(^{29}\)

Additionally, the intuition is more complicated in the multiple type case. For part (1), DQDA still ends under some ceilings and capacities $(U^k, Q^k) \geq (U^K, Q^K)$, and it seems that higher ceilings and capacities should make the students better off. However, the full argument is more complex here. Formally, the set of students chosen from any given set of applications must be weakly smaller (in the set inclusion sense) in later stages compared to earlier stages. With only a single type, this follows immediately, because all students have equal access to all seats, and so less seats is always worse from the perspective of every student.

\(^{28}\)This is a common notion of optimality in the literature when strategyproofness is an important constraint. See, for example, Klaus (2008), Abdulkadiroğlu et al. (2009), and Anno and Kurino (2014).

\(^{29}\)Recall that with multiple types, there is an impossibility result that says that matchings that eliminate justified envy (across types) may not even exist (see Section 3), and so the most we can hope to achieve is elimination of justified envy among same types. If, in addition, we require non-wastefulness, another impossibility result obtains: matchings that simultaneously eliminate justified envy among same types and are non-wasteful may also fail to exist (see Ehlers et al. (2012) and Fragiadakis et al. (2013)). Thus, one of these properties must be weakened. Markets that use ACDA are opting to keep elimination of justified envy among same types and weaken non-wastefulness. However, as Theorem 6 shows, ACDA weakens non-wastefulness more than necessary, as DQDA Pareto dominates ACDA while still satisfying elimination of justified envy among same types and strategyproofness.
student. With multiple types, this is not necessarily true. Consider two students, $i$ and $j$, such that $\tau(i) = \theta$ and $\tau(j) = \theta'$. Lowering $U_{s,\theta}$ alone (without also lowering $Q_s$) might benefit $j$ if it causes $s$ to reject $i$, which then opens up a seat for $j$. This means that $j$ can be better off in later stages, and hence DQDA might not Pareto dominate ACDA. When $\eta$ is a minimal reduction sequence, this cannot occur.

A similar issue arises with strategyproofness: lowering the $(s, \theta)$ ceiling alone could make extensions profitable for $j$. This is because an extension may cause $i$ to be rejected, which gives $j$, who is of a different type than $i$, access to a seat at school $s$ that she did not have previously. The minimal reduction sequence restriction ensures that even if $i$ is rejected, this seat cannot be reassigned to $j$ (or any student), and so any such manipulation is not profitable.

Remark 2. In the appendix, we define a more general model that uses the notion of a school choice function, which is an arbitrary function that assigns to each set of potential applicants the subset of students admitted by the school (the set-up is similar to Abdulkadiroğlu (2005) or Hatfield and Milgrom (2005)). We next define a generalized DQDA algorithm where $\eta$ is a sequence of choice functions, which allows our model to be applied to settings where schools may have more general preferences over sets of students than those given here. While DQDA can be defined quite generally, to ensure good properties such as strategyproofness, additional structure on the school choice functions is needed. The most important of these is monotonicity, which says that, fixing the set of applicants at a school, the set of rejected students expands as we move to later stages. This purpose of this condition is similar to the substitutability condition of Hatfield and Milgrom (2005): both guarantee that as the algorithm progresses, a school will never want to admit a student it has previously rejected. See Appendix A for formal definitions.

6.2 Endogenous-reduction DQDA (EDQDA)

While we have mostly been concerned with strategyproof mechanisms because this is an important constraint for many markets, there is of course a trade-off between strong incentives and efficiency. For example, papers such as Abdulkadiroğlu et al. (2011) and Troyan (2012) show that the (manipulable) Boston mechanism can result in efficiency gains over DA in some settings. However, these results requires students to play complicated best-responses, and so it is unclear if these efficiency gains will actually be realized in equilibrium. On other hand, if it were possible to find a mechanism that increased efficiency while still making truthful reporting “almost” a dominant strategy, we could be confident that the efficiency gains would be realized.

To motivate such a mechanism, we first discuss two features of DQDA that were necessary for strategyproofness, but that intuitively limit its efficiency. First, whenever we lower a type-specific ceiling $U_{s,\theta}$, we must also lower the capacity $Q_s$ of school $s$. This means that when a type $\theta$ student is rejected from $s$, her seat cannot be reassigned to a different type of student, even if all of that type’s floors are already satisfied. The second problematic feature from an efficiency standpoint is the exogenous nature of $\eta$ (see Remark 1), because in moving from $k$ to $k+1$, we may lower the quota of a type that has already filled all floors. Both of these features can be wasteful.

We thus introduce a new mechanism called endogenous-reduction DQDA (EDQDA). It runs similarly to DQDA, with corrections to alleviate the inefficiencies of the previous paragraph. To define
it, we use a slightly different definition of a reduction sequence. Let \( \rho = \{(s^1, \theta^1), \ldots, (s^K, \theta^K)\} \) be a sequence of school-type pairs, where each \((s^k, \theta^k) \in S \times \Theta\). \( \rho \) is a baseline order for reducing the ceilings, but, unlike for DQDA, an entry will be skipped if all floors for the corresponding type have already been met. In addition, we will only reduce the type-specific ceilings, and not the capacities. The same entry may appear multiple times in \( \rho \).\(^{30}\)

**Endogenous-reduction DQDA (EDQDA)**

Set \( U^1 = U \).

**Stage 1** Starting with the empty matching, run DA under \((U^1, Q)\), and set \( \mu^1 = DA(U^1, Q)(P_I) \).

(a) If \( \mu^1 \) is a feasible matching, end the algorithm and output this matching.

(b) Otherwise, let \( \Theta = \{ \theta \in \Theta : |\mu^1_\theta(s)| < L_{s,\theta} \text{ for some } s \in S \} \) be the set of types for which at least one floor constraint is not yet satisfied, and let \( Y = \{(s, \theta) \in S \times \Theta : |\mu^1_\theta(s)| > L_{s,\theta}\} \) be the set schools that have an excess of these types of students. Let \((s, \theta)\) be the element of \( Y \) that occurs earliest in \( \rho \). Set \( U^2_{s,\theta} = U^1_{s,\theta} - 1 \) and \( U^2_{s',\theta'} = U^1_{s',\theta'} \) for all other \((s', \theta')\), and delete the earliest occurrence of \((s, \theta)\) from \( \rho \). Proceed to stage 2.

In general,

**Stage \( k \)** Starting with the empty matching, run DA under \((U^k, Q)\) and set \( \mu^k = DA(U^k, Q)(P_I) \).

(a) If \( \mu^k \) is a feasible matching, end the algorithm and output this matching.

(b) Otherwise, let \( \Theta = \{ \theta \in \Theta : |\mu^k_\theta(s)| < L_{s,\theta} \text{ for some } s \in S \} \) be the set of types for which at least one floor constraint is not yet satisfied, and let \( Y = \{(s, \theta) \in S \times \Theta : |\mu^k_\theta(s)| > L_{s,\theta}\} \) be the set schools that have an excess of these types of students. Let \((s, \theta)\) be the element of \( Y \) that occurs earliest in \( \rho \). Set \( U^{k+1}_{s,\theta} = U^k_{s,\theta} - 1 \) and \( U^{k+1}_{s',\theta'} = U^k_{s',\theta'} \) for all other \((s', \theta')\), and delete the earliest occurrence of \((s, \theta)\) from \( \rho \). Proceed to stage \( k + 1 \).

EDQDA functions similarly to DQDA, except instead of following the reduction sequence in order from start to finish, we find the earliest entry \((s^k, \theta^k)\) for which \( s^k \) has an excess of type \( \theta^k \) students and not all type \( \theta^k \) floors have been met. We then reduce the \( \theta^k \) ceiling at \( s^k \) by one and leave everything else fixed. We thus may skip entries if the types corresponding to those entries have already met all floor constraints, which does not occur in DQDA. In addition, we only lower the type-specific ceilings, and not the capacities, so if a type \( \theta \) ceiling is lowered in order to satisfy a floor elsewhere, that student’s seat is not “wasted”, and can be taken by a student of a different type. To ensure that EDQDA produces a feasible matching, \( \rho \) must be chosen in such a way that \((U^K, Q)\) ensures a feasible match, where \( U^K \) is defined as \( U^K_{s,\theta} = U_{s,\theta} - \sum_{k=1}^{K} 1\{ (s^k, \theta^k) = (s, \theta) \} \), where \( 1\{ \cdot \} \) is an indicator function that takes on a value of 1 if the \( k^{th} \) entry of \( \rho \) is \((s, \theta)\).

\(^{30}\)Note also that it is possible to describe any minimal reduction sequence \( \eta \) from the definition of DQDA by listing the school-type pair whose capacity and ceiling are reduced at each step.
These modifications intuitively make EDQDA a more efficient mechanism. While it turns out that EDQDA will not be more efficient in a Pareto sense (for reasons related to the discussion following Theorem 6), the simulations performed below show that on average, the students will prefer EDQDA to both ACDA and DQDA. The cost of these welfare gains is that EDQDA is no longer strategyproof. However, EDQDA may still be a successful mechanism in practice, provided that the potential manipulations are not too easy to enact. This is formalized in the next section.

Remark 3. The final matching produced by EDQDA is again equivalent to DA under some ceilings and capacities \((U', Q')\), and so EDQDA will eliminate justified envy among same types (the argument is equivalent to the one used to prove Theorem 6, part (2)).

6.3 Large markets

In this section, we show formally that EDQDA has good large market incentive properties. There are many ways to formalize the notion of a large market limit. We will use the concept of **strategyproofness in the large (SPL)** proposed by Azevedo and Budish (2013).\(^{31}\) We choose this particular formalization because it is very broadly applicable (beyond matching algorithms), and whether a mechanism is SPL or not turns out to be a good predictor of whether it is a successful mechanism in practical applications.\(^{32}\)

To show that our mechanism is SPL, we must expand the formal model. We consider a sequence of markets, indexed by \(n \in \mathbb{N}\) (which will grow large), where \(n\) is the number of agents, and \(I^n = \{i_1, \ldots, i_n\}\) is the set of agents in market \(n\). \(\Theta\) is a finite set of **quota types**, and each student is of exactly one type in \(\Theta\). As previously, it may be useful to think of each \(\theta \in \Theta\) as a different socioeconomic tier, though the practical meaning of this set may vary across applications. The set \(\Theta\) is fixed for all \(n\), but the number of students of each type \(\theta\), \(|I^n_\theta|\), grows according to some fixed sequence. The set of schools \(S = \{s_1, \ldots, s_m\}\) is also fixed for all \(n\), but the capacities of the schools increase with \(n\). Specifically, for each market \(n \in \mathbb{N}\), school \(s\) has a capacity of \(Q^n_s\), and type-specific floors and ceilings of \(L^n_s, \theta\) and \(U^n_s, \theta\). As in the original model, we collect these quotas into matrices \(L^n, U^n, Q^n\). We assume that the sequence \((L^n, U^n, Q^n)_{n \in \mathbb{N}}\) is such that at least one feasible matching exists for every market \(n\).

Strategyproofness in the large is a cardinal concept. There is a finite set of **payoff (utility) types** \(T\). Corresponding to each \(t_i \in T\) is a von Neumann-Morgenstern expected utility function \(u_{t_i} : \Delta S \to [0, 1]\), where \(\Delta S\) is the set of lotteries over schools. Preferences are private, in that an agent’s payoff depends only on her payoff type \(t_i\) and outcome (lottery). Each utility type \(t_i\) also has an associated ordinal preference relation over schools, which we denote \(P_{t_i}\).

Each school has a finite number of **priority classes** \(Z_s = \{1, \ldots, |Z_s|\}\). Each student \(i\) is assigned to one priority class at each school. Each \(s\) has a primitive (strict) ranking of priority classes \(\succ_s\), and ranks all students in a higher priority class above all students in a lower. Ties within a priority class will

---

\(^{31}\)For other large market incentive compatibility notions that are closely related to SPL, see Immorlica and Mahdian (2005), Kojima and Pathak (2009), and Kojima et al. (2013), who study the large market properties of DA, or Che and Kojima (2010) and Kojima and Manea (2010), who do the same for the probabilistic serial mechanism. One of the main advantages of SPL is that it is not tailored to a specific mechanism, and so can be applied more broadly.

\(^{32}\)Azevedo and Budish (2013) show that non-SPL mechanisms (e.g., pay-as-bid auctions for Treasury bills, priority matching algorithms in hospital-residency markets) tend to perform poorly in the field and are eventually abandoned, while their SPL counterparts (uniform price auctions, DA) are successful and in continued use.
be broken using a random lottery (see below). For each \( i, z_i \in Z = \times_{s=1}^m Z_s \) is a vector that denotes \( i \)'s priority class at each school. One practical interpretation of priority classes is that each class corresponds to a certain zone, with students living within a certain radius of a school receiving higher priority for their neighborhood school than those living farther away. All students in each zone have equal priority, up to the random lottery used to break ties. Consistent with this interpretation, we will sometimes refer to \( Z \) as a set of zones for concreteness, but we emphasize that the priority classes can be based on factors other than geography.\(^{33}\)

To summarize, let \( \Lambda = \Theta \times Z \times T \). A student’s overall type is then an element \( \lambda_i \in \Lambda \). For each market \( n \), define \( n(\theta, z) \) as the number of students of quota-zone type \((\theta, z)\). We assume that \( n(\theta, z) \to \infty \) for all \((\theta, z)\), so that the number of agents of each type \((\theta, z)\) grows large according to some fixed sequence.

**Definition 4.** A (direct) mechanism \( \{(\psi^n)_{n \in \mathbb{N}}, \Lambda\} \) is a sequence of allocation functions \( \psi^n : \Lambda^n \to \Delta(S^n) \) such that every allocation in the support of \( \psi^n(\lambda) \) is feasible for all \( n \) and all \( \lambda \in \Lambda^n \).

Note that a mechanism as defined here produces a random allocation. For notational purposes, the inputs are vectors of types \( \lambda \in \Lambda^n \), but each student \( i \) is restricted to reporting \( \theta_i \) and \( z_i \) truthfully; the only private information is her payoff type \( t_i \).

Given a distribution of payoff types \( \pi \in \Delta T \), define for a student \( i \) of quota-zone type \((\theta_i, z_i)\) the following quantity:

\[
\phi^n_{(\theta_i, z_i)}(t'_i, \pi) = \sum_{\lambda_i \in \Lambda^{n-1}} \psi^n(\lambda'_i, \lambda_{-i}) \Pr(\lambda_{-i}|t_{-i} \sim iid(\pi)).
\]

where \( \lambda'_i = (\theta_i, z_i, t'_i) \) and \( \Pr(\lambda_{-i}|t_{-i} \sim iid(\pi)) \) gives the probability that the realized type profile of the \( n - 1 \) other agents is \( \lambda_{-i} = (\theta_{-i}, z_{-i}, t_{-i}) \), given that the payoff types are drawn iid from some distribution \( \pi \) (recall that \( \theta_{-i} \) and \( z_{-i} \) are fixed). In words, \( \phi^n_{(\theta_i, z_i)} \) gives the outcome an agent of type \((\theta_i, z_i)\) receives when she reports her payoff type as \( t'_i \) and the payoff type reports of the other students are drawn according to \( \pi \).

We are now ready to formally define strategyproofness in the large. Given a finite set \( X \), let \( \bar{\Delta}X \) denote the set of full-support probability distributions over \( X \).

**Definition 5.** (Azevedo and Budish, 2013) Mechanism \( \{(\psi^n)_{n \in \mathbb{N}}, \Lambda\} \) is strategyproof in the large (SPL) if for any \( \varepsilon > 0 \) and any \( \pi \in \bar{\Delta} T \), there exists an \( n_0 \) such that for all \( n \geq n_0 \), \((\theta, z) \in \Theta \times Z \), and all \( t_i, t'_i \in T \):

\[
\bar{u}_i(\phi^n_{(\theta, z)}(t_i, \pi)) \geq \bar{u}_i(\phi^n_{(\theta, z)}(t'_i, \pi)) - \varepsilon.
\]

Last, we must define the EDQDA mechanism in this setting. Given some collection of reduction sequences \( \{\rho^n\}_{n \in \mathbb{N}} \) defined as above, the mechanism proceeds as follows. First, draw a vector of lottery numbers \( \ell \in [0, 1]^n \) uniformly at random, where \( \ell_i \) denotes the lottery number of student \( i \). Then, create a strict priority relation for each school \( s \), \( \succ^*_s \), as follows:

\[
i \succ^*_j \iff z_{i,s} \succ z_{j,s} \text{ or } [z_{i,s} = z_{j,s} \text{ and } \ell_i > \ell_j],
\]

\(^{33}\)Formally, it is necessary to divide the students into a fixed number of priority classes to ensure semi-anonymity of the algorithms, in the sense of Kalai (2004).
where $z_{i,s}$ is student $i$’s priority class (zone) at $s$. Let $\mu^n(\lambda, \ell)$ be the matching produced by the EDQDA algorithm (as defined the previous section) using type-specific floors $L^n$, type-specific ceilings $U^n$, capacities $Q^n$, priorities $\succ^n_s$, reduction sequence $\rho^n$, and (ordinal) preferences $(P_t)_{i \in I^n}$. Then, define $E^n$ as:

$$E^n(\lambda) = \int_{\ell \in [0,1]} \mu^n(\lambda, \ell) d\ell.$$

Theorem 7. The mechanism $\{(E^n)_{n \in \mathbb{N}}, \Lambda\}$ is strategyproof in the large.

Intuitively, in a large enough market, it is unlikely that student $i$’s report will have an effect on the final stage of the algorithm, and thus it is unlikely that she will be able to profitably manipulate. At a formal level, we prove the result by showing that EDQDA satisfies an envy-freeness condition identified by Azevedo and Budish (2013) as sufficient for strategyproofness in the large.

6.4 Simulations

We have thus far argued from a theoretical perspective that our new mechanisms should lead to improved performance of matching markets with floor constraints, as they increase efficiency while still satisfying good incentive and fairness properties. However, the theoretical results do not say anything about the number of students who are made better off by the use of our mechanisms. To answer this question, we turn to simulations.

The purpose of these simulations is two-fold: first, to approximate an actual school choice market and obtain a sense of the magnitude of the potential gains from our mechanisms; second, to conduct comparative statics with respect to important market parameters. With these dual goals in mind, we use the details of kindergarten assignment in Cambridge, MA (for which limited data is publicly available) as an anchor to set the number of students, schools, and student types, but also structure the simulations with enough flexibility to allow us to conduct comparative statics with respect to preference correlation, quotas, and capacities.34

Simulation parameters

There are $n = 750$ students, $m = 12$ schools, and two possible types $\Theta = \{\ell, h\}$. There are 250 students of type $\ell$ (“low SES”) and 500 students of type $h$ (“high SES”).

Student preferences are determined as follows. Student $i$’s utility for school $s$ is $u_i(s) = \alpha v_c(s) + (1 - \alpha)v_p^\ell(s)$, where $v_c(s)$ is a common utility component that is the same across students, and $v_p^\ell(s)$ is $i$’s private, idiosyncratic utility for school $s$. The common component $v_c(s)$ and all private components $v_p^\ell(s)$ are drawn iid uniformly from $[0,1]$. Ordinal preferences are then created from these cardinal preferences. By varying $\alpha$, we can study how mechanism performance varies as a function of preference correlation. A value of $\alpha = 0$ corresponds to uncorrelated preferences, while $\alpha = 1$ corresponds to common preferences. To get a sense of the degree of preference correlation in a real market, we use the Cambridge data, which

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34 The Cambridge data can be accessed at http://www3.cpsd.us/department/frc/FRC.
35 This method of drawing preferences is common in the literature; see, for example, Hafalir et al. (2013) and Miralles (2009). Using other distributions (e.g., normal) leads to similar results.
lists for each school the number of students who rank it as their first choice. The value of $\alpha$ corresponding to the data is $\alpha = 0.13$.\footnote{This value was obtained by finding the $\alpha$ that minimized the distance between 1000 simulated distributions of first choices and the empirical distribution from the Cambridge data. While indicative of student preferences, care should be taken in interpreting this number, because Cambridge uses a non-strategyproof mechanism (immediate acceptance), and so it is unknown if the reported preferences are truthful (which is one of the common arguments made in support of switching to a strategyproof mechanism). However, because our proposed mechanisms are strategyproof, by varying $\alpha$ we are able to get a sense of how our mechanisms would perform if implemented, even if the true $\alpha$ were different from that implied by the current data.}

School priority vectors are drawn uniformly at random, independently across schools. For the school floors, ceilings, and capacities, we consider two cases: low flexibility and high flexibility. Table 1 gives the type-specific floors, ceilings, and capacities for the two cases. For simplicity, we treat all schools symmetrically, and so the numbers in the table correspond to the floors, ceilings, and capacities for each school. In the high flexibility case, the floors are lower and the ceilings are higher (compared to the low flexibility case), meaning there is a wider range of possible final assignments in the high flexibility case. Given the primitive floors, ceilings, and capacities ($L, U, Q$), the artificial capacities ($\bar{U}, \bar{Q}$) are chosen as the highest symmetric values that ensure a feasible matching.

<table>
<thead>
<tr>
<th>Floors</th>
<th>Ceilings</th>
<th>Capacities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low flexibility</td>
<td>Primitive (14,36)</td>
<td>(39,69)</td>
</tr>
<tr>
<td></td>
<td>Artificial -</td>
<td>(21,42)</td>
</tr>
<tr>
<td>High flexibility</td>
<td>Primitive (14,14)</td>
<td>(76,76)</td>
</tr>
<tr>
<td></td>
<td>Artificial -</td>
<td>(21,44)</td>
</tr>
</tbody>
</table>

Table 1: The floors, ceilings, and capacities at each school for the low and high flexibility cases. For entries $\langle x, y \rangle$, $x$ corresponds to the low type $\ell$ and $y$ corresponds to the high type $h$.

We last must discuss how to construct $\eta$ (for DQDA) and $\rho$ (for EDQDA). As noted before, there are many possible ways to do this. Since it is not obvious ex-ante which should be chosen, we do so randomly and symmetrically. That is, at each stage we randomly choose which quota to reduce, subject to the constraint that all $(s, \theta)$ must be reduced once before any is reduced for a second time (and all must be reduced twice before any is reduced a third time, etc.). For $\eta$, we also reduce the capacity of school $s$ and each stage, while for $\rho$, we do not. For more details on the specification of $\eta$ and $\rho$, see the appendix.

**Simulation results**

We run 150 iterations for each set of parameters. The outcome metric is the average rank distribution over these 150 iterations. The rank distribution plots, for each $x$, the number of students who receive their $x^{th}$ or better choice. It is one of the common metrics publicly reported by school districts when evaluating mechanisms (for example, on its website, the Cambridge school district says that “85% of students receive one of their top 3 choices,” which is a simplified reporting of the rank distribution).\footnote{Motivated by the fact that authorities are often concerned with rank distributions, Featherstone (2011) investigates mechanisms that optimize the rank distribution in the context of object allocation without priorities.}

In Figure 3, we plot the rank distributions for our mechanisms under different choices of parameters. To read the figures, take the top left panel as an example: it says that under ACDA, about 620 students...
on average get their first choice, about 725 get their first or second choice, etc., while under EDQDA, about 700 students get their first choice, 745 get their first or second choice, etc. For clarity, we only plot the beginning of the rank distributions, because at higher ranks \( x \), essentially all students are getting their \( x^{th} \) or better choice under all mechanisms.

The figure includes plots for six parameterizations: three values of \( \alpha (\alpha = 0, 0.13, 0.26) \) and two levels of flexibility. In the appendix, we report different results for other parameter values, but the three main takeaways can be obtained from Figure 3.

1. Comparing ACDA, DQDA, and EDQDA: In every case, the average rank distribution for ACDA is stochastically dominated by DQDA, which in turn is itself stochastically dominated by EDQDA. This confirms the Pareto dominance of DQDA over ACDA (Theorem 6). While there is no analogous Pareto dominance of EDQDA over DQDA, the simulations show that it is the case that the students will on average be better off under EDQDA compared to both DQDA and ACDA. The results suggest that there are significant gains to be had from our mechanisms, with on average over 20% more students receiving their first choice under EDQDA compared to ACDA for some parameterizations (see the appendix).

2. Comparative statics with respect to preference correlation: Within each column, the flexibility is fixed, but the preference correlation increases as we move from the top row to the bottom. As can be seen in the figure, the gains from our mechanism are larger when the preference correlation is smaller. Intuitively, this is because when preferences are uncorrelated, the floors are more likely to be filled in the early stages of the dynamic quotas mechanisms. As the correlation increases, this becomes less likely, and dynamic quotas is more likely to end in later stages, making it closer to artificial caps. However, even at high correlations, there are still gains to be had from our mechanisms.

3. Comparative statics with respect to flexibility: Within each row, the preference correlation \( \alpha \) is fixed, but the flexibility increases as we move from the left column to the right. As can be seen in the figure, the gains from our mechanism are larger when there is more flexibility. Intuitively, when there is less flexibility (i.e., the floors and ceilings are “close”), dynamic quotas becomes more similar to artificial caps. In the extreme case when all ceilings are equal to all floors and there is no flexibility, DQDA is equivalent to ACDA (and to EDQDA). As the flexibility increases, there is more room for our dynamic quotas mechanisms to respond to the submitted preferences of the agents, and hence the gains obtained from using a dynamic quotas mechanism increase.

In summary, while our mechanisms will produce efficiency gains for any parametrization, the potential gains are increasing in flexibility and decreasing in preference correlation. However, we still recommend the use of DQDA or EDQDA even in markets with low flexibility or high preference correlation, because they will make the students better off on average, without sacrificing fairness or incentive properties.
Below we present simulation results for additional parameter values. The first column corresponds to the low flexibility case, while the second corresponds to the high flexibility case. Each row corresponds to different values of the correlation parameter $\alpha$. Note that for image clarity, the vertical axes differ across figures.

![Graphs showing rank distribution for low and high flexibility with different $\alpha$ values](image)

Figure 3: The number of students assigned to their $x^{th}$ or better choice for various parameter values, averaged over 150 iterations. Higher plots correspond to better mechanisms (on average) for the students. For clarity, we only plot the beginning of the distributions, but the first-order stochastic dominance (FOSD) relation $\text{EDQDA} \succ_{\text{FOSD}} \text{DQDA} \succ_{\text{FOSD}} \text{ACDA}$ holds for all values of $x$. 

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7 Extensions

7.1 General mechanisms with dynamic quotas

For most of the paper, we have focused specifically on deferred acceptance. However, the main ideas behind artificial caps and dynamic quotas can in fact be applied much more broadly to any mechanism that has inputs for ceiling constraints but not for floors, including immediate acceptance, the serial dictator, and top trading cycles.

Formally, we define a class of mechanisms called upper quota mechanisms. Upper quota mechanisms are indexed by vectors of ceilings and capacities \((U', Q')\), which we refer to jointly as “upper quotas”. Recall that \(\mathcal{M}(U', Q') = \{\mu \in \mathcal{M} : |\mu_\theta(s)| \leq U'_s, \theta \text{ and } |\mu(s)| \leq Q'_s \text{ for all } (s, \theta)\}\) is the set of all matchings that respect \((U', Q')\). An upper quota mechanism is a function \(\psi(U', Q') : \mathcal{P} \rightarrow \mathcal{M}\) such that \(\psi(U', Q')(P_I) \in \mathcal{M}(U', Q')\) for all \(P_I \in \mathcal{P}\). Note that upper quota mechanisms always satisfy the given ceilings and capacities \((U', Q')\), but need not satisfy any floors.

A collection of mechanisms, one for each \((U', Q')\), is denoted \(\Psi := \{\psi(U', Q')\}_{(U', Q')}\). We refer to \(\Psi\) as a class of upper quota mechanisms. As an example, \(\Psi\) could be the class of DA mechanisms as defined in Section 3: \(\Psi = \{DA(U', Q')\}_{(U', Q')}\).

**Definition 6.** Let \((U', Q')\) and \((U'', Q'')\) be such that \((U', Q') \leq (U'', Q'')\) and \(\sum_{\theta \in \Theta}(U''_{s, \theta} - U'_s, \theta) \leq Q''_s - Q'_s\) for all \(s \in S\). \(\Psi\) is resource monotonic if, for all such \((U', Q')\) and \((U'', Q'')\), \(\psi(U', Q')\) Pareto dominates \(\psi(U'', Q'')\).

Resource monotonicity means that raising the ceilings and capacity of a school makes all students better off, provided that the type-specific ceilings are not raised more than the capacity.\(^{38}\)

**Artificial caps**

Recall that \((\bar{U}, \bar{Q})\) ensures a feasible match if \(\mathcal{M}(\bar{U}, \bar{Q}) \subseteq \mathcal{M}_f\). If \((\bar{U}, \bar{Q})\) ensures a feasible match, \(\psi(\bar{U}, \bar{Q}) : \mathcal{P} \rightarrow \mathcal{M}\) is by definition a feasible mechanism. We call such a mechanism an artificial caps mechanism.

**Remark 4.** An artificial caps mechanism is a particular type of upper quota mechanism, where \((\bar{U}, \bar{Q})\) is chosen to be strict enough to ensure feasibility. Upper quota mechanisms in general may or may not satisfy the floors (depending on the submitted preferences).

We now generalize dynamic quotas to any class of mechanisms \(\Psi\).

**Dynamic quotas \(\Psi\) (DQ\(\Psi\))**

Fix a sequence of ceiling-capacity vectors \(\eta = \{(U^1, Q^1), (U^2, Q^2), \ldots, (U^K, Q^K)\}\) such that: (i) \((U^{k+1}, Q^{k+1}) \leq (U^k, Q^k)\) for all \(k\), and (ii) \((U^K, Q^K)\) ensures a feasible match. The algorithm then proceeds in

---

\(^{38}\)Resource monotonicity has been used as an important axiom in many other allocation settings as well (see, for example, Ehlers and Klaus (2004), Kesten (2006), and Thomson (2005)). The previous works do not have type-specific ceilings, and prior notions of resource monotonicity say that raising just the capacity of a school makes all agents better off (in a Pareto sense). This is implied by Definition 6, but for our purposes, we want to allow the type-specific ceilings to be raised as well. However, we must do this in a “controlled” manner: raising the type-specific ceilings by more than the number of capacity seats may make students of other types whose ceilings were not raised worse off (see the discussion after Theorem 6).
a series of stages.

**Stage 1** Calculate $\psi(U^1, Q^1)(P_I)$. If $\psi(U^1, Q^1)(P_I)$ is a feasible matching, end the algorithm and output this matching. If not, proceed to stage 2.

In general,

**Stage k** Calculate $\psi(U^k, Q^k)(P_I)$. If $\psi(U^k, Q^k)(P_I)$ is a feasible matching, end the algorithm and output this matching. If not, proceed to stage $k+1$.

We define dynamic quotas $\Psi$ as the function $DQ^\Psi : \mathcal{P}^n \rightarrow \mathcal{M}$ that produces, for each input, the matching at the end of the above algorithm.

Note that we use the sequential formulation from Section 4. Sequential DA is a special case of $DQ^\Psi$ when $\Psi$ is the class of deferred acceptance mechanisms, but dynamic quotas can be applied to many other choices of $\Psi$. We discuss common classes of mechanisms used in practice below.

**Theorem 8.** Fix $P_I \in \mathcal{P}^n$. Then, $DQ^\Psi(P_I) = \psi(U^k, Q^k)(P_I)$ for some $k \leq K$. If, in addition, $\Psi$ is resource monotonic and $\eta$ is a minimal reduction sequence, then the dynamic quotas mechanism $DQ^\Psi$ Pareto dominates the artificial caps mechanism $\psi(U^K, Q^K)$.

The above theorem can be thought of as a generalization of the result that DQDA Pareto dominates ACDA to other classes of mechanisms $\Psi$.

**Applying dynamic quotas to other common mechanisms**

We now define three other mechanisms commonly used in the field that can be adapted to our dynamic quotas framework: the serial dictatorship, immediate acceptance (also known as “the Boston mechanism”), and top trading cycles.

**Mechanism 1: Serial dictatorship under $(U', Q')$**

Fix an ordering of the students $\succ^{SD}$ (without loss of generality, we let $i_1 \succ^{SD} \cdots \succ^{SD} i_n$).

**Step 1** Consider student $i_1$. Assign $i_1$ to her most preferred school $s$ according to $P_{i_1}$ that has not yet reached its capacity $Q'_s$ or type-specific ceiling $U'_{s, \tau(i_1)}$.

In general,

**Step k** Consider student $i_k$. Assign $i_k$ to her most preferred school $s$ according to $P_{i_k}$ that has not yet reached its capacity $Q'_s$ or type-specific ceiling $U'_{s, \tau(i_k)}$.

We define $SD(U', Q') : \mathcal{P}^n \rightarrow \mathcal{M}$ as the function that produces, for each input, the matching at the end of the above algorithm. We let $SD = \{SD(U', Q')\}_{(U', Q')}$ denote the class of serial dictatorship mechanisms.

The serial dictatorship fixes some ordering of students and allows the students to pick their favorite school in this order. The serial dictatorship is a commonly used mechanism in assignment problems where the agents do not have priorities over the objects to be assigned.
Mechanism 2: Immediate acceptance under \((U', Q')\)

**Step** 1 Each student applies to the first school on her preference list. Each school \(s\) admits students one-by-one according to \(\succ_s\), unless admitting a student would violate either the capacity \(Q'_s\) or some type-specific ceiling \(U'_{s,t}\). Reduce the capacity of each \(s\) by the total number of students admitted, and reduce each type-specific ceiling by the number of students of that type admitted.

In general,

**Step** \(k\) Each student not already accepted in some previous step applies to her \(k^{th}\) choice school. Each school \(s\) admits students one-by-one according to \(\succ_s\), unless admitting a student would violate either the updated capacity or updated type-specific ceilings from the previous step. Reduce the capacity of each \(s\) by the total number of students admitted, and reduce each type-specific ceiling by the number of students of that type admitted.

We define \(IA^{(U', Q')} : \mathcal{P}^n \to \mathcal{M}\) as the function that produces, for each input, the matching at the end of the above algorithm. We let \(IA = \{IA^{(U', Q')}\}_{(U', Q')}\) denote the class of immediate acceptance algorithms.

The immediate acceptance algorithm runs similarly to the deferred acceptance algorithm with one crucial difference: at the end of each step, the acceptances are permanent (“immediate”), rather than tentative. The immediate acceptance algorithm is popular in many school choice settings. Variants of it are currently used in Cambridge, MA and Minneapolis, MN, and the algorithm has been used previously in Seattle, WA, and Boston, MA. The main drawback of immediate acceptance is that it is not strategyproof (nor is it even strategyproof in large markets and is anecdotally “easy” to manipulate; see Kojima and Pathak (2009) or Azevedo and Budish (2013)). Nevertheless, it is still a commonly used mechanism in the field, and so understanding its properties with regard to dynamic quotas is important.

Mechanism 3: Top trading cycles under \((U', Q')\)

**Step** 1 Each student \(i\) points to her most preferred school which has a seat available for her type (i.e., both \(Q'_s > 0\) and \(U'_{s,t(i)} > 0\)). Each school \(s\) points to the student with highest priority according to \(\succ_s\). There is at least one cycle. Every student in a cycle is assigned a seat at the school she is pointing to and is removed. For the schools that are assigned a student, their capacity and relevant type-specific ceiling are reduced by one. If a school has no more capacity, it is removed from the market. If there is at least one student remaining, move to the next step.

In general,

**Step** \(k\) Each remaining student \(i\) points to her most preferred school which has a seat available for her type. Each school \(s\) points to the student with highest priority according to \(\succ_s\). There is at least one cycle. Every student in a cycle is assigned a seat at the school she is pointing to and is removed. For the schools that are assigned a student, their capacity and relevant type-specific ceiling are reduced by one. If a school has no more capacity, it is removed from the market. If there is at least one student remaining, move to the next step.
We define $TTC(U_0, Q_0) : \mathcal{P}^n \to \mathcal{M}$ as the function that produces, for each input, the matching at the end of the above algorithm. We let $TTC = \{TTC(U_0, Q_0)\}_{(U_0, Q_0)}$ be the class of TTC mechanisms.

TTC mechanisms were first defined by Shapley and Scarf (1974), and were proposed for school choice by Abdulkadiroğlu and Sönmez (2003), who show that TTC is strategyproof and Pareto efficient. The main drawback of TTC is that it will not eliminate justified envy (nor will it eliminate justified envy among same types), because TTC allows students to trade their priorities with other students who may be much lower on the primitive priority ranking for a given school. Versions of TTC are currently in use in school districts in San Francisco and New Orleans.

The serial dictatorship, immediate acceptance, and top trading cycles are all upper quota mechanisms used in practice. As such, they are all natural candidates for use in the presence of floor constraints as well. By imposing sufficiently strict artificial caps, it is possible to ensure a feasible matching by running any of these mechanisms. However, doing so will result in the same inefficiencies as with deferred acceptance, because artificial caps still eliminates seats ex-ante, ignoring important information contained in the student preferences. We thus argue that any market that uses artificial caps SD/IA/TTC would be better served by switching to dynamic quotas SD/IA/TTC. Because $SD$ and $IA$ are indeed resource monotonic, dynamic quotas SD/IA will Pareto dominate artificial caps SD/IA. TTC, on the other hand, is not resource monotonic,\(^{39}\) and so, while the Pareto dominance of dynamic quotas over artificial caps will not hold, it will still be true that the final matching implemented under dynamic quotas will have higher ceilings than artificial caps, which intuitively will make the students better off on average. We thus suggest that policymakers may want to consider dynamic quotas over artificial caps, even if resource monotonicity does not hold formally.

7.2 Outside options

In the main model, we made the assumption that all students find all schools acceptable, and vice-versa. In this section, we allow students to have an outside option, denoted $O$. We let the set of schools now be $S \cup \{O\}$, and students have a strict preference relation $P_i$ over this set. Formally, we treat $O$ as a school just like any other, only with no floors or ceilings and infinite capacity. We say a mechanism $\psi : \mathcal{P}^n \to \mathcal{M}$ is individually rational if $\psi_i(P_i)R_iO$ for all $i \in I$.

**Theorem 9.** Assume that $L_{s, \theta} > 0$ for some $(s, \theta)$. There is no mechanism $\psi$ that is both feasible and individually rational.

The above result says that it is impossible to guarantee an outcome that satisfies all floor constraints if students can take an outside option. The result can be proved by simply considering a preference profile at which every student ranks $O$ as her first choice. Then, individual rationality requires all students be assigned to $O$, and no floor seats at any $s \in S$ will be satisfied. While this is of course formally possible, such an extreme preference profile is unlikely, and the relevant issue in practical applications is how many students have a relevant outside option available to them.

\(^{39}\)See Kesten (2006).
In fact, in many settings, there are no outside options, and individual rationality is not an issue at all. For example, in the Army, all cadets have signed contracts to serve for a specified number of years, and so can be assigned to any branch at which they are needed (though of course the Army still elicits cadet preferences and prefers to respect them as much as possible). Even in public school choice, because all students are legally required to be enrolled in some school, if a student is not assigned any school on her submitted preference list, districts will assign her to some other school, generally the school closest to their home with an available seat (in New York City, this process is called “administrative assignment”). Some students may then choose to attend a private school, but many may not have such a viable outside option.\textsuperscript{40}

While it is impossible to guarantee a matching that is feasible in the formal sense when an outside option is available, the choice of artificial caps or dynamic quotas will still have strong effects on both how close the final assignment is to satisfying all constraints and overall student welfare. In particular, artificial caps may once again go “too far”, and eliminate more seats than necessary. This can end up pushing too many students to leave the school district and take their outside option, thereby making students worse off and harming the long-term viability of the school district.

The next theorem shows that dynamic quotas will again outperform artificial caps, even allowing for outside options. With an outside option, dynamic quotas works similarly to before. The only difference is, if for some preferences $P_I$ the algorithm runs all the way to stage $K$, then the final matching output is $\psi(U^K, Q^K)(P_I)$, whether this matching is feasible or not. Then, we have the following results.

Theorem 10. Assume that $\Psi$ is resource monotonic, every $\psi(U', Q') \in \Psi$ is individually rational, and let $\eta = \{(U^1, Q^1), \ldots, (U^K, Q^K)\}$ be a minimal reduction sequence. Then, the following hold:

1. The dynamic quotas mechanism $DQ^\Psi$ Pareto dominates the artificial caps mechanism $\psi(U^K, Q^K)$.
2. The number of students assigned to schools in $S$ (in the school district) is weakly greater under $DQ^\Psi$ than under the artificial caps mechanism $\psi(U^K, Q^K)$.
3. The number of floor seats left unfilled is weakly less under $DQ^\Psi$ than under the artificial caps mechanism $\psi(U^K, Q^K)$.

Any school district that imposes some type of upper quotas/artificial caps runs the risk of pushing some students to take an outside option. This will be true of any mechanism. How high this cost is, and whether it is outweighed by the benefits of diverse student bodies, are factors that must be determined by each school district based on its own needs and goals. Given that many school districts do indeed impose floors and ceilings because they think there are sufficient benefits to diversity, the results above suggest that dynamic quotas is a better approach to achieving it than artificial caps: all students (weakly) prefer dynamic quotas, more students stay in the public school district, and the diversity goals are not harmed.

Remark 5. The results above may be in particular relevant for the Japan Residency Matching Program, where doctors often submit short preference lists. Rather than imposing artificial caps from the outset and running DA, the JRMP might be better served by running a DQDA type mechanism. In the worst case,\textsuperscript{40} For example, in 2003, before the redesign of the high school assignment mechanism in New York City, over 30,000 of the 90,000 students were assigned to a school they did not list, with the vast majority attending their assigned school. See Abdulkadiroğlu et al. (2005b).
the matching produced by DQDA will be the same as under artificial caps, but DQDA may in fact end earlier, under higher capacities for all hospitals.

8 Conclusion

This paper shows that a common approach used in many matching markets with distributional constraints may result in avoidable inefficiencies. We propose new mechanisms based on a concept of dynamic quotas that recover these inefficiencies by allocating seats more flexibly, based on the submitted preferences of the students. A key feature of our dynamic quotas approach is its straightforward and intuitive nature, which is essential for practical implementation in market design settings where the mechanism used must be easily communicable to participants. We also provide rigorous theoretical analysis with respect to crucial properties such as incentives and efficiency. We show that it is possible to improve upon the existing approaches in a Pareto sense without compromising fairness or incentives, suggesting that policymakers would be better served by the use of our mechanisms. In addition, we analyze the aforementioned incentive/efficiency tradeoff further, and show that by relaxing the incentive constraints to strategyproofness in the large, we can achieve even greater efficiency gains. Methodologically, we identify conditions under which the popular deferred acceptance remains strategyproof even using dynamic quotas, and, in showing this result, introduce techniques which may be helpful in designing strategyproof mechanisms for other settings as well.

The main motivation for most of our formal modeling decisions was diversity goals in school choice, though our mechanisms can be applied to many other markets, including military cadet branching, hospital-resident matching, or other places where distributional constraints are important. We take no position on the merits of imposing distributional constraints; many school districts that impose such constraints do so because they believe that there are social benefits to diverse educational environments. We instead follow the paradigm of market design research advocated by Roth (2002) and take the constraints as empirical realities. We then attempt to provide practical mechanisms within this framework. We do argue, however, that markets that impose artificial caps as a way to satisfy some “implicit” floor constraints (such as the Japan Residency Matching Program or other markets in which the true distributional constraints are not publicly stated) should consider modeling their goals more explicitly and switching to a dynamic quotas mechanism similar to those provided in this paper. Doing so will improve agent welfare without compromising fairness, incentives, or the true distributional goals.

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Appendices - For Online Publication Only

A  A general model of school priorities

In this section, we define a model that allows for more general school choice functions and feasibility constraints. We then introduce some conditions on choice functions and preliminary theorems that will be useful in the proof of strategyproofness (found in the next section), though they also may be of independent interest. To simplify the flow of the argument, the proofs of some lemmas will be found in Appendix C.

A.1 Primitives of the general model

The primitives of the general model once again consist of a set of schools $S = \{s_1, \ldots, s_m\}$ and a set of students $I = \{i_1, \ldots, i_n\}$. Each student has a strict preference relation $P_i$ on the set of schools $S$. The priorities of the schools are now described differently. For any set $X$, let $2^X$ denote the power set of $X$. For each school $s$ we define a choice function $Ch_s : 2^I \rightarrow 2^I$, where $Ch_s(I')$ denotes the set of students admitted to school $s$ when its choice set is $I'$. This set up is similar to the model of Hatfield and Milgrom (2005).41 In the baseline model, $Ch_s(I')$ would be the highest priority subset of students in $I'$ subject to

41Aygün and Sönmez (2013) point out a technical ambiguity in the model of Hatfield and Milgrom (2005), noting that to ensure the choice functions are derived from a well-defined underlying priority relation $>_s$ over sets of students, one must assume a condition called irrelevance of rejected students (IRS), which, in our setting, says that if $Ch_s(I'' \subseteq I' \subseteq I''')$, then $Ch_s(I') = Ch_s(I''')$. We assume below that all of our choice functions satisfy substitutability and the law of aggregate demand, which Aygün and Sönmez (2013) show implies IRS, and so we are justified in working directly with the choice functions, rather than the underlying priority relation.

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the floors and ceilings (see Section 3). Corresponding to each choice function is a rejection function \( \text{Rej}_s(I') = I' \setminus \text{Ch}_s(I') \). Let \( \text{Ch} := \{\text{Ch}_{s_1}, \ldots, \text{Ch}_{s_n}\} \) denote a vector of choice functions, one for each school.

The following two conditions on choice functions were identified by Hatfield and Milgrom (2005) as key for strategyproofness of DA in a model without floor constraints.

**Definition 7.** Choice function \( \text{Ch}_s \) is substitutable if \( I' \subseteq I'' \) implies \( \text{Rej}_s(I') \subseteq \text{Rej}_s(I'') \).

**Definition 8.** Choice function \( \text{Ch}_s \) satisfies the law of aggregate demand if \( I' \subseteq I'' \) implies \( |\text{Ch}_s(I')| \leq |\text{Ch}_s(I'')| \).

We assume that all choice functions defined from here forward satisfy both substitutability and the law of aggregate demand.

Let \( \mathcal{I}(s) = \{I' \subseteq I : I' = \text{Ch}_s(I'') \text{ for some } I'' \in 2^I\} \).\(^{42}\) In words, \( \mathcal{I}(s) \) is a set consisting of all possible assignments for school \( s \), obtained by considering every potential set of applicants \( I'' \) that \( s \) may have the opportunity to choose from. Beyond the individual school choice functions, the school district can also impose additional feasibility constraints. In this more general setting, we assume for each school \( s \) the school district defines a priori a subset \( \mathcal{I}_f(s) \subseteq \mathcal{I}(s) \) of feasible assignments.\(^{43}\) The set of feasible matchings \( \mathcal{M}_f \) is then defined as follows:

\[
\mathcal{M}_f = \{\mu \in \mathcal{M} : \mu(s) \in \mathcal{I}_f(s) \text{ for all } s \in S\}.
\]

Note that the definition of \( \mathcal{I}_f(s) \) and IRS (footnote 41) imply that if \( \mu \in \mathcal{M}_f \), then \( \text{Ch}_s(\mu(s)) = \mu(s) \) for all \( s \). As in the main text, we assume that the set of feasible matchings is nonempty.

In this context, a reduction sequence is now a sequence of vectors of choice functions \( \eta = \{\text{Ch}^1, \ldots, \text{Ch}^K\} \), where each \( \text{Ch}^k = (\text{Ch}^k_s)_{s \in S} \) is a vector of choice functions (one for each school) such that deferred acceptance under choice function vector \( \text{Ch}^K \) results in a feasible match.\(^{44}\)

**Definition 9.** Reduction sequence \( \eta = \{\text{Ch}^1, \ldots, \text{Ch}^K\} \) is monotonic if \( \text{Rej}^k_s(I') \subseteq \text{Rej}^{k''}_s(I') \) for all \( s \in S, I' \subseteq I, \text{ and } k'' \geq k' \).

**Remark 6.** If \( \eta \) is monotonic and \( \text{Ch}^k_s \) is substitutable for all \( s \) and \( k \), we say \( \eta \) is monotonically substitutable. Note that if \( \eta \) is monotonically substitutable, then \( I' \subseteq I'' \) and \( k' \leq k'' \implies \text{Rej}^k_s(I') \subseteq \text{Rej}^{k''}_s(I'') \).\(^{45}\)

**Definition 10.** Reduction sequence \( \eta \) is minimal if the following hold for all \( k \): (i) for exactly one \( s \), \( 0 \leq |\text{Ch}^k_s(I')| - |\text{Ch}^{k+1}_s(I')| \leq 1 \) for all \( I' \subseteq I \) and (ii) for all remaining \( s' \neq s \), \( \text{Ch}^{k+1}_{s'}(\cdot) = \text{Ch}^k_{s'}(\cdot) \).

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\(^{42}\)Note that by IRS (footnote 41), \( I' \in \mathcal{I}(s) \) implies that \( I' = \text{Ch}_s(I') \).

\(^{43}\)In the main text, \( \mathcal{I}_f(s) \) would consist of all assignments that satisfy a school’s type-specific floor and ceiling constraints and capacities. In the standard school choice model (e.g., Abdulkadiroğlu and Sönmez (2003)), \( \mathcal{I}_f(s) = \mathcal{I}(s) \).

\(^{44}\)As in the main text, it is always possible to find such feasibility-ensuring choice functions: simply take any feasible match \( \mu \) and set \( \text{Ch}^k_s(I') = \mu(s) \cap I' \) for all \( I' \). In addition, a (quota) reduction sequence as defined in Definition 2 induces a (choice function) reduction sequence in the natural way.

\(^{45}\)\( \text{Rej}^k_s(I') \subseteq \text{Rej}^{k''}_s(I'') \subseteq \text{Rej}^{k''}_s(I'') \), where the first inclusion is by substitutability and the second is by monotonicity.
Minimality guarantees that when moving from stage $k$ to $k + 1$, at most one student will be rejected. Lemma 2 below shows that any (quota) reduction sequence (as defined in Section 4) that is minimal in the sense of Definition 3 induces a choice function reduction sequence that is monotonic and minimal (in the sense of Definitions 9 and 10), and each $\text{Ch}_s^k(\cdot)$ satisfies substitutability and the law of aggregate demand for all $k$ and all $s$.

### A.2 Generalized DQDA

We now define a generalized version of the DQDA algorithm that takes as an input a reduction sequence of choice functions. As a matter of notation, we use $A_k^s(t)$ to denote the cumulative set of students who have applied to school $s$, up to and including stage $k$, step $t$ of the algorithm (i.e., $A_k^s(t)$ includes all students who ever made an application to $s$ in the algorithm, up to and including stage $k$, step $t$).

**Generalized dynamic quotas deferred acceptance (GDQDA)** Fix a reduction sequence $\eta = \{\text{Ch}^1, \ldots, \text{Ch}^K\}$.

#### Stage 1

**Step 0** Set $A_1^s(0) = \emptyset$ for all $s \in S$.

**Step 1** Choose some student $i^1$ who applies to her favorite school, $s^1$. Let $A_1^s(1) = \{i^1\}$ and $A_1^s(t) = A_1^s(t-1) \cup \{i^1\}$ for all other $s \in S$. Each school $s \in S$ tentatively accepts the students in $\text{Ch}_s^1(A_1^s(1))$ and rejects the rest.

**Step $t$** Choose a student $i^t$ that is not tentatively accepted by any school, and let her apply to her most preferred school $s^t$ that has not yet rejected her. Let $A_s^1(t) = A_s^1(t-1) \cup \{i^t\}$ and $A_s^1(t) = A_s^1(t-1)$ for all $s \neq s^t$. Each school $s \in S$ tentatively accepts the students in $\text{Ch}_s^1(A_s^1(t))$, and rejects all other students.

Stage 1 terminates when every student is either tentatively accepted by some school $s \in S$ or has applied to all schools and been rejected. This happens in a finite number of steps $T^1$. Let the resulting matching be $\nu^1$, where $\nu^1(s) = \text{Ch}_s^1(A_s^1(T^1))$ for all $s \in S$. If $\nu^1 \in \mathcal{M}_f$, end the algorithm and output matching $\nu^1$. If not, proceed to stage 2.

In general,

#### Stage $k$

**Step 0** Set $A_k^s(0) = A_s^{k-1}(T^{k-1})$ for all $s \in S$, and let each school tentatively accept $\text{Ch}_s^k(A_s^k(0))$ and reject all remaining students.

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46 It is well-known that the order in which students are allowed to apply is irrelevant (see, e.g., McVitie and Wilson (1971) or Hatfield and Kojima (2010)).

47 Note that in this definition, we allow the possibility that a student applies to and is rejected from every school, in which case she is unmatched at the end of the algorithm. Under suitable choice functions that ensure feasible matchings, this will not be an issue.
Step 1 Choose a student $i^1$ that is not tentatively accepted by any school, and let her apply to her most preferred school $s^1$ that has not yet rejected her (in this stage, or any previous stages). Let $A_s^k(1) = A_s^k(0) \cup \{i^1\}$ and $A_s^k(1) = A_s^k(0)$ for all other $s \in S$. Each school $s \in S$ tentatively accepts the students in $Ch_s^k(A_s^k(1))$ and rejects the rest.

Step $t$ Choose a student $i^t$ that is not tentatively accepted by any school, and let her apply to her most preferred school $s^t$ that has not yet rejected her (in this stage, or any previous stages). Let $A_s^k(t) = A_s^k(t-1) \cup \{i^t\}$ and $A_s^k(t) = A_s^k(t-1)$ for all other $s \in S$. Each school $s \in S$ tentatively accepts the students in $Ch_s^k(A_s^k(t))$ and rejects the rest.

Stage $k$ terminates when every student is tentatively accepted by some school $s \in S$ or has applied to all schools and been rejected. This happens in a finite number of steps $T^k$. Let the resulting matching be defined by $\nu^k(s) = Ch_s^k(A_s^k(T^k))$ for all $s \in S$. If $\nu^k \in \mathcal{M}_f$, end the algorithm and output matching $\nu^k$. If not, proceed to stage $k + 1$.

The above description of the algorithm makes use of the cumulative offer process of Hatfield and Milgrom (2005) (see also Hatfield and Kojima (2009)). As the cumulative offer process progresses, schools continually accumulate applications from students, and at each point, hold on to their most preferred set students among all of those who have cumulatively applied to it. Students who are not currently held by any school make new applications to their most preferred school that has not yet rejected them. Note that, as it stands, there is no assumption of consistency imposed on the algorithm, since some student could be assigned to more than one school. However, we will show below that under our conditions on $\eta$, this will not be an issue.

Recall that in the main text, we introduced a second way to run a dynamic version of DA that we call sequential deferred acceptance. It is also possible to define a generalized sequential deferred acceptance algorithm in the context of the general model.

**Generalized sequential deferred acceptance (GSDA)**

Fix a reduction sequence $\eta = \{Ch^1, \ldots, Ch^K\}$, and let $DA^{Ch^i} : \mathcal{P}^n \rightarrow \mathcal{M}$ denote the deferred acceptance algorithm under choice function vector $Ch^i$.

Stage 1 Starting with the empty matching, compute $DA^{Ch^1}(P_I)$. If $DA^{Ch^1}(P_I) \in \mathcal{M}_f$, end the algorithm and output this matching. If not, proceed to stage 2.

In general,

Stage $k$ Starting with the empty matching, compute $DA^{Ch^k}(P_I)$. If $DA^{Ch^k}(P_I) \in \mathcal{M}_f$, end the algorithm and output this matching. If not, proceed to stage $k + 1$.

**Theorem 11.** Let $\nu^k$ be the matching at the end of stage $k$ of the GDQDA algorithm, and $\mu^k$ be the matching at the end of stage $k$ of the GSDA algorithm. Assume that $\eta$ is monotonic and minimal and that $Ch_s^k(\cdot)$ satisfies substitutability and the law of aggregate demand for all $k$ and $s$. Then $\nu^k = \mu^k$ for all $k$. 

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Proof. As in the definition of the algorithm above, let $A^k_s(t)$ denote the cumulative offer set of school $s$ at step $t$ of stage $k$ under GDQDA. For each stage $k$, let $T^k$ denote the final step of stage $k$.

Now consider GSDA. Within a stage $k$, we are simply running DA under choice functions $Ch^k$. We can define an analogous within-stage cumulative offer process. Let $B^k_s(t)$ denote the cumulative set of applicants to school $s$ through step $t$ of the within-stage cumulative offer process. Because GSDA starts each stage $k$ from the empty matching, $B^k_s(0) = \emptyset$ for all $s$ and $k$. Let $\hat{T}^k$ denote the last step of stage $k$. Hatfield and Milgrom (2005) show that the matching produced by this cumulative offer process is equivalent to deferred acceptance, i.e. $\mu^k(s) = Ch^k_s(B^k_s(\hat{T}^k))$ for all $s \in S$ and all $k$.

The following lemma, which is proved in Appendix C, is key to the argument.

**Lemma 1.** For all $k$ and all $s$, $A^k_s(T^k) = B^k_s(\hat{T}^k)$.

Note that in general, $T^k \neq \hat{T}^k$, but the result says that the cumulative applicant sets at the end of each stage $k$ are still the same. With this lemma, the result follows easily: $\mu^k(s) = Ch^k_s(B(\hat{T}^k)) = Ch^k_s(A^k_s(T^k)) = \nu^k(s)$ for all $k$ and $s$, where the first and third equalities are by definition, and the second is by Lemma 1. ■

The following result is an immediate corollary of Theorem 11.

**Corollary 1.** If $\eta$ is monotonic and minimal and $Ch^k_s(\cdot)$ satisfies substitutability and the law of aggregate demand for all $k$ and all $s$, then the final matching produced by GDQDA under $\eta$ is equivalent to the final matching produced by GSDA under $\eta$.

The next theorem is a more general statement of the result that dynamic quotas DA Pareto dominates artificial caps. Part (2) of Theorems 4 and 6 follow from this theorem.

**Theorem 12.** Let $\eta = \{Ch^1, \ldots, Ch^K\}$ be monotonic and minimal assume that $Ch^k_s(\cdot)$ satisfies substitutability and the law of aggregate demand for all $k$ and all $s$. Then, GDQDA under $\eta$ Pareto dominates DA under $Ch^K$.

**Proof.** By Corollary 1, the final matching produced by GDQDA is equivalent to the matching produced by DA under choice functions $Ch^{k'}$ for some $k' \leq K$. The outcome of DA in stage $k$ is equivalent to the outcome of the cumulative offer process under $Ch^k$ (Hatfield and Milgrom (2005); Hatfield and Kojima (2010)). By monotonicity, $Ch^K_s(I') \subseteq Ch^{k'}_s(I')$ for all $I' \subseteq I$. We then apply Lemma 1 of Kamada and Kojima (2013) to obtain the result. ■

**B Proofs from the main text**

Proofs of any lemmas not given here can be found in Appendix C.

**Theorem 1**

Consider any feasible matching $\mu \in M_f$, and define $\tilde{Q}_s = |\mu(s)|$ and $\tilde{U}_{s,\theta} = |\mu_{\theta}(s)|$ for all $(s, \theta)$. We show that $(\tilde{U}, \tilde{Q})$ ensures a feasible match. Note that $\sum_{s' \in S} \tilde{U}_{s',\theta} = |I_{\theta}|$. Then, consider any other matching $\nu \in M$ that respects $(\tilde{U}, \tilde{Q})$. $\nu$ respects all true ceilings and capacities $(U, Q)$ by definition. We must show
that $\nu$ also respects all floors. Assume not, i.e., assume there is some pair $(s, \theta)$ such that $|\nu_\theta(s)| < L_{s, \theta}$. Then, $\sum_{s' \in S} |\nu_\theta(s')| < \sum_{s' \in S} \bar{U}_{s', \theta} = |I_\theta|$.\footnote{The first inequality follows from the fact that $\sum_{s' \in S \setminus \{s\}} |\nu_\theta(s')| \leq \sum_{s' \in S \setminus \{s\}} \bar{U}_{s', \theta}$ (because $\nu$ respects $(\bar{U}, \bar{Q})$) and that $|\nu_\theta(s)| < L_{s, \theta} \leq \bar{U}_{s, \theta}$.} But this implies that under $\nu$, there exists some student $i$ of type $\theta$ who is not assigned to any school, and so $\nu \notin M$, which is a contradiction.

**Theorem 2**

Lemma 2 below shows that for any stage $k$ of DQDA, the induced within-stage choice functions of the schools satisfy substitutability and the law of aggregate demand. Since ACDA is equivalent to DA using the stage $K$ choice functions, strategyproofness follows from Hatfield and Milgrom (2005), who show that substitutability and the law of aggregate demand are sufficient for strategyproofness.

To show that ACDA eliminates justified envy among same types, note that ACDA is equivalent to DA under $(U^K, Q^K)$. Denote the matching produced by DA under $(U^K, Q^K)$ as $\mu$. Assume that some student $i$ envies another student $j$ of her same type: $\mu(j) P_i \mu(i)$ and $\tau(i) = \tau(j) = \theta$. Let step $t$ be the step of the DA algorithm at which $i$ is rejected from $\mu(j)$. In step $t$, $i$ is rejected because the type $\theta$ specific seats are filled with $L_{\mu(j), \theta}$ students of type $\theta$ ranked higher than $i$ according to $>_{\mu(j)}$, and the open seats are filled with either: (i) $U^K_{\mu(j), \theta} - L_{\mu(j), \theta}$ students of type $\theta$ ranked higher than $i$ according to $>_{\mu(j)}$, or (ii) $Q^K_{\mu(j)} - \sum_{\theta \in \Theta} L_{\mu(j), \theta}$ students of any type ranked higher than $i$ according to $>_{\mu(j)}$. As the algorithm progresses, a student accepted in step $t$ can only be rejected from the type $\theta$ specific seats only if a higher-ranked student of type $\theta$ applies, and the same is true of the students at the open seats. Thus, at the end of the algorithm, all students assigned to $\mu(j)$ through either the type $\theta$ specific seats or the open seats must be ranked higher than $i$. Since $\tau(j) = \theta$ as well, this implies that $j >_{\mu(j)} i$, i.e., $i$ does not justifiably envy $j$.\footnote{Of course, there may be students of other types $\theta' \neq \theta$ whom $i$ envies and has higher priority over, since these students could be assigned through the type $\theta'$ specific seats.}

**Theorem 3**

We first note the following lemma, which is proved in Appendix C. With slight abuse of notation, let $\eta = \{Ch^1, \ldots, Ch^K\}$ be the sequence of choice functions induced by a reduction sequence $\{(U^1, Q^1), \ldots, (U^K, Q^K)\}$ as in Definition 2.

**Lemma 2.** The reduction sequence $\eta = \{Ch^1, \ldots, Ch^K\}$ is monotonic , and $Ch^k(\cdot)$ satisfies substitutability and the law of aggregate demand for all $k$ and $s$. If, in addition, $\{(U^1, Q^1), \ldots, (U^K, Q^K)\}$ is minimal in the sense of Definition 3, then $\eta = \{Ch^1, \ldots, Ch^K\}$ is minimal in the sense of Definition 10.

Given Lemma 2, Theorem 3 then is a special case of Theorem 11.

**Theorems 4 and 6**

Theorem 4 is a special case of Theorem 6, and so we prove the latter. Again, let $\eta = \{Ch^1, \ldots, Ch^K\}$ be the sequence of choice functions induced by a reduction sequence $\{(U^1, Q^1), \ldots, (U^K, Q^K)\}$ as in Definition 2.
Part (1)
Follows from Lemma 2 and Theorem 12.

Part (2)
By Corollary 1, the final matching produced by DQDA is equivalent to SDA, which in turn is equivalent to DA under $\text{Ch}^k$ for some $k \leq K$. That DA under $\text{Ch}^k$ eliminates justified envy among same types can be shown using the same argument as in the proof of Theorem 2.

Part (3)
Fix the reports of the other students at $P_{-i}$, and let $i$’s true preferences be $P_i$. We will show that there is no report $P_i'$ that will give $i$ a better assignment than reporting the truth.

In the proof, we move back and forth between the GDQDA and GSDA algorithms, which, by Corollary 1, are equivalent. We begin by working with GSDA, and will make use of the fact that the DA algorithm within each stage $k$ is equivalent to the cumulative offer process of Hatfield and Milgrom (2005). It is well-known (see, e.g., McVitie and Wilson (1971), Dubins and Freedman (1981), Hatfield and Milgrom (2005), or Hatfield and Kojima (2010)) that the order in which students are allowed to apply does not matter, as any such order will lead to the same final matching. Consider an ordering in which student $i$ applies last. In stage $k$ of the GSDA algorithm, let $\tilde{B}_s^k$ denote the cumulative set of applicants that school $s$ receives in the cumulative offer process under $\text{Ch}^k$ on all students other than $i$. Then, let $i$ enter the market, which causes a rejection chain, which simply records the action of the algorithm:

<table>
<thead>
<tr>
<th>Step</th>
<th>Action</th>
<th>Cumulative offer sets</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>None</td>
<td>$B_s^k(0) = \tilde{B}_s^k$ for all $\hat{s}$</td>
</tr>
<tr>
<td>1</td>
<td>Student $i$ applies to school $s$</td>
<td>$B_s^k(1) = B_s^k(0) \cup {i}$, $B_s^k(1) = B_s^k(0)$ for all $\hat{s} \neq s$</td>
</tr>
<tr>
<td>2</td>
<td>$s$ rejects $i'$</td>
<td>$B_s^k(2) = B_s^k(1)$ for all $\hat{s}$</td>
</tr>
<tr>
<td>3</td>
<td>$i'$ applies to $s'$</td>
<td>$B_{s'}^k(3) = B_{s'}^k(2) \cup {i'}$, $B_{s'}^k(3) = B_{s'}^k(2)$ for all $\hat{s} \neq s'$</td>
</tr>
</tbody>
</table>

The rejection chain in stage $k$ ends at the first step $\hat{T}_k$ for which some student $i'$ applies to some school $s'$ and $s'$ does not reject a student. At this point, the applicant pools for each school are $B_{s'}^k(\hat{T}_k)$, and, as described above, the stage $k$ output $\mu^k$ is defined by $\mu^k(s) = \text{Ch}_s^k(B_s^k(\hat{T}_k))$. Note that by substitutability, if $i \in \text{Rej}_s^k(B_s^k(t))$ for some step $t$, then $i \in \text{Rej}_s^k(B_s^k(t'))$ for all $t' \geq t$. In particular, $i \in \text{Rej}_s^k(B_s^k(\hat{T}_k))$, which implies that under $\mu^k$, each student is assigned to exactly one school. Also, note that the law of aggregate demand guarantees at each rejection step, at most one student is rejected, which implies that in each application step, there is a unique student $i'$ who applies to the next school on his preference list $s'$.

50 In the above description of the cumulative offer process, both the application and rejection phases occurred in the same “step”. Here, we have broken the steps up into “rejection steps” and “application steps” for clarity. This will change the numbering of the steps, but does not affect the algorithm or results in any other way.
For any school $s$ and applicant pool $\mathcal{B} \subseteq I$, define $\delta_s(\mathcal{B}) = \sum_{B \in \mathcal{B}} \max\{L_{s,B} - |\mathcal{B} \cap I_{\theta}|, 0\}$. In words, $\delta_s(\mathcal{B})$ is the number of floor seats unfulfilled at $s$ when its applicant pool is $\mathcal{B}$.

Let $\Delta^k = \sum_{s \in S} \delta_s(\mathcal{B}^k_s)$. In words, $\Delta^k$ is the total number of floor seats unfulfilled at all schools in stage $k$ after all students but $i$ have applied.

**Lemma 3.** The following hold for all $k$:

(i) If $\Delta^k = 0$, then $DA^{Ch^k}(P^i_1, P_{-i}) \in \mathcal{M}_f$ for all $P^i_1 \in \mathcal{P}$.

(ii) If $\Delta^k > 1$, then $DA^{Ch^k}(P^i_1, P_{-i}) \notin \mathcal{M}_f$ for all $P^i_1 \in \mathcal{P}$.

(iii) $\Delta^k \geq \Delta^{k+1} \geq \Delta^k - 1$.

If $\Delta^1 = 0$, then, all floor seats have been filled before $i$ enters the market in stage 1, and the GDQDA algorithm ends in stage 1 for any report $P^*_i$ of agent $i$. Therefore, from the perspective of agent $i$, the mechanism is equivalent to $DA^{Ch^1}$ which is known to be strategyproof, and so she has no profitable manipulation. So, assume that $\Delta^1 \geq 1$. Lemma 3, part (iii) then implies that there is some critical stage $k^*$ for which $\Delta^{k^*} = 1$ and $\Delta^{k'} > 1$ for all $k' < k^*$. By Lemma 3, part (ii), the algorithm will not end in stage $k' < k^*$ for any report of student $i$, and so we can begin the algorithm at stage $k^*$ using choice functions $Ch^{k^*}$. From here forward we switch from GSDA and work with the GDQDA algorithm definition. Let $\mathcal{A}^{k^*}_s$ be the cumulative offer sets of the schools after running DA under $Ch^{k^*}$ on all students other than $i$.

Then, let $i$ enter the market with some reported preferences $P^i_1$. As above, her entering again causes a rejection chain:

<table>
<thead>
<tr>
<th>Stage $k^*$</th>
<th>Step $0$</th>
<th>Action</th>
<th>Cumulative offer sets $\mathcal{A}^{k^<em>}_s(0) = \mathcal{A}^{k^</em>}_s$ for all $s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$ Student $i$ applies to school $s$</td>
<td>$\mathcal{A}^{k^<em>}_s(1) = \mathcal{A}^{k^</em>}_s(0) \cup {i}$</td>
<td>$\mathcal{A}^{k^<em>}_s(1) = \mathcal{A}^{k^</em>}_s(0)$ for all $s \neq s$</td>
<td></td>
</tr>
<tr>
<td>$2$ School $s' \neq s$ rejects $i'$</td>
<td>$\mathcal{A}^{k^<em>}_s(2) = \mathcal{A}^{k^</em>}_s(1)$ for all $s$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$3$ School $s'' \neq s$ applies to $s'$</td>
<td>$\mathcal{A}^{k^<em>}_s(3) = \mathcal{A}^{k^</em>}_s(2) \cup {i''}$</td>
<td>$\mathcal{A}^{k^<em>}_s(3) = \mathcal{A}^{k^</em>}_s(2)$ for all $s \neq s'$</td>
<td></td>
</tr>
<tr>
<td>$k^* + 1$ Choice functions become $Ch^{k^*+1}$</td>
<td>$\mathcal{A}^{k^<em>+1}_s(0) = \mathcal{A}^{k^</em>}_s(3)$ for all $s$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The rejection chain $\mathcal{R}$ simply records the action of the algorithm, but note that now, we include quota reductions as part of the rejection chain (stage $k^* + 1$, step 0 above).

---

51 Recall that at every stage $k$, each school $s$ always reserves $L_{s,\theta}$ seats for students of type $\theta$, and so if $|\mathcal{B} \cap I_{\theta}| \leq L_{s,\theta}$, then all type $\theta$ students will be chosen, while if $|\mathcal{B} \cap I_{\theta}| > L_{s,\theta}$, at least $L_{s,\theta}$ students of type $\theta$ will be chosen.

52 Because we are now working with the GDQDA algorithm rather than the GSDA algorithm, we switch the notation from $\mathcal{B}^k_s(i)$ to $\mathcal{A}^k_s(i)$ to remain consistent with the notation used above in the definitions of GDQDA and GSDA.
There are several features to note about rejection chains. First, at stage $k$, step $t$ of the rejection chain, the set of applicants being tentatively held by school $s$ is $\text{Ch}_s^k(\mathcal{A}_s^k(t))$. Second, monotonicity and substitutability guarantee that if at some stage and step $(k, t)$ we have $i \in \text{Rej}_s^k(\mathcal{A}_s^k(t))$, then $i \in \text{Rej}_s^{k'}(\mathcal{A}_s^{k'}(t'))$ for all $(k', t')$ such that either (i) $k' > k$ or (ii) $k' = k$ and $t' \geq t$. This, together with the law of aggregate demand and minimality, guarantee that at each step, there is at most one student who is not currently being held by any school, and it is this student who will make the next application according to her preferences.\footnote{The law of aggregate demand guarantees this within a stage, while minimality guarantees it across stages.} Third, within a stage, the school rejecting a student at some step is the same as the last school to receive an application. However, across stages, this may not be true. For example, at stage $k^*$, step 3, $s'$ receives an application. When the choice functions become $\text{Ch}^{k^*+1}$, school $s''$ rejects a student, but it may be that $s'' \neq s'$.

At stage $k^*$, $\Delta^{k^*} = 1$, which means that $\delta_y(\hat{A}_y^{k^*}) = 1$ for some school $y$ and $\delta_s(\hat{A}_s^{k^*}) = 0$ for all $s \neq y$. By definition of $\delta_s(\cdot)$, it must be that $|\hat{A}_s^{k^*} \cap I_\phi| = L_{w,\phi} - 1$ for some type $\phi \in \Theta$, and $|\hat{A}_s^{k^*} \cap I_\theta| \geq L_{s,\theta}$ for all $(s, \theta) \neq (y, \phi)$. In words, this means that every school has enough students in its choice set to fill all type-specific floors except for school $y$, which is one student short of filling its type $\phi$ floor.

We next note the following important fact about rejection chains:

The algorithm ends the next time a type $\phi$ student applies to $y$ \hspace{1cm} (1)

This is an “if and only if” statement: the algorithm ends the next time a type $\phi$ student applies to $y$, and cannot end earlier. When (1) occurs in some step $t$ of some stage $k \geq k^*$, no student is rejected from $y$, and stage $k$ ends. At this point, all schools have filled all floors, and so the algorithm ends and all tentative assignments are made permanent.

The next part of the proof is inspired by the Scenario Lemma of Dubins and Freedman (1981). Define a scenario $S_i$ as a sequence of applications for agent $i$, i.e., a partial rank ordering over $S$ for student $i$. So, a scenario could be $S_i = \{s, u, v\}$, which means that $i$ first applies to $s$, then to $u$, then to $v$. The list need not include all schools. Since the preferences of the other students are fixed at $P_{-i}$, each scenario induces a corresponding rejection chain. We use $\mathcal{R}(S_i)$ to denote the rejection chain corresponding to scenario $S_i$. The rejection begins with $i$ applying to the first school in $S_i$, and then records all subsequent applications, rejections, and quota reductions. The rejection chain for any scenario $S_i$ ends in one of two ways:

(1) some student $j$ of type $\phi$ applies to school $y$

<table>
<thead>
<tr>
<th>Stage</th>
<th>Step</th>
<th>Action</th>
<th>Cumulative offer sets</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>$t$</td>
<td>$j$ applies to $y$</td>
<td>$\mathcal{A}_s^k(t) = \mathcal{A}_s^k(t-1) \cup {j}$</td>
</tr>
</tbody>
</table>

(2) $i$ is rejected by the last school in $S_i$:

<table>
<thead>
<tr>
<th>Stage</th>
<th>Step</th>
<th>Action</th>
<th>Cumulative offer sets</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>$t$</td>
<td>$i$ is rejected by $v$</td>
<td>$\mathcal{A}_s^k(t) = \mathcal{A}_s^k(t-1)$ for all $s$</td>
</tr>
</tbody>
</table>

\footnote{The law of aggregate demand guarantees this within a stage, while minimality guarantees it across stages.}
The following lemma is key to the remainder of the proof.

**Lemma 4.** Consider two scenarios \( S_i \) and \( \hat{S}_i \) such that every school in \( \hat{S}_i \) is also named in \( S_i \) (order is immaterial), and assume that in \( R(S_i) \), student \( i \) applies to every school in \( S_i \). Then, every step in \( R(\hat{S}_i) \) also occurs at some point in \( R(S_i) \).

With this lemma in hand, we can continue the proof. Suppose, without loss of generality, that \( i \)'s true preferences are \( P_i : s_1, s_2, \ldots, s_m \). Say that if \( i \) submits her true preferences, she receives school \( s_h \), and suppose that there is some scenario \( \hat{S}_i = \{u, \ldots, v\} \), where \( i \) gets some school \( v \) such that \( v P_i s_h \). Then, rejection chain \( R(\hat{S}_i) \) must end with some student \( j \) of type \( \phi \) applying to \( y \), while \( i \) is assigned to \( v \).

**Case (i):** \( s P_i v \) for all \( s \in \hat{S}_i \) \( \setminus \{v\} \).

Compare \( \hat{S}_i \) to a scenario \( S_i = \{s_1, \ldots, s_{h-1}\} \). By assumption, \( \hat{S}_i \subseteq S_i \), and in \( R(S_i) \), \( i \) makes every application in \( S_i \). In particular, the last step of \( R(S_i) \) is “\( s_{h-1} \) rejects \( i \”).

By Lemma 4, every application in \( R(\hat{S}_i) \) is also made in \( R(S_i) \). In particular, \( j \) must also apply to \( y \) in \( R(S_i) \), which contradicts the fact that \( R(S_i) \) ends with \( i \) being rejected by \( s_{h-1} \).

**Case (ii):** \( v P_i s \) for at least one \( s \in \hat{S}_i \).

Delete all schools \( s \in \hat{S}_i \) such that \( v P_i s \) to create a smaller scenario \( \hat{S}_i \subset \hat{S}_i \). By case (i), \( R(\hat{S}_i) \) must end with \( i \) rejected by \( v \). Since \( \hat{S}_i \subset \hat{S}_i \), Lemma 4 implies that \( i \) must also be rejected by \( v \) in \( R(\hat{S}_i) \), which is a contradiction. ■

**Theorem 5**

Let \( DQDA \) denote the DQDA mechanism under some reduction sequence \( \eta = \{Q^1, \ldots, Q^K\} \) such that \( Q^1 = Q \), and let \( \psi \) be a mechanism that dominates \( DQDA \). Let \( P_I \) be a preference profile such that \( \psi_i(P_I)P_iDQDA_s(P_I) \) for some \( i \in I \), and \( \psi_i(P_I)R_iDQDA_s(P_I) \) for all \( i \in I \). Let \( i \) be a student that is made strictly better off under \( \psi \). Define \( DQDA_s(P_I) = s_i \) and \( \psi_i(P_I) = \hat{s}_i \) (implying \( \hat{s}_i P_i s_i \)).

Let the students other than \( i \) submit profile \( P_{-i} \), and define \( k^* \) as the critical stage of DQDA such that DQDA will not end before stage \( k^* \) for any preferences of agent \( i \) (as in the proof of strategyproofness), and let \( \hat{\mu} \) be the tentative matching after every student but \( i \) has entered the market. There are two cases.

**Case (i):** \( k^* = 0 \)

In this case, DQDA ends in stage 1 for any report of agent \( i \). Let \( \mathcal{S} = \{s \in S : |\hat{\mu}^i(s) < Q_s\} \) and \( \mathcal{R} \) be the rejection chain initiated by \( i \) entering the market. Note that the rejection chain ends the first time an application is made to a school in \( \mathcal{S} \).

**Claim 1.** Let \( P'_i \) be some report for agent \( i \), and define \( P_i' = (P_i', P_{-i}) \).\(^{55}\) Then, we have \( \psi_s(P'_i) = DQDA_s(P'_i) \) for all \( s \in \mathcal{S} \).

**Proof.** First we show that \( \psi_s(P'_i) \subseteq DQDA_s(P'_i) \). If not, then there exists some \( j \in \psi_s(P'_i) \), but \( j \notin DQDA_s(P'_i) \). However, under DQDA, no student was rejected from \( s \), which means \( DQDA_j(P'_i)P'_js = \psi_j(P'_i) \), contradicting that \( \psi(P'_i) \) dominates \( DQDA(P'_i) \). Second, we show that \( DQDA_s(P'_i) \subseteq \psi_s(P'_i) \).

\(^{54}\) This follows because \( i \) receives \( s_h \) when he submits his true preferences.

\(^{55}\) We allow for the possibility that \( P'_i = P_i \), in which case \( P'_i = P_i \). Also, note that for \( j \neq i \), \( P'_j = P_j \).
If not, then there is some student $j \in DQDA_s(P'_i)$, but $j \notin \psi_s(P'_i)$. By domination, $j$ must have been rejected from $\psi_j(P'_i)$ under DQDA because $\psi_j(P'_i)$ was filled to capacity. Let $\psi_j(P'_i) = s_j$. In order to assign $j$ to $s_j$ under $\psi$, there must be some student $j' \in DQDA_{s_j}(P'_i)$, but $\psi_{j'}(P'_i) \neq DQDA_{j'}(P'_i)$. By domination, $\psi_j(P'_i)P'_iDQDA_{j'}(P'_i)$. By the same argument, there must be some $j'' \in DQDA_{s_j}(P'_i)$, but $\psi_{j''}(P'_i)P'_iDQDA_{j''}(P'_i)$, where $s_j = \psi_{j'}(P'_i)$. Continuing this line of reasoning, we reach a contradiction. Thus, $\psi_s(P'_i) = DQDA_s(P'_i)$ for all $s \in S$. ■

By the above claim, $s_i \notin S$. Fix some $y \in S$, and note that $\hat{s}_iP_is_iP_iy$. Consider an alternative report for $i$, $P'_i$, which ranks $\hat{s}_i$ first, $y$ second, and everything else remains the same. Define $P'_i = (P'_i, P_{-i})$.

Then, $DQDA_i(P'_i) = y$. By Claim 1, $\psi_i(P'_i) = y$. But now, $\psi_i(P_i) = \hat{s}_iP'_iy = \psi_i(P'_i)$, and so $\psi$ is manipulable.

**Case (ii):** $k^* \geq 1$

We begin with the following claim. Define

$$S(P_i) = \{s \in S : |DQDA_s(P_i)| = L_s \text{ and } s \text{ never rejects a student in the running of } DQDA(P_i)\}.$$ 

**Claim 2.** If a matching $\nu$ dominates $DQDA(P_i)$ under preferences $P_i$, then $\nu(s) = DQDA_s(P_i)$ for all $s \in S(P_i)$.

**Proof.** Assume that $\nu(s) \neq DQDA_s(P_i)$ for some $s \in S(P_i)$. Then, since $|DQDA_s(P_i)| = L_s$ and $|\nu(s)| \geq L_s$ (by feasibility), there must be some student $i$ such that $DQDA_i(P_i) \neq s$ and $\nu(i) = s$. However, since $s$ never rejected a student in the running of $DQDA(P_i)$, this implies that $DQDA_i(P_i)P_is_i$, i.e., student $i$ is worse off under $\nu$, which is a contradiction. ■

Similar to the proof of strategyproofness, we can calculate $DQDA(P_i)$ by starting the algorithm in stage $k^*$. Denote $y$ as the unique school such that $|\mu^{k^*}(y)| = L_y - 1$. Allow $i$ to enter the market using preferences $P_i$, and let $R$ denote the rejection chain that occurs. The rejection chain (and the algorithm) end as soon as $y$ gets its next application. Note that it must be that this application is from some student $j \neq i$, which also implies that $\hat{s}_iP_is_iP_iy$.57

Construct an alternative preference profile $P'_i = (P'_i, P_{-i})$, where $P'_i$ is defined such that $\hat{s}_i$ is first, $y$ is second, and the relative ordering of all other schools is the same as in $P_i$:

$$P'_i : \hat{s}_i, y, \ldots$$

**Claim 3.** $DQDA_i(P'_i) = y$ and $y \in S(P'_i)$.

**Proof.** As in the proof of strategyproofness (Theorem 6), we can calculate $DQDA(P'_i)$ starting with $\mu^{k^*}$ in stage $k^*$ and allowing $i$ to apply last. (The tentative matching before $i$ enters, $\mu^{k^*}$, is the same under $P'_i$, since only $i$’s preferences have changed). Let $R'$ denote the rejection chain that occurs upon $i$ entering the market. Once again, the algorithm ends with the next application to $y$, which implies that $|DQDA_y(P'_i)| = L_y$ and that $y$ never rejects a student, i.e., $y \in S(P'_i)$.

To show that $DQDA_i(P'_i) = y$, note that strategyproofness of DQDA gives $DQDA_i(P'_i) \neq \hat{s}_i$. Thus, $i$ must be rejected from $\hat{s}_i$ at some step $t$ of rejection chain $R'$. Step $t + 1$ must then be $i$ applying to $y$.

---

56 If this were not the case, then $|\psi_s(P'_i)| > |DQDA_s(P'_i)| = Q_{s_j}$, i.e., $\psi$ is not a feasible mechanism.

57 Otherwise, $DQDA_i(P_i) = y$. Then, by Claim 2, $\psi_i(P_i) = y$, which contradicts that $\psi_i(P_i)P_iDQDA_i(P_i)$. 

45
Since $y$ has an empty seat, $i$ is accepted, and the algorithm ends. ■

**Claim 4.** $\psi_i(P'_t) = y$

**Proof.** This follows immediately from Claim 2, since by assumption the matching $\psi(P'_t)$ dominates matching $DQDA(P'_t)$ under preferences $P'_t$. ■

Now, consider the market where the true preferences are $P'_t$. The definition of $s_i$ and Claim 4 imply $s_i = \psi_i(P'_t)P'_t\psi_i(P'_t) = y$, which means that $\psi$ is not strategyproof.

**Theorem 7**

To simplify notation, for a student of type $\lambda_i = (\theta_i, z_i, t_i)$, we define $g_i = (\theta_i, z_i)$, and refer to $g_i$ as student $i$’s group. Recall that students cannot misreport their group and, as described in the text, the number of students in each group, $n_g$, grows according to some fixed sequence such that $n_g \to \infty$ for all groups $g$.

We now state a no-envy condition that will be sufficient for a mechanism to be SPL, and then show that EDQDA does indeed satisfy this condition.

**Definition 11.** (Azevedo and Budish, 2013) A mechanism $\{(\psi^n)_{n \in N}, \Lambda\}$ is **envy-free but for tie-breaking (EFTB)** if for each $n$ there exists a function $x^n : (\Lambda \times [0, 1])^n \to \Delta(S^n)$ that is symmetric over its coordinates and such that

$$\psi^n(\lambda) = \int_{\ell \in [0, 1]^n} x^n(\lambda, \ell) d\ell$$

and if $\ell_i \geq \ell_j$ and $i$ and $j$ belong to the same group $g$, then $u_{t_i}(x^n_i(\lambda, \ell)) \geq u_{t_i}(x^n_j(\lambda, \ell))$.

**Lemma 5.** The mechanism $\{(E^n)_{n \in N}, \Lambda\}$ is envy-free but for tie-breaking.

**Proof.** To show this, we must exhibit a function $x^n$ that satisfies the properties of Definition 11. Such a function is given immediately by $\mu^n(\lambda, \ell)$, as defined in Section 6.3. Recall that by definition we have

$$E^n(\lambda) = \int_{\ell \in [0, 1]^n} \mu^n(\lambda, \ell) d\ell$$

for all $n \in \mathbb{N}$. The function $\mu^n$ is clearly symmetric over its coordinates by construction. Thus, the last thing we need to show is that an agent $i$ in group $g$ never envies another agent $j$ in group $g$ with a lower lottery number. Assume that $i$ and $j$ belong to the same group, but $\ell_i \geq \ell_j$. Recall that $>^n_s$ is the (post-lottery) priority relation for school $s$. The fact that $i$ and $j$ belong to the same group and $\ell_i \geq \ell_j$ imply that the priority relations are such that $i >^n_s j$ for all $s \in S$. Now, $\mu^n(\lambda, \ell)$ is equivalent to the matching produced by standard DA under some quotas $(U^{n,k}, Q^n)$, priorities $(>^n_s)_{s \in S}$, and (ordinal) preferences $(P_{t_i})_{i \in I^n}$. This matching eliminates all justified envy for same types with respect to the strict priorities $(>^n_s)_{s \in S}$, which implies that $\mu^n_i(\lambda, \ell)R_{t_i}\mu^n_j(\lambda, \ell)$, and so $u_{t_i}(\mu^n_i(\lambda, \ell)) \geq u_{t_i}(\mu^n_j(\lambda, \ell))$, which proves the lemma. ■

Given this lemma, the theorem follows from Proposition 1 of Azevedo and Budish (2013), which states that if a mechanism satisfies EFTB, it is SPL.\(^{58}\)

\(^{58}\)Our model is slightly different, since the number of students in each group $g$ grows deterministically, rather than stochastically. In Appendix C of their paper, Azevedo and Budish (2013) note that their results hold when the number of students in each group grows deterministically and the utility types are drawn iid within each group.
Theorem 8

Since \((U^K, Q^K)\) ensures a feasible match, we know that \(DQ^\Psi\) ends no later than stage \(K\) for every \(P_I\); for some \(P_I\), it may end in some stage \(k < K\) if \(\psi(U^K, Q^K)(P_I)\) is a feasible match. For the second part, \(\eta\) being a minimal reduction sequence implies \(\sum_{\theta \in \Theta(U^K, Q^K)} U^K - U^K \leq Q^K - Q^K\) for all \(s \in S\). Then, by resource monotonicity, \(DQ^\Psi_i(P_I) = \psi_i(U^K, Q^K)(P_I)\) for all \(i \in I\). Since this holds for all \(P_I\), the result follows.

Theorem 9

Consider a preference profile \(P_I\) such that \(\mu_i\) ranks \(O\) first for every \(i \in I\). Individual rationality requires that \(\psi_i(P_I) = O\) for all \(i \in I\). Otherwise, \(\psi_i(P_I) = \emptyset\), and so \(\psi(P_I)\) is not feasible.

Theorem 10

As in the proof of Theorem 8, \(DQ^\Psi(P_I) = \psi(U^K, Q^K)(P_I)\) for some \(k \leq K\). The same argument there shows part (1). For part (2), assume that \(DQ^\Psi\) ends in some stage \(k < K\) (if not, the result is obvious). By part (1), \(DQ^\Psi_i(P_I)R_i\psi_i(U^K, Q^K)(P_I) = \emptyset\), which means that \(DQ^\Psi_i(P_I) = \emptyset \implies \psi_i(U^K, Q^K)(P_I) = \emptyset\), which delivers the result. For part (3), if \(DQ^\Psi\) ends in stage \(K\), then \(DQ^\Psi(P_I) = \psi(U^K, Q^K)(P_I)\), and so both mechanisms fill the same number of floor seats. If \(DQ^\Psi\) ends in some stage \(k < K\), then the resulting matching \(\psi(U^K, Q^K)(P_I)\) must be feasible, i.e., there are zero floor seats that are left unfilled, and so the result holds.

C Omitted proofs of lemmas

Proof of Lemma 1

We first start with two sub-lemmas.

Lemma 6. If \(k' \geq k\), then \(B^k_s(\hat{T}^k) \subseteq B^{k'}_s(\hat{T}^{k'})\) for all \(s\).

Proof. By monotonicity, \(Ch^k_s(I') \subseteq Ch^{k'}_s(I')\) for all \(s \in S\) and all \(I' \subseteq I\). The statement then follows from Lemma 1, part (2) of Kamada and Kojima (2013).\(^{59}\)

The next sub-lemma makes use of the following definition of stability. Note that this is only a technical definition used to prove the results below, and is not related to the justified envy definitions used in the main text. Let \(Ch' := \{Ch'_{s_1}, \ldots, Ch'_{s_m}\}\) be a vector of choice functions (which again need not be equal to \(Ch\)).

Definition 12. A matching \(\mu\) is stable with respect to \(Ch'\) if:

(i) \(\mu(s) = Ch'_{s}(\mu(s))\) for all \(s \in S\)

(ii) there exist no pair \((i, s)\) such that \(sP_i\mu(i)\) and \(i \notin Ch'_{s}(\mu(s) \cup \{i\})\).

\(^{59}\)See also the “Capacity Lemma” of Konishi and Ünver (2006) for a related result.
Part (i) says that a school does not unilaterally reject any student assigned to it. The corresponding “individual rationality” property holds for students automatically, because we assume that all students find all schools acceptable. Given this definition, we have the following lemma.

**Lemma 7.** Under $\nu^k$, each student is assigned to at most one school, and $\nu^k$ is stable with respect to $Ch^k$.

**Proof.** For the first part, note that within a stage $k$, substitutability implies $Rej^k_s(A^k_s(t) - 1) \subseteq Rej^k_s(A^k_s(t))$ for all $t = 1, \ldots, T^k$, and across stages, monotonicity ensures that $Rej^{k-1}_s(A^{k-1}_s(T^{k-1})) \subseteq Rej^k_s(A^k_s(0))$ for all $s$. Thus, if $i$ is rejected by a school $s$ at some step $t$ of some stage $k$, then $i$ is rejected in all later steps and stages, implying that no student is assigned to more than one school. For stability, first note that by irrelevance of rejected students (see footnote 41), $Ch^k_s(A^k_s(T^k)) = \nu^k(s) = Ch^k_k(\nu^k(s))$. For the second part, if $sP_i \nu^k(i)$, then $i$ was at some point rejected from $s$, which implies that $i \in A^k_s(T^k)$ but $i \notin \nu^k(s)$. This means that $\nu^k(s) = Ch^k_s(A^k_s(T^k)) \subseteq \nu^k(s) \cup \{i\} \subseteq A^k_s(T^k)$, which, together with irrelevance of rejected students, implies that $Ch^k_s(\nu^k(s) \cup \{i\}) = \nu^k(s)$, i.e., $i \notin Ch^k_s(\nu^k(s) \cup \{i\})$. Therefore, $\nu^k$ is stable with respect to $Ch^k$. $\blacksquare$

We now use induction to show

\[ A^k_s(T^k) = B^k_s(T^k) \text{ for all } s \in S \tag{2} \]

holds for all $k$. Since in stage 1, both algorithms are just standard DA using $Ch^1$, (2) holds for $k = 1$. Assume the inductive hypothesis that (2) holds for $1, \ldots, k - 1$. We show that this implies it holds for $k$ as well.

First, note that since stage $k$ of GSDA is simply the DA algorithm under $Ch^k$, we know that $\mu^k$ is the student-optimal stable match with respect to $Ch^k$ (Hatfield and Milgrom (2005)). Further, by Lemma 7, $\nu^k$ is some match that is stable with respect to $Ch^k$. This implies that $\nu^k(i)R_i \nu^k(i)$ for all $i \in I$. This implies that $B^k_s(T^k) \subseteq A^k_s(T^k)$ for all $s \in S$.\(^{60}\)

Last, we must show that $A^k_s(T^k) \subseteq B^k_s(T^k)$ for all $s \in S$. Assume the contrary, i.e., that there exists some $s$ such that $A^k_s(T^k) \not\subseteq B^k_s(T^k)$, and let $t'$ be the first step of stage $k$ of GDQDA such that $A^k_s(t) \subseteq B^k_s(T^k)$ for all $t < t'$ and all $s \in S$, but $A^k_s(t') \not\subseteq B^k_s(T^k)$.\(^{61}\) Let $i$ be the student who applies to $s$ at step $t'$. This means that $i$ is rejected from $\mu^k(i)$ (the school she is matched to under GSDA) in some step of stage $k$ of GDQDA; let the earliest of these steps be $t''$, so that $i \in Rej^{k}_{\mu^k(i)}(A^k_{\mu^k(i)}(t''))$.\(^{62}\) Further, note that $t'' < t'$, which, by the definition of $t'$, implies that $A^k_{\mu^k(i)}(t'') \subseteq B^k_{\mu^k(i)}(T^k)$. Substitutability of the choice functions within stage $k$ then implies that $R^k_{\mu^k(i)}(A^k_{\mu^k(i)}(t'')) \subseteq R^k_{\mu^k(i)}(B^k_{\mu^k(i)}(T^k))$, which means $i \in Rej^k_{\mu^k(i)}(B^k_{\mu^k(i)}(T^k))$, which contradicts the fact that $i$ is assigned to school $\mu^k(i)$ under GSDA in stage $k$.

\(^{60}\)If this were not the case, then there exists some $s$ such that $B^k_s(T^k) \not\subseteq A^k_s(T^k)$. This means that in GSDA in stage $k$, some student $i$ is rejected from $\nu^k(i)$ (her match under GDQDA). But, this contradicts the fact that $\mu^k(i)R_i \nu^k(i)$.

\(^{61}\)Such a $t'$ exists because $A^k_s(0) = A^k_s(T^k - 1) = B^k_s(T^k - 1) \subseteq B^k_s(T^k)$ for all $s \in S$, where the first equality is by definition, the second is by the inductive hypothesis, and the set inclusion is by Lemma 6.

\(^{62}\)Note that $i$ must be rejected at some step $t''$ of stage $k$ (and not in an earlier stage). To see this, assume that $i$ was rejected from $\mu^k(i)$ in some earlier stage $k'$ of GDQDA. This implies that $\mu^k(i)R_i \nu^{k'}(i)$. Since $k' < k$, Lemma 1, part (1) of Kamada and Kojima (2013) implies that $\mu^{k'}(i)R_i \nu^{k'}(i)$. Combining these two inequalities, we conclude that $\mu^{k'}(i)R_i \nu^{k'}(i)$, which contradicts the inductive hypothesis.

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Proof of Lemma 2

Substitutability

Consider a school $s$, stage $k$, and set of students $I'$ such that $i \in \text{Rej}^k_s(I')$, and another set of students $A''$ such that $A' \subseteq A''$. Let $\tau(i) = \theta$. When the set of applicants is $A'$, student $i$ is rejected because the type $\theta$ specific seats are filled with $L_{s,\theta}$ higher ranked type $\theta$ students, and the open seats are filled with either (i) $U^k_{s,\theta} - L_{s,\theta}$ higher ranked type $\theta$ students or (ii) $Q^k_s - \sum_{\theta \in \Theta} L_{s,\theta}$ higher ranked students of any type. In either case, since all students in $A'$ are also in $A''$, when school $s$ is choosing from $A''$, the type $\theta$ specific seats will once again be filled with $L_{s,\theta}$ higher ranked type $\theta$ students, and either condition (i) or (ii) will also still hold. So, $i \in \text{Rej}^k_s(A'')$ as well.

Monotonicity

If $s'$ is not the school whose quotas are reduced in moving from stage $k$ to $k + 1$, then $\text{Rej}^k_s(A) = \text{Rej}^{k+1}_s(A)$ trivially. So, let $s$ be the school whose capacity is reduced in moving from stage $k$ to $k + 1$: $Q^k_s = Q^{k+1}_s - 1$ and $U^{k+1}_{s,\theta} = U^k_{s,\theta} - 1$, while $U^{k+1}_{s,\theta'} = U^k_{s,\theta'}$ for all other $\theta' \neq \theta$.

We want to show that $\text{Rej}^k_s(A) \subseteq \text{Rej}^{k+1}_s(A)$ for all $k$ and all $A \subseteq I$.\footnote{An inductive argument then implies that $\text{Rej}^k_s(A) \subseteq \text{Rej}^{k''}_s(A)$ for all $k'' \geq k'$.} To do so, we show the contrapositive:

$$i \in \text{Ch}^{k+1}_s(A) \implies i \in \text{Ch}^k_s(A).$$

Assume not, and let $i$ be the highest ranked student according to $\succ_s$ such that $i \in \text{Ch}^{k+1}_s(A)$, but $i \notin \text{Ch}^k_s(A)$ (equivalently, $i \in \text{Rej}^k_s(A)$). Let $\tau(i) = \theta'$, which may or may not be equal to $\theta$. If $i$ is admitted through the type $\theta'$ specific seats in stage $k + 1$, then she is one of the $L_{s,\theta'}$ highest ranked type $\theta'$ students in $A$, and so she will be admitted in stage $k$ as well. So, $i$ must be admitted through an open seat in stage $k + 1$. Therefore, when $i$’s application is considered in stage $k + 1$, the following both hold: (a) at most $U^{k+1}_{s,\theta'} - L_{s,\theta'} - 1$ higher ranked type $\theta'$ students have been accepted to the open seats and (b) at most $Q^k_s - \sum_{\theta \in \Theta} L_{s,\theta} - 1$ students in total have been accepted to the open seats. Define $J = \{ j \in \text{Rej}^k_s(A) : j \succ_s i \}$. Note that (b) implies that for all $j \in J$, the type-specific ceiling $U^{k+1}_{s,\tau(j)}$ is reached before $j$’s application is considered in stage $k + 1$.

Since $Q^k_s = Q^{k+1}_s + 1$, for $i$ to be rejected in stage $k$, there must be two $j_1, j_2 \in J$ such that $j_1, j_2 \in \text{Ch}^k_s(A)$; without loss of generality, let $j_1 \succ_s j_2 \succ_s i$. Since $j_1 \in \text{Ch}^k_s(A)$, it must be that $\tau(j_1) = \theta$.\footnote{If $\tau(j_1) \neq \theta$, then $U^k_{s,\tau(j_1)} = U^{k+1}_{s,\tau(j_1)}$. Then, since all $j \succ_s j_1$ such that $j \in \text{Ch}^{k+1}_s(A)$ also satisfy $j \in \text{Ch}^k_s(A)$ (by the assumption that $i$ is the highest-ranked student for which this is not the case), the ceiling for type $\tau(j_1)$ students will already have been reached in stage $k$ when $j_1$’s application is considered, and $j_1$ will be rejected.} Then, after $j_1$ is admitted, the $U^k_{s,\theta}$ ceiling is now binding (since $U^k_{s,\theta} = U^{k+1}_{s,\theta}$ and all $j \succ_s j_1$ who are admitted in stage $k + 1$ are also admitted in stage $k$ by the assumption that $i$ is the highest-ranked such student for which this is not the case). So, when $j_2$’s application is considered, she will be rejected, which is a contradiction.\footnote{If $\tau(j_2) = \theta$, this follows from the previous sentence. If $\tau(j_2) \neq \theta$, it follows from the same argument as in footnote 64.}

Minimality

As for monotonicity, we only need consider the school $s$ whose quotas are reduced in moving from $k$ to $k + 1$. Let $s$ be the school such that $Q^k_s = Q^{k+1}_s - 1$ and $U^{k+1}_{s,\theta} = U^k_{s,\theta} - 1$ for some $\theta$, while $U^{k+1}_{s,\theta'} = U^k_{s,\theta'}$. In this case, student $i$ is rejected because $U^{k+1}_{s,\theta} - 1$ (or $U^{k+1}_{s,\theta'} - 1$) type $\theta$ students have already been admitted in stage $k$, and school $s$ is now choosing from $A''$. If $i$ is admitted in stage $k$, then she is one of the $L_{s,\theta'}$ highest ranked type $\theta'$ students in $A$, and so she will be admitted in stage $k + 1$, and set of students $A''$.
for all $\theta' \neq \theta$. Consider a set of applicants $\mathcal{A} \subseteq I$, with $Ch^k_s(\mathcal{A})$ the set that is admitted in stage $k$. There are four cases.

**Case (i):** $|Ch^k_s(\mathcal{A})| < Q^k_s$ and $|Ch^k_s(\mathcal{A}) \cap I_\theta| < U^k_{s,\theta}$

In this case, $Ch^{k+1}_s(\mathcal{A}) = Ch^k_s(\mathcal{A})$, which implies $|Ch^k_s(\mathcal{A})| - |Ch^{k+1}_s(\mathcal{A})| = 0$.

**Case (ii):** $|Ch^k_s(\mathcal{A})| = Q^k_s$ and $|Ch^k_s(\mathcal{A}) \cap I_\theta| < U^k_{s,\theta}$

Let $i'$ be the lowest-ranked student admitted through the open seats in stage $k$. Then, $Ch^{k+1}_s(\mathcal{A}) = Ch^k_s(\mathcal{A}) \setminus \{i'\}$, which implies $|Ch^k_s(\mathcal{A})| - |Ch^{k+1}_s(\mathcal{A})| = 1$.

**Case (iii):** $|Ch^k_s(\mathcal{A})| < Q^k_s$ and $|Ch^k_s(\mathcal{A}) \cap I_\theta| = U^k_{s,\theta}$

Let $i'$ be the lowest-ranked type $\theta$ student admitted through the open seats in stage $k$. Then, $Ch^{k+1}_s(\mathcal{A}) = Ch^k_s(\mathcal{A}) \setminus \{i'\}$, which implies $|Ch^k_s(\mathcal{A})| - |Ch^{k+1}_s(\mathcal{A})| = 1$.

**Case (iv):** $|Ch^k_s(\mathcal{A})| = Q^k_s$ and $|Ch^k_s(\mathcal{A}) \cap I_\theta| = U^k_{s,\theta}$

Let $i'$ be the lowest-ranked type $\theta$ student admitted through the open seats in stage $k$. Then, $Ch^{k+1}_s(\mathcal{A}) = Ch^k_s(\mathcal{A}) \setminus \{i'\}$, which implies $|Ch^k_s(\mathcal{A})| - |Ch^{k+1}_s(\mathcal{A})| = 1$.

**Law of aggregate demand**

Within a stage, school $s$ admits students one-by-one until either some type-specific ceiling $U^k_{s,\theta}$ or overall capacity $Q^k_s$ has been reached. Therefore, more students in the applicant pool weakly increases the number of students admitted, and the law of aggregate demand is satisfied.

**Proof of Lemma 3**

We will use the following facts about $\delta_s$:

(a) $\mathcal{B} \subseteq \mathcal{B}' \implies \delta_s(\mathcal{B}') \leq \delta_s(\mathcal{B})$

(b) if $\delta_s(\mathcal{B} \cup \{i\}) < \delta_s(\mathcal{B}) \implies \text{Rej}^k_s(\mathcal{B} \cup \{i\}) = \text{Rej}^k_s(\mathcal{B})$ for all $k$.

Fact (a) follows immediately from the definition of $\delta$. Fact (b) follows because $\delta_s(\mathcal{B} \cup \{i\}) < \delta_s(\mathcal{B})$ implies that student $i$ is of some type $\theta$ such that $|\mathcal{B} \cap I_\theta| < L_{s,\theta}$. But this means that when the applicant pool at school $s$ is $\mathcal{B} \cup \{i\}$, student $i$ is accepted through one of the type $\theta$ seats. This does not affect the students accepted by the type $\theta'$ seats for $\theta' \neq \theta$, nor the open seats, and thus $Ch^k_s(\mathcal{B} \cup \{i\}) = Ch^k_s(\mathcal{B})$, or equivalently, $\text{Rej}^k_s(\mathcal{B} \cup \{i\}) = \text{Rej}^k_s(\mathcal{B})$.

**Part (i):** Since $\Delta^k = 0$, the set $\bar{B}^k_s$ contains at least $L_{s,\theta}$ students of type $\theta$ for all schools $s$. Let $\bar{B}^k_s(\hat{T}^k)$ be the cumulative set of applicants to school $s$ at the end of stage $k$. Since $\bar{B}^k_s \subseteq \bar{B}^k_s(\hat{T}^k)$ for any submitted preferences of student $i$, $Ch^k_s(\bar{B}^k_s(\hat{T}^k))$ is a feasible assignment for school $s$, and the algorithm ends in stage $k$.

**Part (ii):** We can have $\Delta^k > 1$ in two ways: either $\delta_s(\bar{B}^k_s) > 1$ for some school, or $\delta_s(\bar{B}^k_s) \geq 1$ for multiple schools. First, consider $\delta_s(\bar{B}^k_s) > 1$ for some school $s$. This means that there are at least 2 floor seats at $s$ that are not yet filled because not enough students have applied to $s$. When student $i$ enters the market in stage $k$, he causes a rejection chain that ends the first time a school gets an application from a student $i'$ and does not reject an additional student. Therefore, at the end of stage $k$, at most one of the unfilled floor seats at $s$ can be filled, and so the assignment of school $s$ will still not be feasible.

The case where $\delta_s(\bar{B}^k_s) \geq 1$ for multiple schools is argued similarly.

**Part (iii):** By Theorem 11, $\bar{B}^{k+1}_s$ can be computed by starting with $\bar{B}^k_s$ and then reducing the choice functions to $Ch^{k+1}_s$. Doing so causes a rejection chain that ends the first time a student $i'$ applies to a
school $s'$ and $s'$ does not reject a student. Since $\bar{B}_s^k \subseteq \bar{B}_s^{k+1}$, we have $\delta_s(\bar{B}_s^{k+1}) \leq \delta_s(\bar{B}_s^k)$, which implies that $\Delta^k \geq \Delta^{k+1}$. To see that $\Delta^{k+1} \geq \Delta^k - 1$, note that in the rejection chain, the first time a student applies to a school and fills a floor, no further student is rejected (by fact (b)) and the rejection chain ends. Thus, at the end of the rejection chain, at most one floor seat that wasn’t filled under $\tilde{\mu}^k$ can be filled under $\tilde{\mu}^{k+1}$, and so $\Delta^{k+1} \geq \Delta^k - 1$.

**Proof of Lemma 4**

We use the notation $(k, t)$ to denote the line corresponding to stage $k$, step $t$ of a rejection chain. Let $A_s^k(t)$ denote the cumulative offer set of school $s$ at line $(k, t)$ of $R(S_i)$, and $\hat{A}_s^k(t)$ denote the corresponding set at line $(k, t)$ of $\hat{R}(\hat{S}_i)$. Similarly, let $i^k$ denote the final step of stage $k$ under scenario $S_i$, and $k^t$ denote the final step of stage $k$ under scenario $\hat{S}_i$. Let $k_{end}$ denote the last stage of $R(S_i)$ and $\hat{k}_{end}$ denote the last stage of $R(\hat{S}_i)$.

We prove the result by induction on the line index $(k, t)$. Line $(k^*, 0)$ of $R(\hat{S}_i)$ is “$i$ applies to $s$”, and this step occurs in $R(S_i)$ by assumption. So, make the inductive assumption that all lines up to $(k, t-1)$ of $R(\hat{S}_i)$ also occur in $R(S_i)$. Then, consider the next line in $R(\hat{S}_i)$. There are three cases:

**Case (i): The next line $(k, t)$ is an application line.**

Line $(k, t)$ then reads “$i'$ applies to $s''$”. There are two cases. If $i'' = i$, then this application also occurs in $R(S_i)$ by assumption. If $i'' \neq i$, then let $u$ be the school immediately before $s'$ on the preference list of $i'$. Because $(k, t)$ is an application line, $(k, t-1)$ must be a rejection line in which student $i'$ is rejected by $u$. Since, by the inductive hypothesis, line $(k, t-1)$ occurs somewhere in $R(S_i)$, student $i'$ must be rejected from $u$ at some point in $R(S_i)$, and will then, according to his preferences, apply to $s'$ in the following line.

**Case (ii): The next line $(k, t)$ is a rejection line.**

Line $(k, t)$ then reads “$s'$ rejects $i''$. Thus, student $i'$ must have already applied to $s'$, either before $i$ entered the market, or somewhere in rejection chain $R(\hat{S}_i)$. The choice function at $s' \text{ when } i'$ is rejected is $Ch_{s'}^k$, and the set of cumulative applicants is $\hat{A}_{s'}^k(t)$, which implies that $i' \in \text{Rej}_{s'}^k(\hat{A}_{s'}^k(t))$. By the inductive hypothesis, all students in $\hat{A}_{s'}^k(t)$ also apply to $s'$ under scenario $S_i$; in other words, $\hat{A}_{s'}^k(t) \subseteq A_{s'}^{k_{end}}(T^{k_{end}})$. Further, the inductive hypothesis plus monotonic substitutability imply that $\text{Rej}_{s'}^k(\hat{A}_{s'}^k(t)) \subseteq \text{Rej}_{s'}^{k_{end}}(A_{s'}^{k_{end}}(T^{k_{end}}))$. So, $i'$ must be rejected from $s'$ at some point under scenario $S_i$, i.e., line $(k, t)$ must occur in $R(S_i)$.

**Case (iii): The next line $(k+1, 0)$ is a choice function reduction line.**

In this case, line $(k+1, 0)$ reads “The choice functions become $Ch^{k+1}$.” Assume to the contrary that this reduction does not occur under $S_i$. Thus, $R(S_i)$ ends in stage $k$ under $Ch^k$ (by the inductive hypothesis, $R(S_i)$ reaches at least stage $k$). By Theorem 11, an alternative way to compute the outcome at the end of stage $k$ under either scenario is to start with the empty matching and run the DA algorithm under $\hat{S}_i$ and $S_i$. Recall that $\hat{A}_{s'}^k$ is the cumulative set of applicants to school $s$ before $i$ enters the market (as before, since the preferences of all agents $-i$ do not change, this is the same under either case). Note that $\delta_s(\hat{A}_{s'}^k) = 0$ for all $s \neq y$, and $\delta_y(\hat{A}_{s'}^k) = 1$, where $y$ has one type $\phi$ floor seat left to be filled (if the

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latter did not hold, all school choice sets would be feasible even before $i$ enters, and the mechanism would not continue to stage $k + 1$ under scenario $\hat{S}_i$). Just as above, we can write a rejection chain for each scenario corresponding to running the DA algorithm within stage $k$. Let $\mathcal{R}^k(S_i)$ and $\mathcal{R}^k(\hat{S}_i)$ denote these two rejection chains.

Since $\mathcal{R}(S_i)$ ends in stage $k$, $\mathcal{R}^k(S_i)$ must end in one of two ways:

1. $y$ gets an application from some student $j$ of type $\phi$:

<table>
<thead>
<tr>
<th>Stage</th>
<th>Step</th>
<th>Action</th>
<th>Offer sets</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>$t$</td>
<td>$j$ applies to $y$</td>
<td>$A^k_y(t) = A^k_y(t - 1) \cup {j}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$A^k_s(t) = A^k_s(t - 1)$ for all $s \neq s$</td>
</tr>
</tbody>
</table>

2. $i$ is rejected by the last school in $S_i$:

<table>
<thead>
<tr>
<th>Stage</th>
<th>Step</th>
<th>Action</th>
<th>Offer sets</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>$t$</td>
<td>$i$ is rejected by $s$</td>
<td>$A^k_s(t) = A^k_s(t - 1)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$A^k_s(t) = A^k_s(t - 1)$ for all $s \neq s$</td>
</tr>
</tbody>
</table>

On the other hand, $\mathcal{R}(\hat{S}_i)$ does not end in stage $k$. This means that $\mathcal{R}^k(\hat{S}_i)$ must end with

<table>
<thead>
<tr>
<th>Stage</th>
<th>Step</th>
<th>Action</th>
<th>Offer sets</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>$t$</td>
<td>Student $i'$ applies to school $v$</td>
<td>$A^k_v(t) = A^k_v(t - 1) \cup {i'}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$A^k_s(t) = A^k_s(t - 1)$ for all $s \neq s$</td>
</tr>
</tbody>
</table>

where either $v \neq y$ or $i'$ is not of type $\phi$.

Now, by the Scenario Lemma of Dubins and Freedman (1981), this last step of $\mathcal{R}^k(\hat{S}_i)$ must also occur in $\mathcal{R}^k(S_i)$. But, this means that this is the last step of $\mathcal{R}^k(S_i)$, contradicting the above.

### D Simulation appendix

#### Constructing $\eta$ and $\rho$

In this section, we describe in detail how we choose $\eta$ and $\rho$ to run the simulations in Section 6. Essentially, at each stage, we randomly choose one school-type pair $(s, \theta)$ for which the ceiling/capacity will be lowered, subject to feasibility constraints. More specifically, to construct $\eta$, we start by setting $(U^K, Q^K) = (\bar{U}, \bar{Q})$. Then, we randomly choose some pair $(s, \theta)$ such that $U^K_{s, \theta} < U^K_{s', \theta}$ and $Q^K_s < Q^K_{s'}$ and set $U^K_{s, \theta} = U^K_{s, \theta} + 1$ and $Q^K_{s'} = Q^K_{s'} = Q^K + 1$. For the remaining $(s', \theta')$, $U^K_{s', \theta'} = U^K_{s', \theta'}$ and $Q^K_{s'} = Q^K_{s'}$. For $(U^{K-2}, Q^{K-2})$, we again randomly select another school-type pair different from $(s, \theta)$, and raise its ceiling and capacity by one (again, subject to the constraint that doing so does not violate the true ceilings and capacities).

We continue in this manner until it is impossible to raise capacities any further without violating the true $(U, Q)$. This produces a sequence $\eta = \{(U^1, Q^1), \ldots, (U^K, Q^K)\}$ which can then be used to run DQDA. We construct $\eta$ “backwards”, starting from $(U^K, Q^K)$, only to simplify the coding. It can be done “forwards” as well.

To construct $\rho$ for use in EDQDA, we start with the $\eta$ just constructed (we begin with this $\eta$ in order to make a fair comparison with DQDA). Starting from where we left off in the previous paragraph, we
choose a pair \((s, \theta)\) randomly from those that have not yet reached their true type-specific ceiling, and raise this type-specific ceiling by 1 (note that just the ceilings are considered, not the capacities, since they are fixed in EDQDA). We continue to do so until we can no longer raise any \((s, \theta)\) ceiling further without violating the true \(U_{s, \rho}\). We then convert this sequence of ceiling-capacity vectors into the corresponding sequence of school-type pairs \(\rho\), as described in Section 6, which is then used to run EDQDA.
Additional simulation results

Below we present simulation results for additional parameter values. The first column corresponds to the low flexibility case, while the second corresponds to the high flexibility case. Each row corresponds to different values of the correlation parameter $\alpha$. Note that for image clarity, the vertical axes differ across figures.
Low flexibility

High flexibility

$\alpha = 0.195$

$\alpha = 0.26$

$\alpha = 0.325$
Low flexibility

Rank distribution for low flexibility and $\alpha = 0.39$

High flexibility

Rank distribution for high flexibility and $\alpha = 0.39$

$\alpha = 0.39$

Rank distribution for low flexibility and $\alpha = 0.455$

Rank distribution for high flexibility and $\alpha = 0.455$

$\alpha = 0.455$

Rank distribution for low flexibility and $\alpha = 0.52$

Rank distribution for high flexibility and $\alpha = 0.52$

$\alpha = 0.52$