Abstract

We propose a notion of selecting rationalizable actions by perturbing players’ higher-order beliefs, which we call robust selection. Like WY selection (Weinstein and Yildiz (2007)), robust selection generalizes the idea behind the equilibrium selection in the email game (Rubinstein (1989)) and the global game (Carlsson and Van Damme (1993)). In contrast to WY selection, robust selection is robust to misspecifications of best replies and to small measurement errors in payoffs, just like the selection in the email game and the global game. Robust selection is a strong notion in the sense that, among types with multiple rationalizable actions, almost all WY selections are not robust; but it is also a weak notion in the sense that any strictly rationalizable action can be robustly selected. We show that robust selection is fully characterized by the curb collection, a notion that generalizes the curb set in Basu and Weibull (1991). Moreover, we also use the curb collection to characterize critical types (Ely and Pêski (2011)) in any fixed finite game.
1 Introduction

One challenge that game theory faces is the prevalence of multiple equilibria, which substantially limits the theory’s predictive power. Two seminal papers introduced the idea of refining predictions by perturbing players’ higher-order beliefs: Rubinstein (1989) on the e-mail game and Carlsson and Van Damme (1993) on the global game. Both papers demonstrate that in a complete-information coordination game with two strict equilibria, we can make one of the two equilibria uniquely rationalizable by perturbing players’ higher-order beliefs.\(^1\)

Weinstein and Yildiz (2007) (hereafter, WY) substantially generalize this observation and prove the following surprising result: under a richness assumption on payoffs, every rationalizable action can be selected as the unique rationalizable action for any (Harsanyi) type by perturbing players’ higher-order beliefs. WY conclude from their result that no refinement of rationalizability retains its validity when we have only partial knowledge of the players’ incomplete information.\(^2\) More precisely, any rationalizable action (which may be ruled out by some refinement) could be the unique prediction, provided that the implicit belief assumptions of a model are relaxed in the “right” way. Consequently, the multiplicity of equilibria is an artifact of the modeling assumptions.

In this paper, we identify an essential difference between the selection notions adopted in WY (hereafter, WY selection) and in Carlsson and Van Damme (1993) (hereafter, global-game selection). Specifically, the global-game selection exhibits two robustness features that are not shared by all WY selections, namely, the robustness to slight misspecifications of best replies, as well as the robustness to small measurement errors in payoffs. We generalize the global-game selection by proposing a new selection notion that we call robust selection. We say that an action can be robustly selected for a type if we can select the action as the unique \(\epsilon\)-rationalizable action by perturbing the higher-order beliefs of the type. Robust selection strengthens WY selection in requiring that \(\epsilon\) be positive and

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\(^1\)The idea has inspired a variety of applied works using robustness to incomplete information as a criterion for equilibrium selection. See, for instance, Morris and Shin (1998), Chamley (1999), Frankel, Morris, and Pauzner (2003), and Goldstein and Pauzner (2004). See also Morris and Shin (2003) for an extensive literature review.

fixed along the perturbations. With the strengthening, we argue that every robust selection exhibits the two robustness features in the global-game selection (Definition 3 and Proposition 1).³

The two robustness features are important because every game in economic models is at best an approximation of reality. Like the precise specification of players’ higher-order beliefs, the exact specification of best replies and the precise measurement of payoffs are postulated only for analytical tractability. It would be unrealistic to assume that players can keep track precisely of all the payoffs or figure out their exact best replies, especially when the game is complicated.⁴,⁵

Based on the notion of robust selection, our results differ substantially from those in WY. First, contrary to their result that any rationalizable action of any type can be WY-selected, we prove that robust selection is generically impossible among types with multiple rationalizable actions (Theorem 1). Furthermore, an action can be robustly selected if it is strictly rationalizable, and only if it is a strict best reply (Proposition 2). Finally, we prove that robust selection is fully characterized by a novel solution concept that we call the curb collection (Theorem 2), which generalizes the curb set defined in Basu and Weibull (1991).

Our results convey two economic messages: one is aligned with the message in WY and the other goes in the opposite direction. First, like WY, we show that it is problematic to refine predictions by perturbing players’ higher-order beliefs, but our reason is different from that of WY. WY prove that any action can be WY-selected; hence, there are too many refinements and selecting any of them seems ad hoc. In contrast, we show here that, generically, no refinement through robust selection exists; hence, refining predictions by perturbing higher-order beliefs has a limited scope.

Second, WY show that any rationalizable action can be WY-selected, and hence, the global-game selection is not different from selecting any other rationalizable action. In contrast, our results demonstrate that the actions selected in global games exhibit some

³In Section 2, we use examples to illustrate robust selection and the two robustness features.
⁴Measurement errors are bound to occur because of (i) the limited cognitive ability of human beings and (ii) our current technology. For instance, a precise measurement of a real number could conceivably be used to store an infinite amount of digital data; however, because such a storage technology is not yet available, we have to rely on traditional storage devices that have finite limits.
⁵Recently, Levine and Zheng (2010) make the forceful statement that “the only meaningful theory of Nash equilibrium is Radner’s notion of ε-equilibrium.”
desirable features that are not shared by arbitrary rationalizable actions. In particular, our Proposition 2 implies that a weakly dominated action can never be robustly selected, despite being rationalizable. For instance, in a second-price auction with discrete bids, although any bid can be WY-selected, bidding the true value is the unique prediction that can be robustly selected.\(^6\)

The remainder of the paper is organized as follows. Section 2 presents examples to illustrate robust selection and the two robustness features. Section 3 defines preliminaries. Section 4 proves the generic impossibility of robust selection. Section 5 fully characterizes robust selection. Section 6 discusses the implications of our results and related issues. The appendix contains all the proofs omitted from the main text.

## 2 The two robustness features of robust selection

In this section, we illustrate robust selection as well as its two robustness features. Like WY, we use (Harsanyi) types to formulate incomplete-information scenarios. Recall that each type of a player in a type space specifies a belief (possibly, but not necessarily, derived from a common prior) about the payoff-relevant states and the opponents’ types. The belief of a player’s type thus encodes the player’s belief about the payoff-relevant states (i.e., the first-order belief), the player’s belief about the other players’ beliefs about the payoff-relevant states (i.e., the second-order belief), and so on.

To capture the two robustness features, we adopt the notion of \(\varepsilon\)-best replies in defining robust selection.\(^7\) Specifically, for any \(\varepsilon \geq 0\), we say an action is \(\varepsilon\)-rationalizable for a type \(t\) if it survives the iterated deletion of actions which are never \(\varepsilon\)-best replies.\(^8\) An

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\(^6\)Consider a concrete second-price auction, where the values are uniformly distributed on the set \(\{0, 1, 2, ..., 10\}\), and bidders simultaneously bid integers between 0 and 10. First, any bid between 0 and 9 is a best reply if an opponent bids 10. Second, bidding 10 is a best reply if all opponents bid 0. As a result, any bid is rationalizable, and hence can be WY-selected. Furthermore, truthful bidding forms a strict equilibrium, and any other bid is weakly dominated by truthful bidding. Hence, by Proposition 2, bidding the true value is the unique prediction that can be robustly selected.


\(^8\)Bernheim (1984) and Pearce (1984) define (0-)rationalizable actions in complete-information games.
action $a$ is said to be *robustly selected* for type $t$ if there exist $\epsilon > 0$ and a sequence of types $\{t_m\}$ such that (i) the sequence $\{t_m\}$ (weak*-)-approximates $t$ up to any finite order; and (ii) $a$ is the unique $\epsilon$-rationalizable action for every $t_m$.

We interpret robust selection as follows. For tractability, we use a simple type $t$ to represent a player’s higher-order beliefs. The type $t$ lives in a simple type space, e.g., a type space with common knowledge of payoffs, a finite type space, or a type space with a common prior. Because of the implicit assumptions associated with the simple type space (e.g., the common knowledge of payoffs or the common prior), $t$ has multiple rationalizable actions, including action $a$. However, we admit that $t$ is merely an idealization of the reality denoted by $t'$, which is a slightly perturbed version of $t$. We study whether a rationalizable action of $t$ can be made a uniquely rationalizable action for $t'$ which is one type in a sequence of types $\{t_m\}$ converging to $t$. Since $a$ is the unique $\epsilon$-rationalizable action for every $t_m$ (hence, also for $t'$), we say $a$ is robustly selected for $t$.

In contrast, an action $a$ is said to be *WY-selected* for type $t$, if there exists a sequence of types $\{t_m\}$ such that (i) the sequence $\{t_m\}$ (weak*-)-approximates $t$ up to any finite order; and (ii) $a$ is the unique 0-rationalizable action for every $t_m$. In other words, WY selection replaces $\epsilon > 0$ with $\epsilon = 0$ in the definition of robust selection. Our view is that, just as a simple type space is an idealization, the adoption of exact best replies (more precisely, 0-rationalizability) as a solution concept is also an idealization of practical decision-making. Because we share WY’s concern about treating these idealizations as modeling artifacts, we take our analysis one step further by considering the selection notion which does not rely on exact best replies.

We now illustrate the two robustness features. First, robust selection is robust to small misspecifications of best replies. This robustness feature is embedded in the definition of robust selection. To see the idea, consider the global-game selection:

**Example 1—The Global Game** There are two players whose payoffs depend on an un-
known parameter $\theta \in \mathbb{R}$, as summarized in the following matrix:

$$
\begin{array}{c|cc}
\text{Action} & \text{Attack} & \text{No Attack} \\
\hline
\text{Attack} & \theta, \theta & \theta - 1, 0 \\
\text{No Attack} & 0, \theta - 1 & 0, 0 \\
\end{array}
$$

Suppose that player $i$ receives a noisy signal $x_i \equiv \theta + \xi_i$, where $\xi_i$ is i.i.d. uniformly distributed on the interval $[-1/m, 1/m]$. Let $t_{i,m}$ denote the type of player $i$ who observes $x_i = \frac{3}{4}$, given $m$. The type $t_{i,\infty}$ has common knowledge of $\theta = \frac{3}{4}$ and it has two rationalizable actions, “Attack” and “No Attack.” When $m \to \infty$, type $t_{i,m}$ can be viewed as a small perturbation of the common knowledge scenario, and “Attack” is the unique rationalizable action for every $t_{i,m}$ with a finite $m$. In this sense, we say that “Attack” is selected for type $t_i$. Indeed, this global-game selection is robust to small misspecifications of best replies, i.e., for any $\epsilon' \in \left(0, \frac{1}{10}\right)$, “Attack” is the unique $\epsilon'$-rationalizable action for $t_{i,m}$ with $m$ sufficiently large. However, this may not be the case for the WY selection, as illustrated by the following example.

**Example 2—Tie-Breaking** There is only one player, who chooses between two actions $a$ and $b$. The player’s payoff depends on an unknown parameter $\theta \in \{\theta_0, \theta_a\}$ and on the action chosen, as specified below.$^{10}$

$$
\begin{array}{c|c|c}
\text{Action} & \text{Payoff} & \text{Action} \\
\hline
a & 0 & a & 0 \\
b & 0 & b & -1 \\
\end{array}
$$

In this single-agent decision problem, a type is simply the player’s belief about $\theta$. For $m = 1, 2, \ldots, \infty$, define type $t_m$ as

$$
t_m[\theta_0] = 1 - \frac{1}{m} \quad \text{and} \quad t_m[\theta_a] = \frac{1}{m},
$$

where $t_m[\theta]$ denotes the probability that type $t_m$ assigns to $\theta$. Clearly, actions $a$ and $b$ are both rationalizable for $t_\infty$, the sequence of types $\{t_m\}$ converges to $t_\infty$, and $a$ is uniquely rationalizable for every $t_m$. That is, $a$ is WY-selected for $t_\infty$. However, for arbitrarily small $\epsilon > 0$, action $b$ remains $\epsilon$-rationalizable for $t_m$ with sufficiently large $m$. That is, the WY selection of $a$ for $t_\infty$ is not robust to small misspecifications of best replies.$^{11}$

$^{10}$For simplicity, we consider here a single-agent game, but the idea can be easily extended to multi-player games.

$^{11}$The “richness” assumption on payoffs does not play a role in our example. We could add $\theta_b$ under which playing $b$ is strictly dominant so that “richness” is satisfied, and our example would remain valid.
Second, robust selection is also robust to small measurement errors in payoffs. To see this, for games $G$ and $G'$ with the same set of outcomes, we say that $G$ is a $\gamma$-approximation to $G'$ if the payoffs of any outcome in $G$ and $G'$ differ at most by $\gamma$. We view $\gamma$ as a measurement error. Due to the measurement error, a game in an economic model may only be a $\gamma$-approximation of the strategic situation being modeled. The modeler may improve the approximation by reducing $\gamma$, but a perfect modeling (where $\gamma = 0$) may never be feasible. By saying that a selection is robust to measurement errors, we mean that an action being selected for a type in a game $G$ continues to be selected for the type in any $\gamma$-approximation of $G$ with a small $\gamma > 0$. Again, the global-game selection is robust to measurement errors, while this may not be the case for some WY selections.

**Example 1** (continued) Suppose that we modify the payoffs in Example 1 as follows (where $\gamma_l \in \left(-\frac{1}{10}, \frac{1}{10}\right)$ and $l = 1, 2, \ldots, 8$):

<table>
<thead>
<tr>
<th>$G'_1$</th>
<th>Attack</th>
<th>No Attack</th>
</tr>
</thead>
<tbody>
<tr>
<td>Attack</td>
<td>$\theta + \gamma_1, \theta + \gamma_2, \theta - 1 + \gamma_3, \gamma_4$</td>
<td></td>
</tr>
<tr>
<td>No Attack</td>
<td>$\gamma_5, \theta - 1 + \gamma_6, \gamma_7, \gamma_8$</td>
<td></td>
</tr>
</tbody>
</table>

Both “Attack” and “No Attack” are still rationalizable for $t_{i,\infty}$, and “Attack” is the unique rationalizable action for $t_{i,m}$ with $m$ sufficiently large. That is, (robustly) selecting “Attack” is robust to measurement errors represented by $(\gamma_l)_{l=1}^8$.

In Proposition 1, we generalize the observation in Example 1 to prove that every robust selection is robust to small measurement errors in payoffs.

**Example 2** (continued) Suppose that we modify the payoffs in Example 2 as follows (where $\gamma_9 > 0$):

<table>
<thead>
<tr>
<th>$G'_2$</th>
<th>action</th>
<th>payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$b$</td>
<td>$\gamma_9$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$G'_2$</th>
<th>action</th>
<th>payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$b$</td>
<td>$-1$</td>
<td></td>
</tr>
</tbody>
</table>

Clearly, $a$ cannot be WY-selected for $t_\infty$. That is, the WY selection of $a$ is not robust to the measurement error represented by $\gamma_9$. 

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3 Preliminaries

Throughout the paper, for any metrizable space $Y$, we use $\Delta(Y)$ to denote the space of all probability measures on the Borel $\sigma$-algebra of $Y$. We endow $\Delta(Y)$ with the weak* topology. Moreover, we endow a product space with the product topology, a subspace with the relative topology, and a finite set with the discrete topology. Let $|E|$ denote the cardinality of a finite set $E$.

3.1 The model of incomplete information

Fix a finite set of players $N$ and a finite set of payoff-relevant parameters $\Theta$. By a model, we mean a pair $(T, \kappa)$, where $T = \prod_{i \in N} T_i$ is a compact metric space. Each $t_i \in T_i$ is called a type of player $i$ and it is associated with a belief $\kappa_{t_i} \in \Delta(\Theta \times T_{-i})$. Assume that $t_i \mapsto \kappa_{t_i}$ is a continuous mapping.

Given a type $t_i$ in a model $(T, \kappa)$, we can compute the first-order belief of $t_i$ (i.e., his belief about $\Theta$) by setting $t_i^2$ equal to the marginal distribution of $\kappa_{t_i}$ on $\Theta$. We can also compute the second-order belief of $t_i$ (i.e., his belief about $(\theta, t_{-i}^1)$) by setting

$$t_i^2[F] = \kappa_{t_i} \left[ \left\{ (\theta, t_{-i}) : (\theta, t_{-i}^1) \in F \right\} \right], \forall F \subset \Theta \times (\Delta(\Theta))^{|N|-1}.$$ 

We can compute the entire hierarchy of beliefs $(t_i^1, t_i^2, \ldots, t_i^k, \ldots)$ by proceeding in this way.

We will work with the universal type space $T_i^*$ constructed in Mertens and Zamir (1985) which contains all such hierarchies. That is, for any type $t_i$ in any model, there is some $t_i' \in T_i^*$ such that $t_i^k = t_i'^k$ for every $k$. We endow $T_i^*$ with the product topology so that a sequence of types $\{t_{i,m}\}$ converges to a type $t_i$ (denoted as $t_{i,m} \rightarrow t_i$) if and only if, for every $k$, $t_{i,m}^k \rightarrow t_i^k$ in the weak*-topology. Moreover, we will continue to write $t_{i,m} \rightarrow t_i$ whenever $t_{i,m}^k \rightarrow t_i^k$ for every $k$, even if $t_i$ and $t_{i,m}$ live in some models different from $T_i^*$. Mertens and Zamir (1985) show that $T_i^*$ is a compact metric space and homeomorphic to $\Delta(\Theta \times T_{-i}^*)$. We use $\kappa_i^*$ to denote the homeomorphism. Then, $(T^*, \kappa^*)$ becomes a model, where $\kappa_i^* \equiv \kappa_i^*(t_i)$ for every $t_i \in T_i^*$.

A type $t_i \in T_i^*$ is said to be a complete-information type if there exist some $\theta_0 \in \Theta$ and some $t_{-i} \in T_{-i}^*$ such that $\kappa_i^*[\theta_0, t_{-j}] = 1$ for every $j \in N$. That is, a complete-
information type has common knowledge of \( \theta = \theta_0 \). We use \( t_i^{\theta_0} \) to denote the complete-information type for which \( \theta = \theta_0 \) is common knowledge.

### 3.2 The game and solution concept

Each player \( i \) has a finite set of actions \( A_i \) with \( |A_i| \geq 2 \), which we fix throughout the paper. Let \( A = \prod_{i \in N} A_i \) denote the set of action profiles. A game is a tuple \( G = (u_i)_{i \in N} \), where \( u_i : \Theta \times A \to \mathbb{R} \) is player \( i \)'s payoff function. WY’s richness assumption can be stated as follows.

**Definition 1** A game \( G = (u_i)_{i \in N} \) satisfies the richness assumption if for every \( i \in N \) and every \( a_i \in A_i \), there exists \( \theta^{a_i} \in \Theta \) such that \( u_i (\theta^{a_i}, a_i, a_{-i}) > u_i (\theta^{a_i}, a'_i, a_{-i}), \forall a'_i \neq a_i, \forall a_{-i} \).

Following WY, we adopt the solution concept of interim correlated rationalizability (ICR) proposed in Dekel, Fudenberg, and Morris (2006, 2007). For any \( \pi \in \Delta (\Theta \times A_{-i}) \) and any \( \varepsilon \in \mathbb{R} \), we use \( \text{BR}_i (\pi, G, \varepsilon) \) to denote the set of \( \varepsilon \)-best replies to \( \pi \) in game \( G = (u_i)_{i \in N} \). That is,

\[
\text{BR}_i (\pi, G, \varepsilon) = \left\{ a_i \in A_i : \sum_{\theta, a_{-i}} [u_i (\theta, a_i, a_{-i}) - u_i (\theta, a'_i, a_{-i})] \pi [\theta, a_{-i}] \geq -\varepsilon, \forall a'_i \neq a_i \right\}.
\]

In a model \( (T, \kappa) \), a conjecture of player \( i \) is a measurable function \( \sigma_{-i} : \Theta \times T_{-i} \to \Delta (A_{-i}) \). Based on \( \sigma_{-i} \), a type \( t_i \) forms a distribution on \( \Theta \times A_{-i} \), denoted by \( \pi_{t_i, \sigma_{-i}} \), where

\[
\pi_{t_i, \sigma_{-i}} [\theta, a_{-i}] = \int_{T_{-i}} \sigma_{-i} (\theta, t_{-i}) [a_{-i}] \kappa_{t_i} [\theta, dt_{-i}], \forall (\theta, a_{-i}) \in \Theta \times A_{-i}.
\]

Given any model \( (T, \kappa) \) and any \( \varepsilon \in \mathbb{R} \), the \( \varepsilon \)-ICR actions of type \( t_i \) in game \( G \), denoted by \( S_i^\infty [t_i, G, \varepsilon] \), is defined as

\[
S_i^\infty [t_i, G, \varepsilon] = \bigcap_{k=0}^{\infty} S_i^k [t_i, G, \varepsilon],
\]

where \( S_i^0 [t_i, G, \varepsilon] = A_i \), and inductively, for each integer \( k \geq 1 \), \( a_i \in S_i^k [t_i, G, \varepsilon] \) if and only if there is a measurable function \( \sigma_{-i} : \Theta \times T_{-i} \to \Delta (A_{-i}) \) such that

(a) \( \sigma_{-i} (\theta, t_{-i}) [a_{-i}] > 0 \Rightarrow a_j \in S_j^{k-1} [t_j, G, \varepsilon], \forall j \neq i; \)
(b) \( a_i \in \text{BR}_i (\pi_{t_i, \sigma_{-i}}, G, \epsilon) \).

For \( t_{-i} = (t_j)_{j \neq i} \) we write \( S_{-i}^\infty [t_{-i}, G, \epsilon] \) for \( \prod_{j \neq i} S_j^\infty [t_j, G, \epsilon] \). We say that \( \sigma_{-i} \) is an \( \epsilon \)-valid conjecture if \( \sigma_{-i} (\theta, t_{-i}) [S_{-i}^\infty [t_{-i}, G, \epsilon]] = 1 \). Dekel, Fudenberg, and Morris (2006, 2007) prove the following properties for any \( \epsilon \)-ICR with \( \epsilon \geq 0 \), which we will use later in the paper: (i) \( S_i^\infty [t_i, G, \epsilon] \) is nonempty for any \( t_i \); (ii) \( S_i^\infty [t_i, G, \epsilon] \subset S_i^\infty [t_i, G, \epsilon'] \) if \( \epsilon \leq \epsilon' \); (iii) \( a_i \in S_i^\infty [t_i, G, \epsilon] \) if and only if \( a_i \in \text{BR}_i (\pi_{t_i, \sigma_{-i}}, G, \epsilon) \) for some \( \epsilon \)-valid conjecture \( \sigma_{-i} \); (iv) the set of \( \epsilon \)-ICR actions of a type is fully determined by its belief hierarchy. By (iv), we will identify a type with its belief hierarchy.

### 3.3 WY selection and robust selection

We now define WY selection and robust selection as follows.

**Definition 2** Given any model \((T, \kappa)\), an action \( a_i \) can be WY-selected for \( t_i \in T_i \) in \( G \) if there exists a sequence of types \( \{t_{i,m}\} \) on \( T_i^* \) such that \( t_{i,m} \to t_i \) and \( S_i^\infty [t_{i,m}, G, 0] = \{a_i\} \) for every \( m \).

WY show that every rationalizable action can be WY-selected. Conversely, by the upper hemicontinuity of the rationalizable correspondence (see (Dekel, Fudenberg, and Morris, 2006, Theorem 2)), every action that is WY-selected must also be rationalizable. That is, an action is WY-selected if and only if it is rationalizable.

**Definition 3** Given any model \((T, \kappa)\), an action \( a_i \) can be robustly selected for \( t_i \in T_i \) in \( G \) if there exist \( \epsilon > 0 \) and a sequence of types \( \{t_{i,m}\} \) on \( T_i^* \) such that \( t_{i,m} \to t_i \) and \( S_i^\infty [t_{i,m}, G, \epsilon] = \{a_i\} \) for every \( m \).

Whether an action can be WY-/robustly selected or not for a type depends only on its belief hierarchy. Note that in Definition 3, we take \( \epsilon > 0 \) uniformly along the sequence of types; if instead we required \( S_i^\infty [t_{i,m}, G, \epsilon_m] = \{a_i\} \) for some \( \epsilon_m > 0 \), this notion would be equivalent to WY selection. Consequently, an action that is robustly selected is also WY-selected and thus rationalizable. However, we will show in Section 4 that robust selection is generically impossible among types with multiple rationalizable actions.
We say type \( t_i \) admits a robust selection (resp. WY selection) if there is some action which can be robustly selected (resp. WY-selected) for \( t_i \). The following proposition shows that robust selection is robust to measurement errors. Recall from Example 2 that WY selection does not satisfy this robustness property.

**Proposition 1** If \( a_i \) can be robustly selected for \( t_i \) in \( G = (u_j)_{j \in N} \) then for some \( \gamma > 0 \), \( a_i \) can also be robustly selected for \( t_i \) in any \( G' = (u'_j)_{j \in N} \) such that

\[
|u'_j(\theta, a) - u_j(\theta, a)| \leq \gamma, \forall (j, \theta, a) \in N \times \Theta \times A.
\]

### 4 Generic impossibility

Hereafter, we fix a game \( G = (u_i)_{i \in N} \) which satisfies the richness assumption. We then simplify the notation by writing \( S^\infty_i [t_i, \varepsilon] \) for \( S^\infty_i [t_i, G, \varepsilon] \). Furthermore, if \( \varepsilon = 0 \), we write \( S^\infty_i [t_i] \) for \( S^\infty_i [t_i, 0] \). Similarly, all notations without a reference to \( \varepsilon \) should be understood as an implicit reference to \( \varepsilon = 0 \).

In this section, we demonstrate that robust selection is generically impossible among types with multiple rationalizable actions. To achieve this, we first follow WY in partitioning the universal type space into two parts: the set of types with multiple rationalizable actions, denoted by \( M_i \), and the set of types with a unique rationalizable action, denoted by \( U_i \).

\[
M_i = \{ t_i \in T^*_i : |S^\infty_i [t_i]| > 1 \} \quad \text{and} \quad U_i = \{ t_i \in T^*_i : |S^\infty_i [t_i]| = 1 \}.
\]

Note that \( M_i \neq \emptyset \) because of \( |A_i| \geq 2 \) and the richness assumption. Here we focus on \( M_i \) because there is no need to refine the prediction for types with a unique rationalizable action.\(^{12} \) WY prove that every type in \( M_i \) admits a WY selection. Here we show that except for a meager set, types in \( M_i \) do not admit any robust selection.

A **meager** set is a countable union of nowhere dense sets and the complement of a meager set is called a **residual** set. That is, a residual set contains a countable intersection of open and dense sets. A topological space is called a Baire space if every intersection of

\(^{12} \text{In fact, by (Dekel, Fudenberg, and Morris, 2006, Lemma 1 and Theorem 2), if } S^\infty_i [t_i] = \{a_i\}, \text{ then there is some } \varepsilon > 0 \text{ such that for any sequence of types } \{t_{i,m}\} \text{ converging to } t_i, S^\infty_i [t_{i,m}, \varepsilon] = \{a_i\} \text{ for all sufficiently large } m, \text{ i.e., } a_i \text{ is robustly selected for } t_i. \)
countably many open and dense sets is dense. In a Baire space, residual sets and meager sets are the usual notions of genericity and non-genericity, respectively. Specifically, a residual set in a Baire space is dense and not meager. In our case, since $M_i$ is a closed subset of the compact metric space $T_i^*$ (see Proposition 2 in WY), $M_i$ is a Baire space (see (Aliprantis and Border, 2006, 3.47 Theorem)). Theorem 1 thus implies that for “almost all” types in $M_i$, WY selections are not robust.

**Theorem 1** There is a residual set $B_i \subset M_i$ such that types in $B_i$ do not admit any robust selection.

To prove Theorem 1, consider the following sets.

$$B_i \equiv \left\{ t_i \in M_i : \kappa_{i,i}^i [\Theta \times U_{-i}] = 1 \right\}$$
\hspace{10cm} (1)

$$B_{i,n} \equiv \left\{ t_i \in M_i : \kappa_{i,i}^i [\Theta \times U_{-i}] > 1 - \frac{1}{n} \right\}, \forall n \in \mathbb{N}.$$

Theorem 1 is then a direct consequence of the following three lemmas. We will later prove a stronger result, Proposition 9 in Section 6.3, which implies Lemma 3. The proofs of Lemmas 1 and 2 can be found in the Appendix.

**Lemma 1** $B_{i,n}$ is open in $M_i$.

**Lemma 2** $B_i$ is dense in $M_i$.

**Lemma 3** No type in $B_i$ admits a robust selection.

Since $B_i \subset B_{i,n}$, by Lemmas 1 and 2, $B_{i,n}$ is open and dense in $M_i$. Moreover, since $\bigcap_{n=1}^{\infty} B_{i,n} = B_i$, it follows that $B_i$ is a residual set in $M_i$. Theorem 1 thus follows from Lemma 3.

Lemma 1 is a technical result. The intuition of Lemma 3 is similar to the idea of Example 2 in Section 2. More precisely, facing opponents with a unique rationalizable action, every type in $B_i$ has a unique valid conjecture about their opponents’ rationalizable
actions. Thus, a type in \( B_i \) has multiple rationalizable actions only because of payoff ties, as in Example 2. As a result, these types do not admit any robust selection.\(^\text{13}\)

To illustrate Lemma 2, we revisit \( G_1 \) in Example 1 with \( \theta \in \{-2/5, 2/5, 6/5\}.\(^\text{14}\)

\[
G_1:
\begin{array}{|c|c|c|}
\hline
& \text{Attack (A)} & \text{No Attack (NA)} \\
\hline
\text{Attack (A)} & \theta, \theta & \theta - 1, 0 \\
\hline
\text{No Attack (NA)} & 0, \theta - 1 & 0, 0 \\
\hline
\end{array}
\]

Recall that we use \( t^\theta_j \) to denote the complete-information type for which \( \theta \) is common knowledge. First, observe that both actions are rationalizable for \( t^{2/5}_i \). In their Example 3, WY recap the idea of Rubinstein (1989) to show that we can select either “Attack” or “No Attack” for \( t^{2/5}_i \). These are in fact robust selections.

We show below that there exists a sequence of types \( \{t_{i,k}\} \) such that \( t_{i,k} \rightarrow t^{2/5}_i \) and \( t_{i,k} \in B_i \) for every \( k \). To achieve this, we define six sequences of types, \( \{t_{i,k}\}, \{t^A_{i,k}\}, \{t^A_{i,k}\} \) (with \( i = 1, 2 \)), inductively. For \( i = 1, 2 \),

\[
\kappa^*_{t_{i,1}} \left[ \theta = 2/5, t^{-2/5}_{-i} \right] = 2/5 \text{ and } \kappa^*_{t_{i,1}} \left[ \theta = 2/5, t^{6/5}_{-i} \right] = 3/5;
\]

\[
\kappa^*_{t_{i,1}} \left[ \theta = 2/5, t^{6/5}_{-i} \right] = 1;
\]

\[
\kappa^*_{t^A_{i,1}} \left[ \theta = 2/5, t^{-2/5}_{-i} \right] = 1.
\]

For \( k \geq 2 \) and \( i \in \{1, 2\} \),

\[
\kappa^*_{t_{i,k}} \left[ \theta = 2/5, t^{NA}_{-i,k-1} \right] = 2/5 \text{ and } \kappa^*_{t_{i,k}} \left[ \theta = 2/5, t^A_{-i,k-1} \right] = 3/5;
\]

\[
\kappa^*_{t^A_{i,k}} \left[ \theta = 2/5, t^A_{-i,k-1} \right] = 1;
\]

\[
\kappa^*_{t^A_{i,k}} \left[ \theta = 2/5, t^{NA}_{-i,k-1} \right] = 1.
\]

It is easy to draw the following two observations.

1. \( t^{2/5}_i, t_{i,k}, t^A_{i,k}, \) and \( t^{NA}_{i,k} \) share the same \( k \)-th order belief for every \( (i,k) \);

\(^\text{13}\)We note that having a payoff tie is not the only way not to have a robust selection. More generally, Proposition 9 in Subsection 6.3 shows that types of player \( i \) who assign probability one to \( B_{-i} \) (and who may not have a payoff tie) also admit no robust selection.

\(^\text{14}\)This is Example 3 in WY which is modified from Rubinstein (1989) and Carlsson and Van Damme (1993).
2. \( S^\infty_i \left[ t_{i,k}^A \right] = \{ A \} \), \( S^\infty_i \left[ t_{i,k}^{NA} \right] = \{ NA \} \), and \( S^\infty_i [t_{i,k}] = \{ A, NA \} \) for every \((i, k)\).

As a result, \( \{ t_{i,k} \} \to t_i^{2/5} \) and \( t_{i,k} \in B_i \) for every \( k \).

It is worth noting that the multiplicity of rationalizable actions of \( t_i^{2/5} \) and \( t_{i,k} \) occurs for different reasons: for \( t_i^{2/5} \), there is a coordination problem because his opponent also has two rationalizable actions; for \( t_{i,k} \), all types of his opponents have a unique rationalizable action and multiplicity occurs simply due to a payoff tie. Nonetheless, \( t_i^{2/5} \) and \( t_{i,k} \) are indistinguishable for a modeler who knows the players’ beliefs only up to order \( k \).

5 Characterization

In light of the negative result that “almost all” WY selections are not robust, we may wonder when we can obtain a robust selection. In this section, we provide a full characterization of robust selection. We first draw connections between robust selection and the notions of strict best replies and strict rationalizability. We then propose a notion called a curb collection and prove that this notion fully characterizes robust selection.

5.1 Strict rationalizability

We first define a strict best reply and a strictly rationalizable action as follows. For complete-information games, our definition of strictly rationalizability is equivalent to the standard one (i.e., iterated deletion of never-strict best replies).

**Definition 4** Given a model \((T, \kappa)\) and \( t_i \in T_i \), an action \( a_i \) is a strict best reply for a type \( t_i \) if \( a_i \in S^1_i [t_i, -\epsilon] \) for some \( \epsilon > 0 \); an action \( a_i \) is strictly rationalizable for a type \( t_i \) if \( a_i \in S^\infty_i [t_i, -\epsilon] \) for some \( \epsilon > 0 \).

We will prove the following result in the Appendix as an immediate consequence of our main characterization of robust selection.
Proposition 2 Given a model \((T, \kappa)\) and \(t_i \in T_i\), (i) an action \(a_i\) can be robustly selected for \(t_i\) if \(a_i\) is strictly rationalizable for \(t_i\); (ii) an action \(a_i\) can be robustly selected for \(t_i\) only if \(a_i\) is a strict best reply for \(t_i\).

However, there are actions which can be robustly selected without being strictly rationalizable, as shown in the following example.\(^{15}\)

Example 3 Consider a two-player game:

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D</th>
<th>D'</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1,1</td>
<td>0,0</td>
<td>0,0</td>
</tr>
<tr>
<td>B</td>
<td>0,0</td>
<td>1,1</td>
<td>1,1</td>
</tr>
</tbody>
</table>

\(\theta = \theta_0\)

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D</th>
<th>D'</th>
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</thead>
<tbody>
<tr>
<td>A</td>
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</tr>
<tr>
<td>B</td>
<td>0,0</td>
<td>0,1</td>
<td>0,0</td>
</tr>
</tbody>
</table>

\(\theta = \theta_D\)

When it is common knowledge between the two players that \(\theta = \theta_0\), neither \(D\) nor \(D'\) is strictly rationalizable for player 2, and hence \(B\) is not strictly rationalizable for player 1. Nevertheless, \(B\) can be robustly selected for \(t_1^{\theta_0}\). To see this, define a sequence of types \(\{t_m\}\) as follows, with \(t_m\) being player 1’s (resp. player 2’s) type if \(m\) is odd (resp. even).

\(t_0 \equiv t_2^{\theta_D}\) and \(\kappa_{t_m}^\bullet[\theta_0, t_{m-1}] \equiv 1, \forall m \geq 1.\)

Clearly, \(t_{2m-1} \rightarrow t_1^{\theta_0}\) and \(t_{2m} \rightarrow t_2^{\theta_0}\). Moreover, \(S_1^\circ [t_{2m-1}, 1/2] = \{B\}\) and \(S_2^\circ [t_{2m}, 1/2] = \{D,D'\}\) for any positive integer \(m\). Therefore, \(B\) is robustly selected for \(t_1^{\theta_0}\).

5.2 Characterization: complete-information models

Our characterization of robust selection is based on a notion that we call a curb collection. Since this notion requires mathematical technicalities in general models, we first present the definition for complete-information models. Fix a state \(\theta_0 \in \Theta\) and focus on the complete-information type \(t_i^{\theta_0}\).

Let \(A_i\) be the collection of all nonempty subsets of \(A_i\), i.e., \(A_i = 2^{A_i} \setminus \{\emptyset\}\). For each profile \((R_j)_{j \neq i}\) with \(R_j \subset A_j\), we denote by \(R_{-i} \equiv \{R_{-i} = \prod_{j \neq i} R_j : R_j \in R_j\}\) the collection of all product subsets of \(A_{-i}\) generated by \((R_j)_{j \neq i}\).

---

\(^{15}\)In Subsection 6.1, we show that in a complete-information game with generic payoffs, an action can be robustly selected if and only if it is strictly rationalizable.
To motivate the notion of curb collections, consider the collection of sets of \( \epsilon \)-ICR actions for types close to the complete-information type \( t_{i_0}^{\theta_0} \):

\[
\mathcal{R}_i^{\theta_0} \equiv \left\{ R_i \in \mathcal{A}_i : \exists \epsilon > 0, \exists \text{a sequence} \{ t_{i,m} \} \text{ on } T_i^* \text{ s.t. } t_{i,m} \rightarrow t_i^{\theta_0} \text{ and } R_i = S_i^\infty [t_{i,m}, \epsilon], \forall m \right\}.
\]

Clearly, \( a_i \) can be robustly selected for \( t_i^{\theta_0} \) if and only if \( \{ a_i \} \in \mathcal{R}_i^{\theta_0} \). Observe that the profile of collections \( \left( \mathcal{R}_i^{\theta_0} \right)_{i \in N} \) exhibits the following property. For each \( R_i \in \mathcal{R}_i^{\theta_0} \), let \( t_{R_i} \) be a type close to \( t_i^{\theta_0} \) such that \( R_i = S_i^\infty [t_{R_i}, \epsilon] \). Since \( t_{R_i} \) believes with high probability that \( \theta = \theta_0 \) and ICR sets for his opponents belong to \( \mathcal{R}_i^{\theta_0} \), we can define a belief \( \mu_{R_i} \in \Delta(\mathcal{R}_i^{\theta_0}) \) as his (approximate) belief over opponents’ ICR sets. By the definition of \( \epsilon \)-ICR, \( a_i \in R_i \) if \( a_i \) is a best reply to a conjecture (approximately) consistent with \( \mu_{R_i} \).

Our notion of curb collection is to abstract the above property from \( \mathcal{R}_i^{\theta_0} \). Formally, a function \( \varphi_{-i} : \mathcal{A}_{-i} \rightarrow \Delta(\mathcal{A}_{-i}) \) is a conditional conjecture if

\[
\varphi_{-i}(R_{-i})[a_{-i}] > 0 \Rightarrow a_{-i} \in R_{-i}.
\]

Combining \( \mu \in \Delta(\mathcal{A}_{-i}) \) with \( \varphi_{-i} \), we define a distribution \( \pi_{\mu,\varphi_{-i}} \in \Delta(\Theta \times \mathcal{A}_{-i}) \) as

\[
\pi_{\mu,\varphi_{-i}}[\theta_0, a_{-i}] = \sum_{R_{-i} \in \mathcal{A}_{-i}} \mu[R_{-i}] \times \varphi_{-i}(R_{-i})[a_{-i}], \forall a_{-i} \in \mathcal{A}_{-i}.
\]

**Definition 5** A profile of nonempty collections \( (\mathcal{R}_i)_{i \in N} \subset \prod_{i \in N} \mathcal{A}_i \) is called a curb collection at state \( \theta_0 \) if for every \( i \in N \) and every \( R_i \in \mathcal{R}_i \), there exists \( \mu_{R_i} \in \Delta(\mathcal{A}_{-i}) \) such that

1. \( \mu_{R_i}[R_{-i}] > 0 \Rightarrow R_{-i} \in \mathcal{R}_{-i} \), and
2. \( R_i \supset B_{R_i}(\pi_{\mu_{R_i},\varphi_{-i}}) \) for any conditional conjecture \( \varphi_{-i} \).

A curb set as defined in Basu and Weibull (1991) can be identified with a singleton curb collection.\(^{16}\)

We can compute the largest (i.e., finest) curb collection iteratively by letting \( \mathcal{R}_i^0 = \mathcal{A}_i \), and for \( k = 0, 1, \ldots, \mathcal{R}_i^{k+1} \) be the collection of all \( R_i \in \mathcal{A}_i \) such that \( R_i \supseteq \bigcup_{\varphi_{-i}} B_{R_i}(\pi_{\mu,\varphi_{-i}}) \) for some \( \mu \in \Delta(\mathcal{R}_i^k) \). Thus, we have a weakly decreasing sequence \( \mathcal{R}_i^0 \supseteq \mathcal{R}_i^1 \supseteq \cdots \). By the finiteness of actions, there exists \( k^* \) such that \( \mathcal{R}_i^{k^*+1} = \mathcal{R}_i^{k^*} = \mathcal{R}_i^* \) for all \( i \in N \).

\(^{16}\)More precisely, \( (R_i)_{i \in N} \in \prod_{i \in N} \mathcal{A}_i \) is a curb set at \( \theta_0 \) if and only if \( (\mathcal{R}_i = \{ R_i \})_{i \in N} \) is a curb collection at \( \theta_0 \).
The collection \((\mathcal{R}_i^*)_{i \in N}\) thus obtained is the largest curb collection.\(^{17}\) For instance, in Example 3, consider the complete-information model defined by \(\theta = \theta_0\). Let \(\mathcal{R}_1^0 = \{\{A\}, \{B\}, \{A, B\}\}\) and \(\mathcal{R}_2^0 = \{\{C\}, \{D\}, \{D'\}, \ldots, \{C, D, D'\}\}\). Then the process stops in one step with \(\mathcal{R}_1^* \equiv \mathcal{R}_1^1 = \{\{A\}, \{B\}, \{A, B\}\}\) and \(\mathcal{R}_2^* \equiv \mathcal{R}_2^1 = \{\{C\}, \{D, D'\}, \{C, D, D'\}\}\).

The following result, which is a special case of Theorem 2, characterizes robust selection based on curb collections.

**Proposition 3** An action \(a_i\) can be robustly selected for complete-information type \(t_i^{\theta_0}\) if and only if \(\{a_i\} \in \mathcal{R}_i^*\) for the largest curb collection \((\mathcal{R}_i^*)_{i \in N}\) at state \(\theta_0\).

The “if” direction follows by showing that there exist \(\varepsilon > 0\) and sequences of types \(\{\tau_{R_i, m}\}\) for all \(R_i \in \mathcal{R}_i^*\) such that \(\tau_{R_i, m} \rightarrow t_i^{\theta_0}\) as \(m \rightarrow \infty\) and \(S_i^\varepsilon[\tau_{R_i, m}, \varepsilon] \subset R_i\) for all \(m\). The “only if” direction follows by showing that the collection of \(\varepsilon\)-ICR action sets \((\mathcal{R}_i^{\theta_0})_{i \in N}\) is indeed a curb collection. (Recall the argument before Definition 5.) In the next subsection, we formalize these directions for general models in Propositions 4 and 5, respectively.

### 5.3 Characterization: general models

We now extend the notion of curb collections to a general model \((T, \kappa)\). We say that a measurable function \(\varphi^{-i} : \Theta \times T^{-i} \times A^{-i} \rightarrow \Delta (A^{-i})\) is a conditional conjecture if

\[
\varphi^{-i}(\theta, t^{-i}, R^{-i})[a^{-i}] > 0 \Rightarrow a^{-i} \in R^{-i}.
\]

Given \(\mu \in \Delta (\Theta \times T^{-i} \times A^{-i})\) and \(\varphi^{-i}\), we define \(\pi_{\mu, \varphi^{-i}} \in \Delta (\Theta \times A^{-i})\) as

\[
\pi_{\mu, \varphi^{-i}}[\theta, a^{-i}] = \int_{T^{-i} \times A^{-i}} \varphi^{-i}(\theta, t^{-i}, R^{-i})[a^{-i}] \mu[\theta, dt^{-i}, dR^{-i}], \forall (\theta, a^{-i}) \in \Theta \times A^{-i}.
\]

**Definition 6** Given a model \((T, \kappa)\), a profile of measurable functions \((\mathcal{R}_i : T_i \rightarrow 2^{A_i} \setminus \{\emptyset\})_{i \in N}\) is called an \(\varepsilon\)-curb collection in \((T, \kappa)\) if for every \(i \in N\), there exists a measurable function

\[
T_i \times A_i \ni (t_i, R_i) \mapsto \mu_{t_i, R_i} \in \Delta (\Theta \times T^{-i} \times A^{-i})
\]

such that for each \(t_i \in T_i\) and \(R_i \in \mathcal{R}_i(t_i)\),

\(^{17}\)If we begin with the collection of all singleton sets \(\tilde{\mathcal{R}}_i^0 = \{\{a_i\} : a_i \in A_i\}\), then this process is equivalent to the iterated deletion of never strict best replies, and the resulting collection \(\tilde{\mathcal{R}}_i^*\) is equal to \(\{\{a_i\} : a_i\text{ is strictly rationalizable}\}\).
(a) the marginal distribution of $\mu_{t_i,R_i}$ on $\Theta \times T_{-i}$ is equal to $\kappa_{t_i}$,

(b) $\mu_{t_i,R_i} \left[ \{ (\theta, t_{-i}, R_{-i}) : R_{-i} \in R_{-i}(t_{-i}) \} \right] = 1$, and

(c) $R_i \supset BR_i(\pi_{\mu_{t_i,R_i},\varphi_{-i}}, \epsilon)$ for any conditional conjecture $\varphi_{-i}$.

The following result is our full characterization of robust selection.

**Theorem 2** Given a model $(T, \kappa)$, an action $a_i$ can be robustly selected for $t_i \in T_i$ if and only if there exists $\epsilon > 0$ such that $\{a_i\} \in R_i(t_i)$ for some $\epsilon$-curb collection $(R_j)_{j \in N}$ in $(T, \kappa)$.

Note that $\epsilon$-curb collections are defined within the model $(T, \kappa)$, and yet the notion of robust selection requires examining all types in the neighborhoods surrounding $(T, \kappa)$. Nonetheless, Theorem 2 shows that the singleton sets in $\epsilon$-curb collections in $(T, \kappa)$ pin down exactly the actions that can be robustly selected for any type in $(T, \kappa)$.

The “if” and “only if” directions of Theorem 2 are immediate consequences of Propositions 4 and 5, respectively. We prove these two propositions in the next two subsections.

**Proposition 4** Let $(R_j)_{j \in N}$ be an $\epsilon$-curb collection in a model $(T, \kappa)$ with $\epsilon > 0$. If $R_i \in R_i(t_i)$ for some $t_i \in T_i$, then there exist $\gamma > 0$ and a sequence of types $\{t_{i,m}\}$ on $T_i$ such that $t_{i,m} \rightarrow t_i$ and $S_i^\infty [t_{i,m}, \gamma] \subset R_i$ for every $m$.

**Proposition 5** Let $(T, \kappa)$ be a model and $\epsilon > 0$. Then, $(R_i^T : T_i \rightarrow 2^{A_i} \setminus \{\emptyset\})_{i \in N}$ defined below is an $\frac{\epsilon}{2}$-curb collection:

$$R_i^T (t_i) \equiv \{ R_i \in A_i : \exists a \text{ sequence } \{t_{i,m}\} \text{ on } T_i \text{ s.t. } t_{i,m} \rightarrow t_i \text{ and } R_i = S_i^\infty [t_{i,m}, \epsilon], \forall m \} .$$

These results generate interesting corollaries. First, in the Appendix, we show that Proposition 2 immediately follows from Theorem 2. Second, in Subsection 6.3, we employ Propositions 4 and 5 to fully characterize the notion of critical types studied in Ely and Peški (2011) and Chen and Xiong (2013) in any finite-action game that satisfies the richness assumption.

---

18By the finiteness of action sets and the upper hemicontinuity of best replies, a collection $(R_i)_{i \in N}$ is a 0-curp collection in finite model $(T, \kappa)$ (i.e., a model with $|T_i| < \infty$ for all $i \in N$) if and only if it is an $\epsilon$-curb collection in $(T, \kappa)$ for some $\epsilon > 0$. This is why we focus on the simpler notion of (0)-curp collection in Subsection 5.2. However, it can be shown that the equivalence between 0- and $\epsilon$-curp collections no longer holds for infinite models.
5.4 Proof of Proposition 4

Suppose that \( (\mathcal{R}_j : T_j \rightarrow 2^{A_j} \setminus \{\emptyset\})_{j \in \mathbb{N}} \) is an \( \varepsilon \)-curb collection in a model \((T, \kappa)\). By the richness assumption, for every action \( a_i \), there exists some \( \theta^{a_i} \) such that \( a_i \) is strictly dominant for the type \( t_i^{\theta^{a_i}} \) which has common knowledge of \( \theta^{a_i} \). Since the action set is finite, there exists \( \varepsilon' > 0 \) such that \( S_j^\infty \left[ t_i^{\theta^{a_i}}, \varepsilon' \right] = \{a_i\} \) for any \( a_i \). Let \( \gamma = \min(\varepsilon, \varepsilon') \). Then, Proposition 4 is implied by Claim 1 below.

**Claim 1** For any \( j \in \mathbb{N} \) and any integer \( m \geq 0 \), there exists a measurable function

\[
T_j \times A_j \ni (t, R) \mapsto \tau_{t,R,m} \in T_j^*
\]

such that for any \( t_j \in T_j \) and any \( R_j \in \mathcal{R}_j(t_j) \), we have \( \tau_{t_j,R_j,m}^{m'} = t_j^{m'} \) for all \( m' = 1, \ldots, m \) and \( S_j^\infty \left[ \tau_{t_j,R_j,m}, \gamma \right] \subset R_j \).

**Proof** By the definition of \( \varepsilon \)-curb collection, for each \( j \in \mathbb{N} \), there exists a measurable function \((t_j, R_j) \mapsto \mu_{t_j,R_j} \in \Delta(\Theta \times T_{-j} \times A_{-j})\) that satisfies all conditions in Definition 6.

We prove the claim by induction on \( m \). For \( m = 0 \), fix an arbitrary element \( a(R_j) \) of \( R_j \) for each \( R_j \in A_j \). Then, for each \( t_j \in T_j \) and each \( R_j \in \mathcal{R}_j(t_j) \), define \( \tau_{t_j,R_j,0} = t_j^{\theta^{a(R_j)}} \). Obviously, \( \tau_{t_j,R_j,0} \) is measurable in \((t_j, R_j)\). Also, \( S_j^\infty \left[ \tau_{t_j,R_j,0}, \gamma \right] = \{a_i\} \subset R_j \).

We now assume that the claim is true for \( m \) and prove the case for \( m + 1 \). We define \( \tau_{t_j,R_j,m+1} \) as follows.

\[
\kappa_{t_j,R_j,m+1}^* [F] \equiv \mu_{t_j,R_j} \left[ \{ \theta, (t_{-j}, R_{-j}) : (\theta, \tau_{t_{-j},R_{-j},m}) \in F \} \right], \forall F \subset \Theta \times T_{-j}^*
\]

where \( \tau_{t_{-j},R_{-j},m} = (\tau_{t_k,R_k,m})_{k \neq j} \). By the measurability of \( \mu_{t_j,R_j} \) and \( \tau_{t_{-j},R_{-j},m} \), \( \tau_{t_j,R_j,m+1} \) is well-defined and measurable in \((t_j, R_j)\). Also, since the marginal distribution of \( \mu_{t_j,R_j} \) on \( \Theta \times T_{-j} \) is equal to \( \kappa_{t_j} \), the induction hypothesis implies that \( \tau_{t_j,R_j,m+1}^{m'} = t_j^{m'} \) for all \( m' = 1, \ldots, m + 1 \).

Now fix a \( \gamma \)-valid conjecture \( \sigma_{-j} : \Theta \times T_{-j}^* \rightarrow \Delta(A_{-j}) \) for \( \tau_{t_j,R_j,m+1} \) arbitrarily. Let \( \varphi_{-j} : \Theta \times T_{-j} \times A_{-j} \rightarrow \Delta(A_{-j}) \) be given by

\[
\varphi_{-j}(\theta,t_{-j},R_{-j})[a_{-j}] = \sigma_{-j}(\theta, \tau_{t_{-j},R_{-j},m})[a_{-j}].
\]
Since \( \sigma_{-j} \) is \( \gamma \)-valid and \( S^\infty_j \left[ \tau_{t_j,R_j,m}, \gamma \right] \subset R_j \) by the induction hypothesis, \( \varphi_{-j} \) is a conditional conjecture. Also, we have

\[
\pi_{\mu_{t_j,R_j}^*, \varphi_{-j}} [\theta, a_{-j}] = \int_{T_j \times A_{-i}} \varphi_{-j}(\theta, t_{-j}, R_{-j})[a_{-j}]\mu_{t_j,R_j}[\theta, dt_{-j}, dR_{-j}]
= \int_{T_j \times A_{-i}} \sigma_{-j}(\theta, t_{-j}, R_{-j}, m)[a_{-j}]\mu_{t_j,R_j}[\theta, dt_{-j}, dR_{-j}]
= \int_{T_j^*} \sigma_{-j}(\theta, t^*_{-j})[a_{-j}]\kappa_{t_{t_j,R_j,m+1}}^*[\theta, dt^*_{-j}] = \pi_{\mu_{t_j,R_j,m+1}^*, \varphi_{-j}} [\theta, a_{-j}].
\]

Therefore, \( S^\infty_j \left[ \tau_{t_j,R_j,m+1}, \gamma \right] \subset \bigcup_{\varphi_{-j}} BR_j \left( \pi_{\mu_{t_j,R_j}^*, \varphi_{-j}}, \gamma \right) \subset R_j. \]

5.5 Proof of Proposition 5

We assume without loss of generality that \( (T, \kappa) \) is embedded in the universal type space \( (T^*, \kappa^*) \).

For each \( t_i \in T_i \) and each \( R_i \in \mathcal{R}_i^T (t_i) \), let \( \Delta_{t_i,R_i} \) be the set of weak* limits of \( \mu_{t_i,m} \in \Delta(\Theta \times T^*_{-i} \times A_{-i}) \), where

\[
\mu_{t_i,m} [F] = \kappa_{t_i,m}^* (\{(\theta, t_{-i}) : (\theta, t_{-i}, S^\infty_{-i}[t_{-i}, \varepsilon]) \in F\}), \forall F \subset \Theta \times T^*_{-i} \times A_{-i}
\]

(2)

for some sequence of types \( \{t_i,m\} \) such that \( t_{i,m} \rightarrow t_i \) and \( S^\infty_i [t_{i,m}, \varepsilon] = R_i \) for all \( m \). By the compactness of \( \Delta(\Theta \times T^*_{-i} \times A_{-i}) \), we have \( \Delta_{t_i,R_i} \neq \emptyset \). Also \( \Delta_{t_i,R_i} \) depends on \( (t_i, R_i) \) upper hemicontinuously. Thus, it follows from the Kuratowski–Ryll-Nardzewski selection theorem that we have a measurable function \( (t_i, R_i) \mapsto \mu_{t_i,R_i} \in \Delta(\Theta \times T^*_{-i} \times A_{-i}) \) such that \( \mu_{t_i,R_i} \in \Delta_{t_i,R_i} \) whenever \( R_i \in \mathcal{R}_i^T (t_i) \).

Hereafter, we fix any \( t_i \in T_i \) and any \( R_i \in \mathcal{R}_i^T (t_i) \), and prove (a), (b), and (c) in Definition 6. Let \( \{t_i,m\} \) be a sequence of types such that \( t_{i,m} \rightarrow t_i \) and \( \mu_{t_i,m} \rightarrow \mu_{t_i,R_i} \) as \( m \rightarrow \infty \), and \( S^\infty_i [t_{i,m}, \varepsilon] = R_i \) for all \( m \).

Since the marginal distribution of \( \mu_{t_i,m} \) on \( \Theta \times T^*_{-i} \) is equal to \( \kappa^*_{t_i,m} \) for all \( m \), \( t_{i,m} \rightarrow t_i \) as \( m \rightarrow \infty \), and \( \kappa^*_i \) is continuous, the marginal distribution of \( \mu_{t_i,R_i} \) on \( \Theta \times T^*_{-i} \) is equal to \( \kappa_{t_i} \). Thus (a) holds.
For each $\ell \in \mathbb{N}$, let
\[
F_\ell \equiv \{ (\theta, t_{-i}, R_{-i}) : \exists t'_{-i} \in T^*_{-i} \text{ s.t. } d_{-i}(t_{-i}, t'_{-i}) \leq 1/\ell \text{ and } S^\infty_{-i}[t_{-i}, \epsilon] = R_{-i} \},
\]
\[
F_\infty \equiv (\Theta \times T_{-i} \times A_{-i}) \cap \bigcap_{\ell \in \mathbb{N}} F_\ell
\]
where $d_{-i}$ is the metric on $T^*_{-i}$. Note that
\[
F_\infty \subset \{ (\theta, t_{-i}, R_{-i}) : t_{-i} \in T_{-i} \text{ and } R_{-i} \in \mathcal{R}_{T_{-i}}(t_{-i}) \},
\]
because for each $(\theta, t_{-i}, R_{-i}) \in F_\infty$, we have $t_{-i} \in T_{-i}$ and there exists $\{t_{-i,\ell}\}_{\ell \in \mathbb{N}} \rightarrow t_{-i}$ and $S^\infty_{-i}[t_{-i,\ell}, \epsilon] = R_{-i}$ for all $\ell \in \mathbb{N}$.

Furthermore,
\[
\{ (\theta, t_{-i}, R_{-i}) : S^\infty_{-i}[t_{-i}, \epsilon] = R_{-i} \} \subset F_\ell, \forall \ell \in \mathbb{N}.
\]
Hence, (2) and (4) imply that
\[
1 = \mu_{t_{i,m}} \{ (\theta, t_{-i}, R_{-i}) : S^\infty_{-i}[t_{-i}, \epsilon] = R_{-i} \} \leq \mu_{t_{i,m}} [F_\ell],
\]
i.e., $\mu_{t_{i,m}} [F_\ell] = 1$. Since $\mu_{t_{i,m}} \rightarrow \mu_{t_{i,R_i}}$ as $m \rightarrow \infty$, we have $\mu_{t_{i,R_i}} [F_\ell] = 1$ for all $\ell$. As a result, $\mu_{t_{i,R_i}} [\bigcap_{\ell \in \mathbb{N}} F_\ell] = 1$. Combining this with $\mu_{t_{i,R_i}} [\Theta \times T_{-i} \times A_{-i}] = 1$ by (a), we have $\mu_{t_{i,R_i}} [F_\infty] = 1$, which, together with (3) implies (b).

We now prove (c). Fix $a_i \in A_i$ such that $a_i \in BR_i \left( \pi_{\mu_{t_{i,R_i}, \varphi_{-i}}}, \epsilon/2 \right)$ for some conditional conjecture $\varphi_{-i}$. It suffices to show $a_i \in R_i$.

Pick $m$ sufficiently large so that
\[
\sum_{\theta, R_{-i}} |\mu_{t_{i,m}}[\{\theta\} \times T^*_{-i} \times \{R_{-i}\}] - \mu_{t_{i,R_i}}[\{\theta\} \times T^*_{-i} \times \{R_{-i}\}]| \leq \frac{\epsilon}{4 \max_{\theta, a_i, a_{-i}} |u_i(\theta, a_i, a_{-i})|}.
\]

(5)

For each $a_i \in \Delta(A_i \setminus \{a_i\})$, let $\psi_{-i}^{a_i}(\theta, R_{-i}) \in R_{-i}$ be one of the action profiles of player $i$’s opponents that favor action $a_i$ most relative to $a_i$, i.e.,
\[
\psi_{-i}^{a_i}(\theta, R_{-i}) \in \arg \max_{a_{-i} \in R_{-i}} [u_i(\theta, a_i, a_{-i}) - u_i(\theta, a_i, a_{-i})].
\]

Since we have $a_i \in BR_i \left( \pi_{\mu_{t_{i,R_i}, \varphi_{-i}}}, \epsilon/2 \right)$ for some conditional conjecture $\varphi_{-i}$, we have
\[
\int_{\Theta \times T^*_{-i} \times A_{-i}} [u_i(\theta, a_i, \psi_{-i}^{a_i}(\theta, R_{-i})) - u_i(\theta, a_i, \psi_{-i}^{a_i}(\theta, R_{-i}))] d\mu_{t_{i,R_i}} \geq -\frac{\epsilon}{2}.
\]

(6)
Now we define $\sigma_{-i}^{a_i} : \Theta \times T_{-i}^* \to A_{-i}$ as
\[
\sigma_{-i}^{a_i}(\theta, t_{-i}) = \psi_{-i}^{a_i}(\theta, S_{-i}^\infty[t_{-i}, \varepsilon]) \in S_{-i}^\infty[t_{-i}, \varepsilon],
\]
which implies that $\sigma_{-i}^{a_i}$ is an $\varepsilon$-valid (pure-action) conjecture. We then have
\[
\sum_{\theta, a_{-i}} \left[ u_i(\theta, a_i, a_{-i}) - u_i(\theta, a_{i}, a_{-i}) \right] \tau_{t_{i,m}, a_{-i}}^{a_i}[\theta, a_{-i}]
\]
\[
= \int_{\Theta \times T_{-i}^* \times A_{-i}} \left[ u_i(\theta, a_i, \psi_{-i}^{a_i}(\theta, R_{-i})) - u_i(\theta, a_i, \psi_{-i}^{a_i}(\theta, R_{-i})) \right] d\mu_{t_{i,m}}
\]
\[
\geq \int_{\Theta \times T_{-i}^* \times A_{-i}} \left[ u_i(\theta, a_i, \psi_{-i}^{a_i}(\theta, R_{-i})) - u_i(\theta, a_i, \psi_{-i}^{a_i}(\theta, R_{-i})) \right] d\mu_{t_{i,m}, R_i} - \frac{\varepsilon}{2} \geq -\varepsilon,
\]
where the inequalities follows from (5) and (6), respectively.

Therefore, for each $\alpha_i \in \Delta(A_i \setminus \{a_i\})$, there exists an $\varepsilon$-valid conjecture $\sigma_{-i}^{a_i}$ against which $a_i$ is an $\varepsilon$-better reply for $t_{i,m}$ than $\alpha_i$. Then it follows from the usual duality argument that we can find an $\varepsilon$-valid conjecture, independent of $\alpha_i$, against which $a_i$ is an $\varepsilon$-best reply for $t_{i,m}$. Thus, $a_i \in S_{i}^\infty[t_{i,m}, \varepsilon] = R_i. \square$

### 6 Discussion

#### 6.1 Generic complete-information games

In our setup, we fixed the payoffs of a game. A complete-information game can be identified with a profile of utility functions $(u_j : A \to \mathbb{R})_{j \in N}$. Let $\mathcal{G}^c$ be the set of all such complete-information games endowed with the Euclidean topology. For simplicity, we write $S_j^\infty[G]$ and $S_j^\infty[G, \varepsilon]$ for the set of rationalizable actions and the set of $\varepsilon$-rationalizable actions, respectively. We define the set of games $\mathcal{G}^*$ as follows.
\[
\mathcal{G}^* = \left\{ G \in \mathcal{G}^c : \exists \varepsilon > 0 \text{ s.t. } S_j^\infty[G] = S_j^\infty[G, -\varepsilon], \forall j \in N \right\}.
\]
That is, $\mathcal{G}^*$ is the set of games in which the set of strictly rationalizable actions coincides with the set of rationalizable actions in $G$. We prove the following result in the Appendix.\(^{19}\)

\(^{19}\)The result has been mentioned in (Weinstein and Yildiz, 2004, footnote 15) and (Oury and Tercieux, 2011, footnote 20).
Proposition 6 \( G^* \) is open and dense in \( G^c \).

In light of Proposition 2, Proposition 6 implies that, in generic complete-information games, rationalizable actions, strictly rationalizable actions, actions that can be WY-selected, and actions that can be robustly selected all coincide. While this seems to contradict the generic impossibility result in Theorem 1, the setups in which the two results are derived are different. For Theorem 1, we fix the payoffs and vary the hierarchies of beliefs, while for Proposition 6, we vary the payoffs and fix the (complete-information) hierarchy of belief.

This contrast raises the question whether the generic impossibility of robust selection in Theorem 1 still holds if we vary both payoffs and beliefs. To study this issue, fix a subset of parameters \( \{ \theta^j \} \) and assume that \( \Theta \equiv \Theta \setminus \{ \theta^j \} \neq \emptyset \). Define \( G^r = \mathbb{R}^{[N] | \Theta \times A}|. \) Note that each game \( (\bar{u}_j)_{j \in N} \in \mathbb{R}^{[N] | \Theta \times A}| \) can be identified with a game \( (u_j : \Theta \times A \to \mathbb{R})_{j \in N} \) which satisfies the richness assumption by setting

\[
  u_j (\theta^j, a_j, a_{-j}) > u_j (\theta^j, a'_j, a_{-j}), \forall a'_j \neq a_j, \forall a_{-j}.
\]

Let \( \Gamma_i = G^r \times T_i^* \) and define

\[
  M_i = \{(G, t_i) \in \Gamma_i : |S_i \tau [t_i, G, 0]| > 1\} \quad \text{and} \quad U_i = \{(G, t_i) \in \Gamma_i : |S_i \tau [t_i, G, 0]| = 1\}.
\]

The following result (proved in the Appendix) extends the generic impossibility of robust selection in Theorem 1 to the space \( M_i \).

\[ \text{Proposition 7} \] There is a residual set \( B_i \subset M_i \) such that for every \( (G, t_i) \in B_i, t_i \) do not admit any robust selection in \( G \).

6.2 Robust selection and robust prediction

WY show that every rationalizable action can be WY-selected. Conversely, by the upper hemicontinuity of the rationalizable correspondence (see (Dekel, Fudenberg, and Morris, 2006, Theorem 2)), every action that can be WY-selected must also be rationalizable. That is, an action can be WY-selected if and only if it is rationalizable. Our results show that

\[ \text{It can be shown that } M_i \text{ is a Polish space and thus a Baire space.} \]
robust selection is strictly stronger than rationalizability and strictly weaker than strict rationalizability.

Though robust selection refines rationalizability, some types may still have multiple actions which can be robustly selected. For example, the complete-information type in the global game has two strict equilibrium actions, both of which can be robustly selected. In the absence of the exact specification of best replies or the precise measurement of payoffs, the idea of robust selection is that actions which are not robustly selected are fragile predictions. They are predictions inferior to those actions that can be robustly selected and equipped with the two robustness features.

Among those actions that can be robustly selected, which one should we pick? Equivalently, what perturbation of higher-order beliefs is the most sensible approximation of the strategic situation that we are analyzing? The answer to this question is subject to the judgment of the modeler. For example, if we think that the global game (i.e., Example 1 in Section 2) is a good approximation of the strategic situation that we are facing, we would predict that player $i$ who observes $x_i = \frac{3}{4}$ will choose “Attack,” whereas if we consider the alternative perturbation in WY’s Example 2 is more sensible, we would predict “No Attack.” If no perturbation is more appealing than the other, we may follow WY in saying that a prediction is appealing only when it holds under both actions.

### 6.3 Robust selection and critical types

In Ely and Peški (2011) and Chen and Xiong (2013), a type $t_i$ is said to be $G$-critical (or display strategic discontinuity in $G$) if there exist $\epsilon > 0$, an action $a_i$, and a sequence of types $\{t_{i,m}\}$ with $t_{i,m} \rightarrow t_i$ such that $a_i \in S_i^\infty [t_i, G, 0]$ and $a_i \notin S_i^\infty [t_{i,m}, G, \epsilon]$ for every $m$. A type is critical if it is a $G$-critical type for some finite-action game $G$.

For instance, by the upper hemicontinuity of ICR, a type who has a unique rationalizable action in $G$ is not $G$-critical despite having a robust selection. In contrast, a type who has multiple rationalizable actions and admits a robust selection in game $G$ must be $G$-critical. Indeed, the complete-information type in Rubinstein (1989) is a prominent example of $G$-critical types.

Ely and Peški (2011) characterize critical types. Here we fix a finite-action game $G$ that satisfies the richness assumption, and we characterize $G$-critical types by means of
the \( \varepsilon \)-curb collection in \( G \), as follows (the proof is in the Appendix):

**Proposition 8** Given a model \((T, \kappa)\), a type \( t_i \in T_i \) is \( G \)-critical if and only if there exists some \( \varepsilon \)-curb collection \((R_i)_{j \in \mathbb{N}}\) in \((T, \kappa)\) with \( \varepsilon > 0 \) such that \( S_i^{\infty} [t_i] \setminus R_i \neq \emptyset \) for some \( R_i \in R_i (t_i) \).

Moreover, let \( B_i \) be defined as in (1) and \( NC_i \) be the set of types in \( T_i^* \) which are not \( G \)-critical. Note that \( NC_i \) is measurable.\(^{21}\) We prove the following proposition in the Appendix.

**Proposition 9** If \( \kappa^*_{i_1} [\Theta \times NC_{-i}] = 1 \), then \( t_i \in NC_i \). In particular, \( B_i \subseteq NC_i \).

In other words, types in \( B_i \) defined in Section 4 are not \( G \)-critical, although they admit WY selections. Furthermore, since every type that has multiple rationalizable actions and admits a robust selection must be critical, Proposition 9 implies Lemma 3 and Theorem 1 implies that \( G \)-critical types are non-generic in the set of types with multiple rationalizable actions.\(^{22}\)

### 6.4 Set-valued robust selection

We can also define robust selection for a set of actions. We say that a set of actions \( R_i \) can be robustly selected for type \( t_i \), if there is an \( \varepsilon > 0 \) and a sequence of types \( \{t_{i,m}\} \) such that \( t_{i,m} \to t_i \) and \( S_i^{\infty} [t_{i,m}, G, \varepsilon] = R_i \) for every \( m \). Our results can accommodate the set version of robust selection with minor modifications. In particular, Proposition 9 shows

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\(^{21}\)To see this, for each \( R_i \in A_i \) and \( \varepsilon \geq 0 \), let \( T_i (R_i, \varepsilon) = \{ t_i \in T_i^* : S_i^{\infty} [t_i, G, \varepsilon] = R_i \} \). Since \( S_i^{\infty} [\cdot, G, \varepsilon] \) is upper hemicontinuous, \( T_i (R_i, \varepsilon) \) is a finite intersection of open or closed subsets of \( T_i^* \), and hence measurable. By the definition of \( NC_i \), we have

\[
NC_i = \bigcup_{R_i \in A_i} \left( T_i (R_i, 0) \setminus \bigcup_{\ell \in \mathbb{N}} \bigcup_{a_i \in R_i} \bigcup_{S_i} \text{cl}(T_i (S_i, 1/\ell)) \right),
\]

which is measurable.

\(^{22}\)Without imposing the richness assumption, Ely and Peski (2011) show that critical types, i.e., types that are \( G \)-critical for some \( G \), are non-generic in the universal type space. Under the richness assumption, Ely and Peski’s non-genericity result also follows from the non-genericity of types with multiple rationalizable actions proved by Weinstein and Yildiz (2007).
that no proper subset of rationalizable actions of any type in $B_i$ can be robustly selected. Moreover, Proposition 5 shows that every set of actions that can be robustly selected must be belong to a curb collection. Conversely, Proposition 4 shows that every set of actions in a curb collection contains a subset that can be robustly selected.

6.5 Ex ante selection versus interim selection

Both WY selection and robust selection are interim notions. That is, an action is selected according to the interim (hierarchy of) beliefs of a type. An alternative approach is to study selections from an ex ante viewpoint. In this case, a type is replaced by a type space which is usually associated with a common prior. For example, Engl (1995) considered perturbations of priors and proved that for any sequence of ex ante strategy profiles $\{\sigma_m\}$ that converges to an equilibrium in the original game, the strategy profile $\sigma_m$ must be an ex ante $\epsilon_m$-equilibrium in the perturbed games with $\epsilon_m \to 0$.23

A recent paper that is closer to our work is Jackson, Rodriguez-Barraquer, and Tan (2012). The paper obtains two main results: (1) for any ex ante equilibrium and any sequence of ex ante perturbations of the game (including perturbations of priors, type spaces, and payoff functions), there is a corresponding sequence of ex ante $\epsilon_m$-equilibria with $\epsilon_m \to 0$ converging to that equilibrium; and (2) under a stronger notion of perturbation, the approximating ex ante $\epsilon_m$-equilibria can be strengthened to interim $\epsilon_m$-equilibria in which almost all types choose $\epsilon_m$-best replies. Jackson, Rodriguez-Barraquer, and Tan (2012) conclude from their results that any refinement of equilibrium is not robust to slight perturbations in best-reply behavior or to underlying preferences.

Both Jackson, Rodriguez-Barraquer, and Tan (2012) and our paper employ solution concepts based upon $\epsilon$-best replies to approximate solution concepts based upon 0-best replies. However, Jackson, Rodriguez-Barraquer, and Tan (2012) can approximate any equilibrium in any incomplete-information game with a sequence of $\epsilon_m$-equilibria along any sequence of their interim perturbations, while such an approximation is only generically true in our setup. This discrepancy occurs for two reasons. First, we require that all types play interim $\epsilon$-best replies, while their first result allows a set of types with positive measure to play actions different from their interim $\epsilon$-best replies (as long as they

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aggregate to an ex ante $\epsilon$-best reply). Second, the stronger notion of perturbations in their second result excludes “crazy types” that behave very differently from types in the original game. In contrast, WY and we adopt the finite-order perturbations which allow any belief in the tail of a perturbed hierarchy; hence, the type space of the perturbed type may include “crazy types.”

The ex ante perspective is also adopted by Kajii and Morris (1997). They define a robust equilibrium as a Nash equilibrium in a complete-information game which is played with sufficiently large probability in some Bayesian Nash equilibrium on any common priors which are sufficiently close to the complete-information scenario. This approach has been extended by Oyama and Tercieux (2010) to allow for perturbations with non-common priors. The solution concept in Kajii and Morris (1997) and Oyama and Tercieux (2010) is based upon 0-best replies. Haimanko and Kajii (2012) (Chassang and Takahashi (2011), resp.) extend Kajii and Morris (1997) to study the notion of an approximately (strictly, resp.) robust equilibrium based upon $\epsilon$-best replies (strict best replies, resp.).

A Appendix

In the appendix, we first prove Propositions 1 and 6 and Lemmas 1 and 2 which hold independently of other results in the paper. We then use Theorem 2 to derive Proposition 2 and use Propositions 4 and 5 to derive Propositions 8 and 9 which imply Lemma 3 as we discussed in the main text. Finally, we use Lemmas 1–3 to derive Proposition 7.

A.1 Proof of Proposition 1

**Proposition 1** If $a_i$ can be robustly selected for $t_i$ in $G = (u_j)_{j \in N'}$ then for some $\gamma > 0$, $a_i$ can also be robustly selected for $t_i$ in any $G' = (u'_j)_{j \in N}$ such that

$$|u'_j(\theta, a) - u_j(\theta, a)| \leq \gamma, \forall (j, \theta, a) \in N \times \Theta \times A. \quad (7)$$

**Proof** Since $a_i$ is robustly selected for $t_i$ in $G$, there exists some $\epsilon > 0$ and some sequence of

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24 This is also why the lower hemicontinuity result in Engl (1995) fails to hold when he considers the notion of weak convergence of priors which allows “crazy types” to occur with small probability.
types \{t_{i,m}\} such that \(t_{i,m} \rightarrow t_i\) and \(\{a_i\} = S_i^\infty [t_{i,m}, G, \varepsilon]\) for every \(m\). Hence, every \(a'_i \neq a_i\) is not \(\varepsilon\)-rationalizable for \(t_{i,m}\) in \(G\). Let \(\gamma = \varepsilon/3\). By (Dekel, Fudenberg, and Morris, 2006, Lemma 10), every \(a'_i \neq a_i\) is not \(\gamma\)-rationalizable for \(t_{i,m}\) in any \(G'\) satisfying (7). Since \(S_i^\infty [t_{i,m}, G', \gamma] \neq \varnothing\), it follows that \(S_i^\infty [t_{i,m}, G', \gamma] = \{a_i\}\) for every \(m\). Thus, \(a_i\) is robustly selected for \(t_i\) in \(G'\).

\[\text{Claim 9} \quad \text{We conclude that} \quad \text{for every} \quad \text{such that} \quad \text{there is some} \quad \text{finiteness of actions} \quad \text{there is some} \quad \text{any} \]

\[\text{Proposition 3} \quad \text{Proof of Proposition 6} \]

For any \(G = (u_j)_{j \in N}\) and \(G' = (u'_j)_{j \in N}\) in \(G^c\), define \(\|G - G'\| = \max_{j \in N, a \in A} |u_j(a) - u'_j(a)|\).

\[\text{Proposition 6} \quad G^* \text{ is open and dense in } G^c.\]

\[\text{Proof} \quad \text{In this proof, we write} \quad S^\infty [G] = \prod_{j \in N} S_j^\infty [G]. \text{ We first prove that } G^* \text{ is open. Fix any } G \in G^*. \text{ Thus, for some } \varepsilon > 0, S_j^\infty [G] = S_j^\infty [G, -\varepsilon] \text{ for every } j. \text{ Furthermore, by the finiteness of actions, there is some } \varepsilon' > 0 \text{ such that } S_j^\infty [G] = S_j^\infty [G, \varepsilon'] \text{ for every } j. \text{ Hence, there is some } \varepsilon^* > 0, \text{ such that } S_j^\infty [G, -\varepsilon^*] = S_j^\infty [G, \varepsilon^*]. \text{ (8)}\]

Then for any \(G'\) with \(\|G' - G\| < \varepsilon^*/4\), we have

\[S_j^\infty [G, -\varepsilon^*] \subset S_j^\infty [G', -\varepsilon^*/2] \subset S_j^\infty [G'] \subset S_j^\infty [G, \varepsilon^*]. \text{ (9)}\]

Then, (8) and (9) implies that \(S_j^\infty [G'] = S_j^\infty [G', -\varepsilon^*/2]. \text{ Thus,} \)

\[\|G' - G\| < \frac{\varepsilon^*}{4} \Rightarrow G' \in G^*. \]

We conclude that \(G^*\) is open.

\[\text{We next prove that} \quad G^* \text{ is dense by the following claim.} \]

\[\text{Claim 2} \quad \text{For any } \varepsilon > 0 \text{ and any } G = (u_j)_{j \in N} \notin G^*, \text{ there exits } G' \in G^c \text{ with } \|G' - G\| < \varepsilon \text{ such that } S_j^\infty [G] \supseteq S_j^\infty [G']. \]

To see that Claim 2 implies that \(G^*\) is dense, pick any \(G \in G^c\). Construct a sequence of games in \(G^c\) as follows: (1) set \(G^1 = G\); (2) for any \(k \geq 1\), if \(G^k \in G^*\), we set \(G^{k+1} = G^k\);
if \( G^k \notin G^* \), by Claim 2, we can find some \( G^{k+1} \) with \( \| G^{k+1} - G^k \| < \epsilon \) such that \( S^\infty[G^k] \not\supseteq S^\infty[G^{k+1}] \).

Thus, as long as, \( G^k \notin G^* \) for all \( k' < k \), we construct a sequence of games \( \{G^k\} \) such that

\[
S^\infty[G^1] \not\supseteq S^\infty[G^2] \supseteq \cdots \not\supseteq S^\infty[G^k].
\]

The sets of rationalizable actions cannot shrink for more than \( \sum_{i \in N} |A_i| \) times. Hence, \( G^k \in G^* \) for some \( k \leq \sum_{i \in N} |A_i| \). Moreover, \( \| G - G^k \| = \| G^1 - G^k \| \leq \epsilon \times \sum_{i \in N} |A_i| \). Since \( \epsilon \) is arbitrary, we conclude that \( G^* \) is dense.

We now turn to prove Claim 2. Fix any \( \epsilon > 0 \) and \( G = (u_{ij})_{i \in N} \notin G^* \). First, observe that \( G' \in G^* \) if and only if the set \( \prod_{j \in N} S^\infty_j [G'] \) satisfies the strict best reply property, i.e., for any \( j \in N \) and \( a_j \in S^\infty_j [G'] \), there exists some \( \pi \in \Delta \left( S^\infty_{-j} [G'] \right) \) such that \( \textup{BR}_j(\pi, G') = \{a_j\} \). Since \( G = (u_{ij})_{j \in N} \notin G^* \), it follows that there exist \( i \in N \) and \( a_i^* \in S^\infty_i [G] \) such that \( a_i^* \) is not a strict best reply to any \( \pi \in \Delta \left( S^\infty_{-j} [G] \right) \). By the usual duality argument, there exists \( a_i \in \Delta \left( A_i \setminus \{a_i^*\} \right) \) such that

\[
 u_i(a_i, a_{-i}) - u_i(a_i^*, a_{-i}) \geq 0, \forall a_{-i} \in S^\infty_{-i} [G], \quad (10)
\]

where \( u_i(a_i, a_{-i}) = \sum_{a_i \in A_i} a_i[a_i] u_i(a_i, a_{-i}) \). Second, by finiteness of actions, there is some \( \epsilon' > 0 \) such that

\[
 S^\infty_{-i} [G] = S^\infty_{-i} [G, \epsilon'] \quad (11)
\]

We define \( G' = (u'_{ij})_{j \in N} \) as follows.

\[
u'_{ij}(a_j, a_{-j}) = \begin{cases} u_i(a_i^*, a_{-i}) - \gamma, & \text{if } j = i \text{ and } a_j = a_i^*; \\ u_j(a_j, a_{-j}), & \text{otherwise}, \end{cases} \quad (12)
\]

where \( \gamma = \min \{\epsilon, \epsilon'\} / 2 \). Clearly, \( \| G - G' \| = \gamma < \epsilon \). By (11), we have

\[
 S^\infty_j [G'] \subset S^\infty_j [G, 2\gamma] \subset S^\infty_j [G, \epsilon'] = S^\infty_j [G], \forall j \in N. \quad (13)
\]

Then, (10), (12), and (13) imply that

\[
u'_{ij}(a_i, a_{-i}) - \nu'_{ij}(a_i^*, a_{-i}) > 0, \forall a_{-i} \in S^\infty_{-i} [G'].
\]

As a result, \( a_i^* \) is not rationalizable in \( G' \). This together with (13) for \( j = i \) implies that \( S^\infty_i [G] \not\supseteq S^\infty_i [G'] \). Thus, Claim 2 follows.
A.3 Proof of Lemma 1

Lemma 1 \( B_{i,n} \) is open in \( M_i \).

Proof Recall
\[
B_{i,n} \equiv \left\{ t_i \in M_i : \kappa_{i_i}^* [\Theta \times U_{-i}] > 1 - \frac{1}{n} \right\}.
\]

By WY’s Proposition 2, \( U_{-i} \) is open. Then, by the definition of the weak* topology, \( \left\{ \mu \in \Delta (\Theta \times T^*_i) : \mu [\Theta \times U_{-i}] > 1 - \frac{1}{n} \right\} \) is open. Since \( \kappa_{i_i}^* = \kappa_i^*(\cdot) \) and \( \kappa_i^* \) is a homeomorphism between \( T^*_i \) and \( \Delta (\Theta \times T^*_i) \), it follows that \( B_{i,n} \) is open. ■

A.4 Proof of Lemma 2

Lemma 2 \( B_i \) is dense in \( M_i \).

Proof Recall that \( T^*_i \) is a compact metric space and \( d_i \) is the metric on \( T^*_i \). We now divide the proof into three steps.

We say that \( t_i \in T^*_i \) is a finite type if there exists \( t'_i \) in a model \( (T, \kappa^*) \) with \( T_j \subset T^*_j \) and \( |T_j| < \infty \) for every \( j \in N \).

Step 1 For any finite type \( t_i \in M_i \) in \( (T, \kappa^*) \), there is some conjecture \( \sigma_{-i} : \Theta \times T_{-i} \rightarrow \Delta (A_{-i}) \) which is valid for \( t_i \) and \( BR_i (\pi_{t_i,\sigma_{-i}}) \) contains more than one action.

Define
\[
\Sigma_{-i} \equiv \left\{ \sigma_{-i} : \Theta \times T_{-i} \rightarrow \Delta (A_{-i}) : \sigma_{-i}(\theta, t_{-i}) \in \Delta (S^\infty_{t_{-i}} [t_{-i}]), \forall (\theta, t_{-i}) \right\};
\]
\[
\Sigma_{-i}^{a_i} \equiv \left\{ \sigma_{-i} \in \Sigma_{-i} : \{a_i\} = BR_i (\pi_{t_i,\sigma_{-i}}) \right\}, \forall a_i \in A_i.
\]
\( \Sigma_{-i} \) is the set of all valid conjectures, and \( \Sigma_{-i}^{a_i} \) is the set of valid conjectures to which \( a_i \) is the unique best reply for \( t_i \). Observe that \( \Sigma_{-i} \) is a connected subset of \( \mathbb{R}^{\Theta \times T_{-i} \times A_{-i}} \) (recall that \( T_{-i} \) is a finite set). Moreover, for every \( a_i \in A_i \), \( \Sigma_{-i}^{a_i} \) is an open subset of \( \Sigma_{-i} \) and \( \Sigma_{-i} \cap \Sigma_{-i}^{a_i} = \emptyset \) if \( a_i \neq a'_i \).

Now suppose that \( |BR_i (\pi_{t_i,\sigma_{-i}})| = 1 \) for any \( \sigma_{-i} \in \Sigma_{-i} \). Then we have \( \Sigma_{-i} = \bigcup_{a_i \in A_i} \Sigma_{-i}^{a_i} \). Since \( \Sigma_{-i} \) is connected, it cannot be equal to the disjoint union of multiple
nonempty open subsets. Thus, \( \Sigma_{m_i} = \emptyset \) for all but one action. This is a contradiction to \( t_i \in M_i \).

**Step 2** For any finite type \( t_i \in M_i \) in \( (T, \kappa^*) \), there is a sequence of finite types \( \{ t_{i,m} \} \) such that \( t_{i,m} \to t_i \) and \( t_{i,m} \in B_i \) for all \( m \).

By step 1, there is some valid conjecture \( \sigma_{-i} : \Theta \times T_{-i} \to \Delta (A_{-i}) \) for \( t_i \) such that \( BR_i (\pi_{t_i,\sigma_{-i}}) \) has at least two actions.

By Proposition 1 in Weinstein and Yildiz (2007), for each \( t_{-i} \in T_{-i} \) and each \( a_{-i} \in S_{-i}^{\infty} [t_{-i}] \), there exists a sequence of types \( \{ \tau_{t_{-i},a_{-i},m} \} \) on \( T_{-i}^* \) such that \( \tau_{t_{-i},a_{-i},m} \to t_{-i} \) and \( S_{-i}^{\infty} [\tau_{t_{-i},a_{-i},m}] = \{ a_{-i} \} \). We define \( t_{i,m} \in T_i^* \) as follows.

\[
\kappa^*_{t_{i,m}} [\theta, \tau_{t_{-i},a_{-i},m}] = \kappa^*_{t_{i,m}} [\theta, t_{-i}] \times \sigma_{-i} (\theta, t_{-i}) [a_{-i}] , \forall [\theta, t_{-i}] \in \Theta \times T_{-i} , \forall a_{-i} \in A_{-i} .
\]

Then, \( t_{i,m} \) is a finite type and \( \kappa^*_{t_{i,m}} (\Theta \times U_{-i}) = 1 \). Also, since \( \tau_{t_{-i},a_{-i},m} \to t_{-i} \) for each \( t_{-i} \in T_{-i} \) and each \( a_{-i} \in S_{-i}^{\infty} [t_{-i}] \), we have \( \kappa^*_{t_{i,m}} \to \kappa^*_i \). Since \( (\kappa^*)^{-1} \) is continuous, we have \( t_{i,m} \to t_i \).

We now show that \( t_{i,m} \) has multiple rationalizable actions. By our construction, the valid conjecture \( \sigma_{-i,m} : \Theta \times T_{-i}^* \to \Delta (A_{-i}) \) for \( t_{i,m} \) is uniquely determined by

\[
\sigma_{-i,m} (\theta, \tau_{t_{-i},a_{-i},m}) [a_{-i}] = 1
\]

(15)

on the support of \( \kappa^*_{t_{i,m}} \). Thus, for any \( \theta \in \Theta \) and any \( a_{-i} \in A_{-i} \), we have

\[
\pi_{t_{i,m},\sigma_{-i,m}} [\theta, a_{-i}] = \sum_{t_{-i} \in T_{-i}} \kappa^*_{t_{i,m}} [\theta, \tau_{t_{-i},a_{-i},m}] \times \sigma_{-i,m} (\theta, \tau_{t_{-i},a_{-i},m}) [a_{-i}]
\]

\[
= \sum_{t_{-i} \in T_{-i}} \kappa^*_{t_{i,m}} [\theta, \tau_{t_{-i},a_{-i},m}]
\]

\[
= \sum_{t_{-i} \in T_{-i}} \kappa^*_i [\theta, t_{-i}] \times \sigma_{-i} (\theta, t_{-i}) [a_{-i}]
\]

\[
= \pi_{t_i,\sigma_{-i}} [\theta, a_{-i}],
\]

where the first equality follows from the definition of \( \pi_{t_{i,m},\sigma_{-i,m}} \); the second equality follows from (15); the third equality follows from (14); the last equality follows from the definition of \( \pi_{t_i,\sigma_{-i}} \). Thus we have \( \pi_{t_{i,m},\sigma_{-i,m}} = \pi_{t_i,\sigma_{-i}} \). Therefore, \(|BR_i (\pi_{t_{i,m},\sigma_{-i,m}})| = |BR_i (\pi_{t_i,\sigma_{-i}})| > 1\).

To sum up, we find a sequence of finite types \( \{ t_{i,m} \} \) such that \( t_{i,m} \to t_i \) and \( t_{i,m} \in M_i \), \( t_{i,m} \in B_i \), and \( \kappa^*_{t_{i,m}} (\Theta \times U_{-i}) = 1 \), i.e., \( t_{i,m} \in B_i \) for all \( m \).
Step 3 B_i is dense in M_i.

Let d_i be the metric on T_i*. Take any t_i ∈ M_i and any ε > 0. First, by (Chen, 2012, Lemma 3), there is some finite type t_i' ∈ T_i such that S_i∞ [t_i] = S_i∞ [t_i'] and d_i (t_i, t_i') < ε/2. Then, for any ε > 0, by step 2, there is some t_i'' ∈ B_i such that d_i (t_i', t_i'') < ε/2. Hence, d_i (t_i, t_i'') < ε. Therefore, B_i is dense in M_i. ■

A.5 Proof of Proposition 2

Proposition 2 Given a model (T, κ) and t_i ∈ T_i, (i) an action a_i can be robustly selected for t_i if a_i is strictly rationalizable for t_i; (ii) an action a_i can be robustly selected for t_i only if a_i is a strict best reply for t_i.

Proof For (i), suppose that a_i ∈ S_i∞ [t_i, −ε] for some ε > 0. Then, it follows from the Kuratowski–Ryll-Nardzewski selection theorem that

\[ R_j (t_i) = \{ \{ a_j \} : a_j ∈ S_j^\infty [t_j, −ε] \} , \forall t_j ∈ T_j \]

defines an ε-curb collection. Thus, it follows from the “if” part of Theorem 2 that a_i can be robustly selected for t_i.

For (ii), suppose that a_i can be robustly selected for t_i. Then, by the “only if” part of Theorem 2, \{ a_i \} ∈ R_i (t_i) for some ε-curb collection \( R_j \) \( j \in N \). Thus, there exists \( μ_{t_i,R_i} ∈ Δ(Θ × T_{−i} × A_{−i}) \) such that the marginal distribution of \( μ_{t_i,R_i} \) on \( Θ × T_{−i} \) is equal to \( κ_{t_i} \) and \( BR_i (π_{μ_{t_i,R_i}, q_{−i}, ε}) = \{ a_i \} \) for any conditional conjecture \( q_{−i} \). Since at least one such conditional conjecture exists, a_i is a strict best reply for t_i. ■

A.6 Proof of Proposition 8

Proposition 8 Given a model (T, κ), a type t_i ∈ T_i is G-critical if and only if there exists some ε-curb collection \( R_j \) \( j \in N \) in (T, κ) with ε > 0 such that \( S_i^\infty [t_i] \setminus R_i \neq ∅ \) for some \( R_i ∈ R_i (t_i) \).

Proof Suppose that there is some ε-curb collection \( R_j \) \( j \in N \) and some \( R_i ∈ R_i (t_i) \) such that some action \( a_i ∈ S_i^\infty [t_i] \) and \( a_i \notin R_i \). Then, by Proposition 4, there exist γ > 0 and a sequence of types \( \{ t_{i,m} \} \) such that \( t_{i,m} → t_i \) and \( S_i^\infty [t_{i,m}, γ] \subset R_i \) for every m. Since
$S^\infty_i \{t_{i,m}, \gamma\} \subset R_i$ for every $m$, it follows that $a_i \in S^\infty_i \{t_i\}$ and $a_i \notin S^\infty_i \{t_{i,m}, \gamma\}$ for every $m$. That is, $t_i$ is $G$-critical.

Conversely, suppose that $t_i$ is $G$-critical. Then, there exist $\varepsilon > 0$, an action $a_i$, and a sequence of types $\{t_{i,m}\}$ with $t_{i,m} \rightarrow t_i$ such that $a_i \in S^\infty_i \{t_i\}$ and $a_i \notin S^\infty_i \{t_{i,m}, \varepsilon\}$ for every $m$. By finiteness of actions, there exist some $R_i \subset A_i$ and some subsequence of $\{t_{i,m_k}\}$ such that $t_{i,m_k} \rightarrow t_i$ and $S^\infty_i \{t_{i,m_k}, \varepsilon\} = R_i$ for all $k$. Hence, by Proposition 5, $R_i \in \mathcal{R}_i(t_i)$ for the $\frac{\varepsilon}{2}$-curb collection $(\mathcal{R}_i)_{j \in \mathbb{N}}$, and moreover, $a_i \in S^\infty_i \{t_i\}$ and $a_i \notin R_i$. \[\square\]

A.7 Proof of Proposition 9

**Proposition 9** If $\kappa^*_i [\Theta \times NC_{-i}] = 1$, then $t_i \in NC_i$. In particular, $B_i \subset NC_i$.

**Proof** By Proposition 8, we prove the proposition by showing that for any $\varepsilon > 0$ and any $\varepsilon$-curb collection $(\mathcal{R}_j)_{j \in \mathbb{N}}$ in $(T^*, \kappa^*)$,

$$R_i \in \mathcal{R}_i(t_i) \Rightarrow S^\infty_i \{t_i\} \subset R_i. \quad (16)$$

Fix an $\varepsilon > 0$ and an $\varepsilon$-curb collection $(\mathcal{R}_j)_{j \in \mathbb{N}}$ in $(T^*, \kappa^*)$. Then, there exists a measurable function $(t_j, R_j) \mapsto \mu_{t_jR_j} \in \Delta(\Theta \times T^*_{-j} \times A_{-j})$ that satisfies all conditions in Definition 6.

Fix a valid conjecture $\sigma_{-i} : \Theta \times T^*_{-i} \rightarrow \Delta(A_{-i})$ for $t_i$ arbitrarily. By Proposition 8, for each $t_{-i} \in NC_{-i}$ and $R_{-i} \in \mathcal{R}_{-i}(t_{-i})$, we have $S^\infty_{-i}[t_{-i}] \subset R_{-i}$. Let $\varphi_{-i} : \Theta \times T^*_{-i} \times A_{-i} \rightarrow \Delta(A_{-i})$ be a conditional conjecture that satisfies $\varphi_{-i}(\theta, t_{-i}, R_{-i})[a_{-i}] = \sigma_{-i}(\theta, t_{-i})[a_{-i}]$ for any $t_{-i} \in NC_{-i}$ and any $R_{-i} \in \mathcal{R}_{-i}(t_{-i})$. Then it follows that $\pi_{\mu_{t_iR_i}, \varphi_{-i}} = \pi_{t_i\sigma_{-i}}$, and thus $S^\infty_i \{t_i\} \subset \bigcup_{\varphi_{-i}} BR_i(\pi_{\mu_{t_iR_i}, \varphi_{-i}}) \subset \bigcup_{\varphi_{-i}} BR_i(\pi_{\mu_{t_iR_i}, \varphi_{-i}}, \varepsilon) \subset R_i$. Thus (16) holds.

Finally, by the upper hemicontinuity of $S^\infty_j \{\cdot\}$ for $j \neq i$, we have $U_{-i} \subset NC_{-i}$. It follows that $B_i \subset NC_i$. \[\square\]

A.8 Proof of Proposition 7

**Proposition 7** There is a residual set $B_i \subset M_i$ such that for every $(G, t_i) \in B_i$, $t_i$ do not admit any robust selection in $G$. 

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Proof First, we define for every $G \in \mathcal{G}^r$,

$$M_{i,G} = \{ t_i \in T_i^* : |S_i^\infty [t_i, G, 0]| > 1 \} \text{ and } U_{i,G} = \{ t_i \in T_i^* : |S_i^\infty [t_i, G, 0]| = 1 \}.$$ 

It follows that

$$\mathcal{M}_i = \{(G, t_i) \in \Gamma : t_i \in M_{i,G}\} \text{ and } U_i = \{(G, t_i) \in \Gamma : t_i \in U_{i,G}\}.$$ 

We prove the result in two steps:

Step 1 $B_{i,n} \equiv \{ (G, t_i) \in \mathcal{M}_i : k_{t_i}^* [\Theta \times U_{-i,G}] > 1 - 1/n \}$ is open in $\Gamma_i$.

Suppose that $k_{t_i}^* [\Theta \times U_{-i,G}] > 1 - 1/n$. For every $t_{-i} \in U_{-i,G}$, we know that $|S_{-i}^\infty [t_{-i}, G, 0]| = 1$; then, there is some $\epsilon(t_{-i}) > 0$ such that $|S_{-i}^\infty [t_{-i}, G, \epsilon(t_{-i})]| = 1$. Thus, $U_{-i,G} = \bigcup_{m=1}^\infty U_{-i,G,m}$ where

$$U_{-i,G,m} \equiv \{ t_{-i} \in U_{-i,G} : |S_{-i}^\infty [t_{-i}, G, 1/m]| = 1 \}.$$ 

Since $k_{t_i}^* [\Theta \times U_{-i,G}] > 1 - 1/n$, it follows that for some $m$,

$$k_{t_i}^* [\Theta \times U_{-i,G,m}] > 1 - 1/n. \tag{17}$$

Moreover, by upper hemicontinuity of ICR, for each $t_{-i} \in U_{-i,G,m}$, there is an open set $O_{t_{-i}}$ such that $t_{-i} \in O_{t_{-i}} \subset U_{-i,G,m}$. Thus, $\Theta \times U_{-i,G,m}$ is an open set. By (17) and the proof of Lemma 1, there is some an open set $O_{t_i}$ such that $t_i \in O_{t_i}$ and

$$k_{t_i}^* [\Theta \times U_{-i,G,m}] > 1/n, \forall t_i \in O_{t_i}. \tag{18}$$

By (Dekel, Fudenberg, and Morris, 2006, Lemma 10), there is an open set $O_{G}$ such that $G \in O_{G}$ and $|S_{-i}^\infty [t_{-i}, G', 0]| = 1$ for every $t_{-i} \in U_{-i,G,m}$ and $G' \in O_{G}$. That is,

$$U_{-i,G,m} \subset U_{-i,G'}, \forall G' \in O_{G}. \tag{19}$$

By (18) and (19), we conclude that $(G, t_i) \in O_{G} \times O_{t_i}$ and

$$k_{t_i}^* [\Theta \times U_{-i,G'}] > 1/n, \forall (G', t_i) \in O_{G} \times O_{t_i}.$$ 

Hence, $B_{i,n}$ is open in $\Gamma_i$.

Step 2 There is a residual set $B_i \subset \mathcal{M}_i$ such that for every $(G, t_i) \in B_i$, $t_i$ do not admit any robust selection in $G$. 

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Define

\[ B_i \equiv \{(G, t_i) \in M_i : \kappa^*_i [\Theta \times U_{-i,G}] = 1\}. \]

By Lemma 2, for each \((G, t_i) \in M_i\) with \((G, t_i) \notin B_i\), there is a sequence of types \(\{t_{i,m}\}\) such that \(t_{i,m} \to t_i\) and \(\kappa^*_{i,m} [\Theta \times U_{-i,G}] = 1\) for each \(m\), i.e., \((G, t_{i,m}) \in B_i\). Hence, \(B_i\) is dense in \(M_i\). Moreover, by step 1, \(B_{i,n}\) is open in \(\Gamma_i\). Since \(B_i \subseteq B_{i,n}\), it follows that \(B_{i,n}\) is open and dense in \(M_i\). Since \(\bigcap_{n=1}^{\infty} B_{i,n} = B_i\), it follows that \(B_i\) is a residual set in \(M_i\). Finally, by Lemma 3, for every \((G, t_i) \in B_i\), \(t_i\) do not admit any robust selection in \(G\). Thus, step 2 follows. ■

References


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